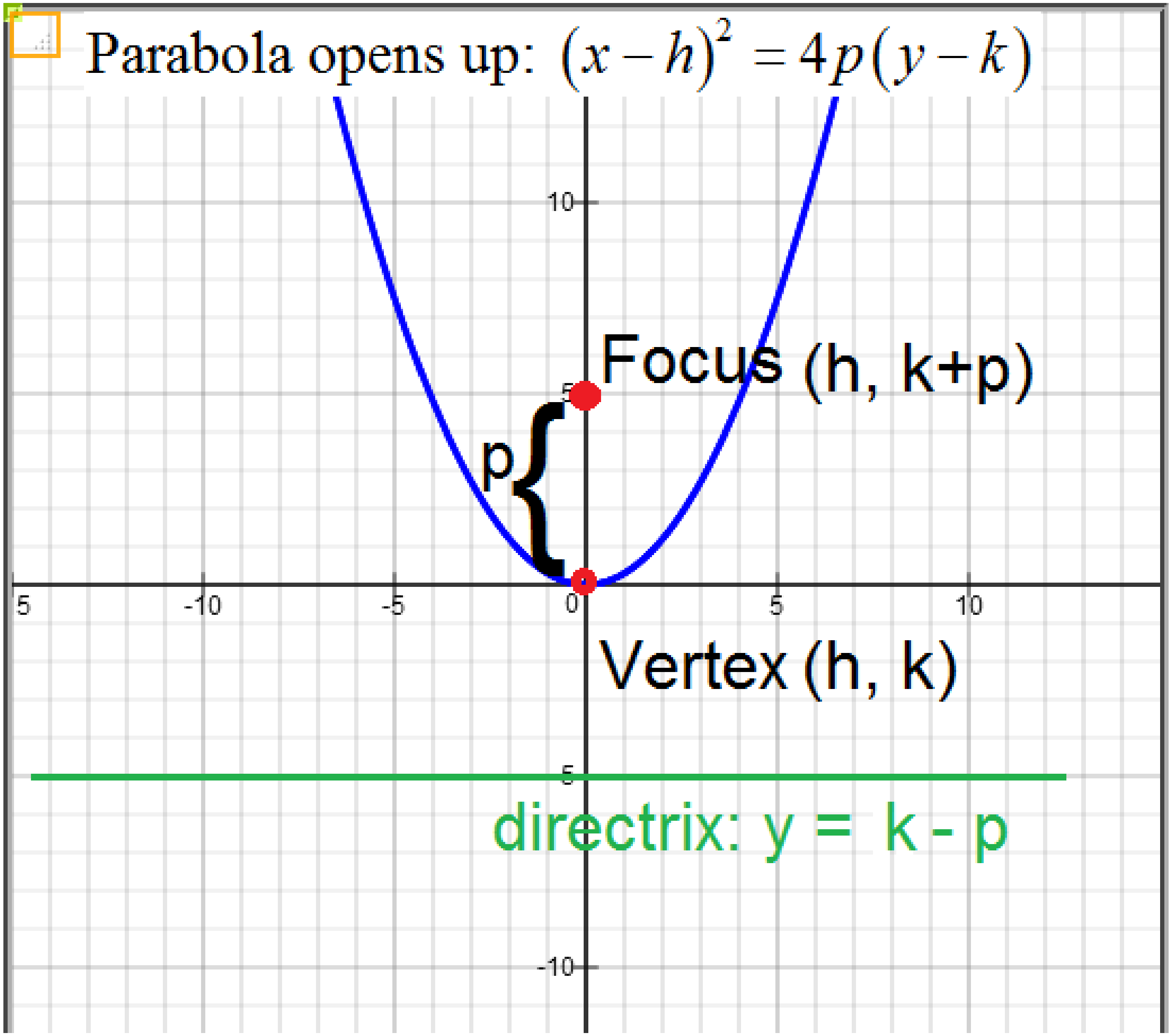
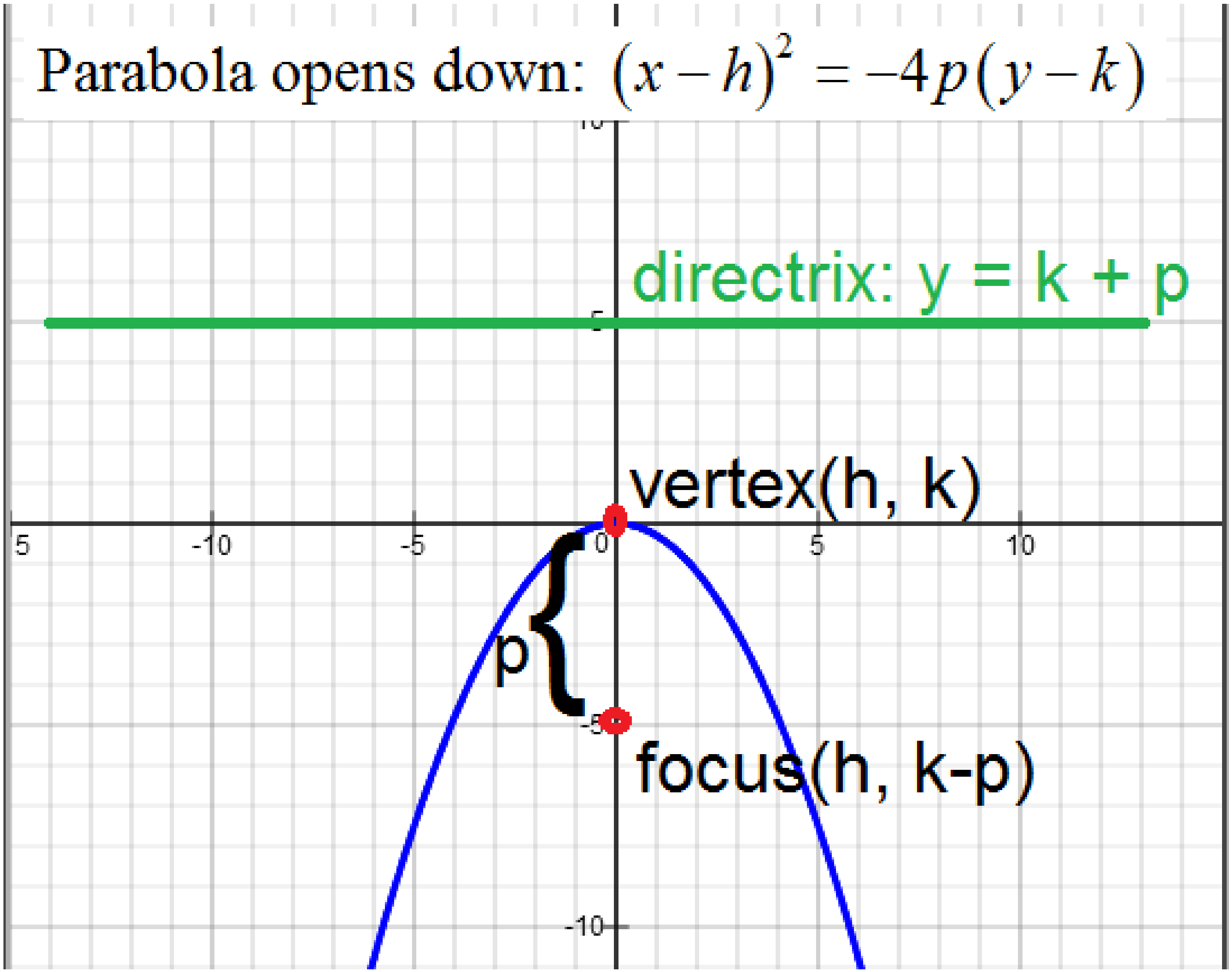
Calculus II Section 10.1 Notes

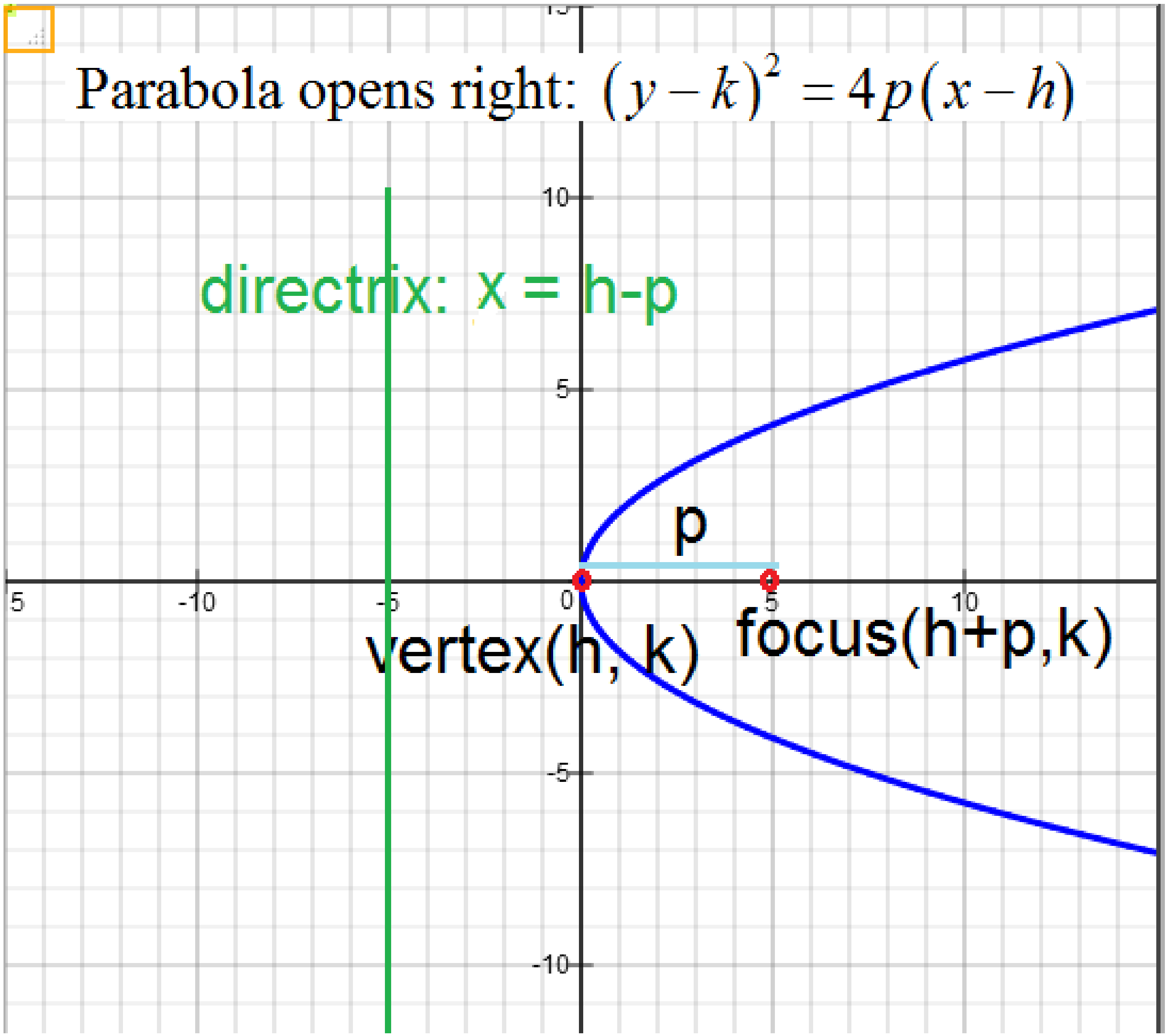
Conic Sections: Parabola, Ellipse, and Hyperbola



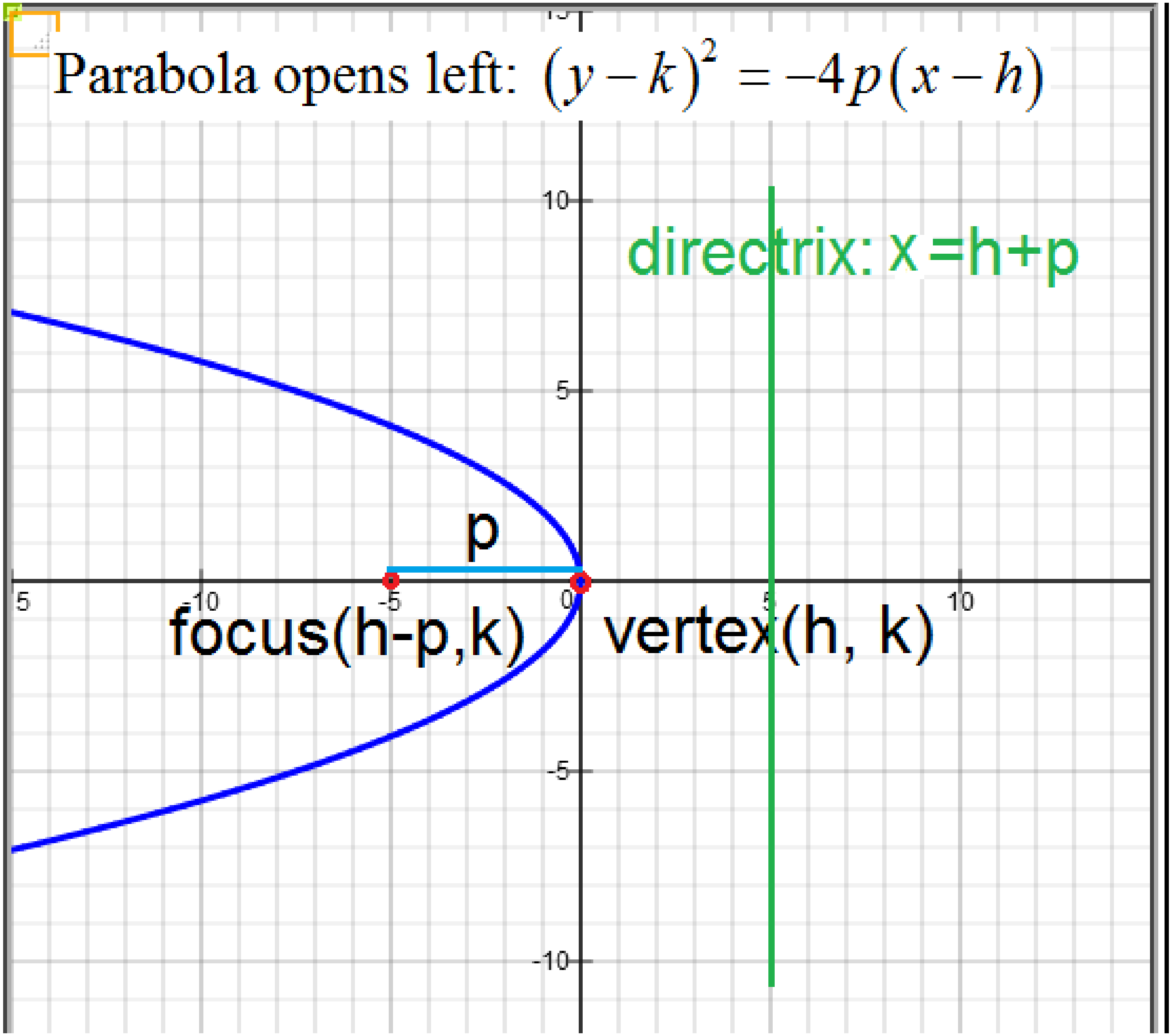
Parabola opens up: $(x-h)^2 = 4p(y-k)$
10
5
-10
-5
0
5
10
-5
-10
Focus (h, k+p)
p
Vertex (h, k)
directrix: y = k - p



Parabola opens down: $(x-h)^2 = -4p(y-k)$
directrix: y = k + p
vertex(h, k)
p
focus(h, k-p)
-10
-5
0
5
10



Parabola opens right: $(y-k)^2 = 4p(x-h)$
directrix: x = h-p
p
vertex(h, k)
focus(h+p,k)
10
5
-5
-10
5
-10
-5
0
5
10


Parabola opens left: $(y-k)^2 = -4p(x-h)$
directrix: x =h+p
p
focus(h-p,k)
vertex(h, k)
10
5
-5
-10
0
10

Equations of Parabola:

1) Parabola opens up: $(x-h)^2 = 4p(y-k)$

vertex $=(h,k)$; focus $= (h, k+p)$; directrix: $y = k - p$

2) Parabola opens down: $(x-h)^2 = -4p(y-k)$

vertex $=(h,k)$; focus $= (h, k-p)$; directrix: $y = k + p$

3) Parabola opens right: $(y-k)^2 = 4p(x-h)$

vertex $=(h,k)$; focus $= (h+p, k)$; directrix: $y = h - p$

4) Parabola opens left: $(y-k)^2 = -4p(x-h)$

vertex $=(h,k)$; focus $= (h-p, k)$; directrix: $y = h + p$

Let $y^2 = 8x$. Find vertex, focus, and directrix.

Stander Form: $(y-0)^2 = 4 \cdot 2(x-0)$

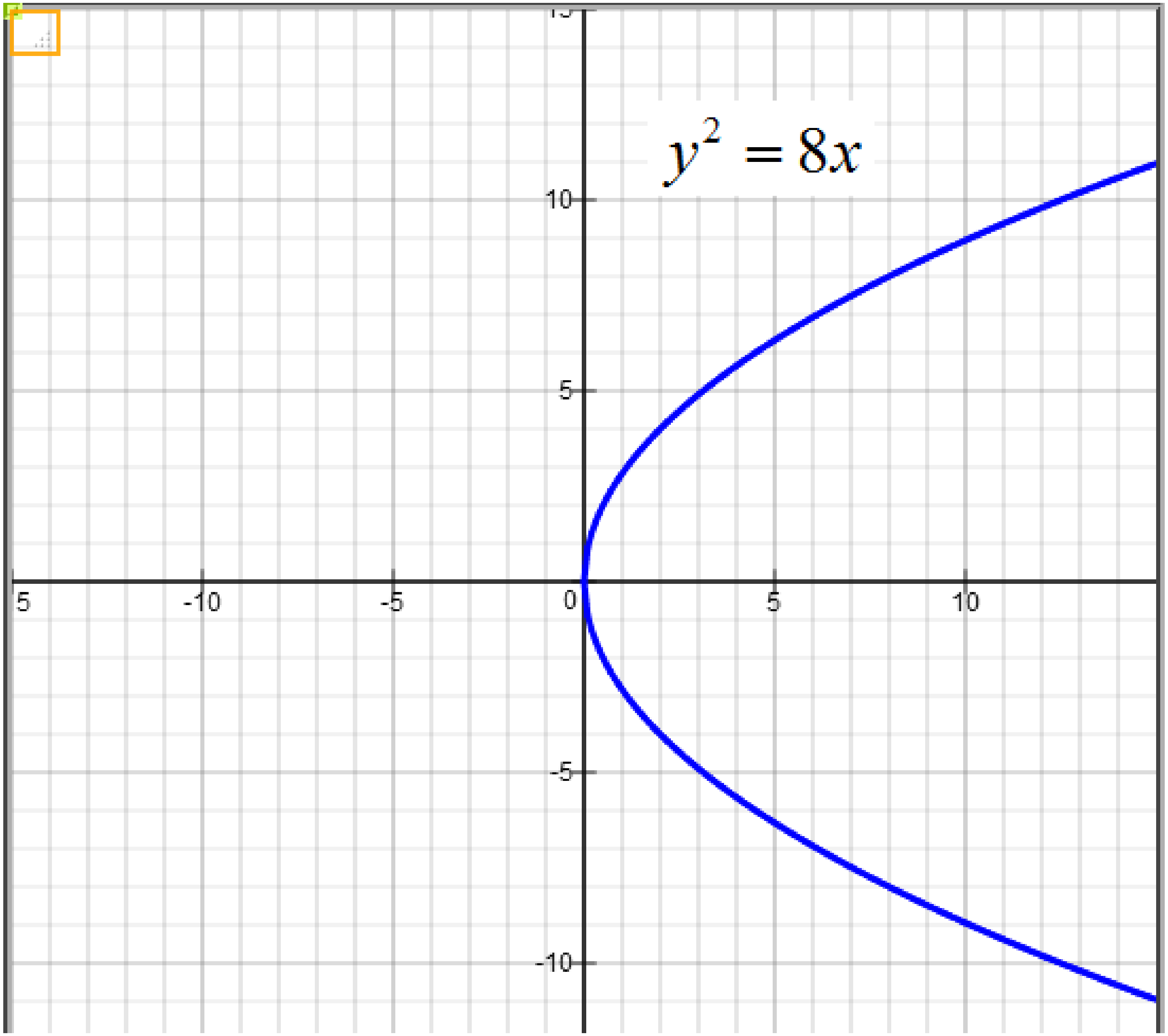
Parabola opens to the right

$p = 2$

a) Find the vertex: $(h, k) = (0, 0)$

b) Find the focus: $(h+p, k) = (0+2, 0)$

c) Find the directrix: $x = h - p \quad \Rightarrow \quad x = 0 - 2 = -2$



$y^2 = 8x$
10
5
-5
-10
5
-10
-5
0
5
10

Let $y^2 - 8y = 12x$. Find vertex, focus, and directrix.

Half of -8 = -4; $\left(-4\right)^2 = 16$

Add 16 to both sides:

$$y^2 - 8y + 16 = 12x + 16$$

$$\left(y-4\right)\left(y-4\right) = 12(x + 16/12)$$

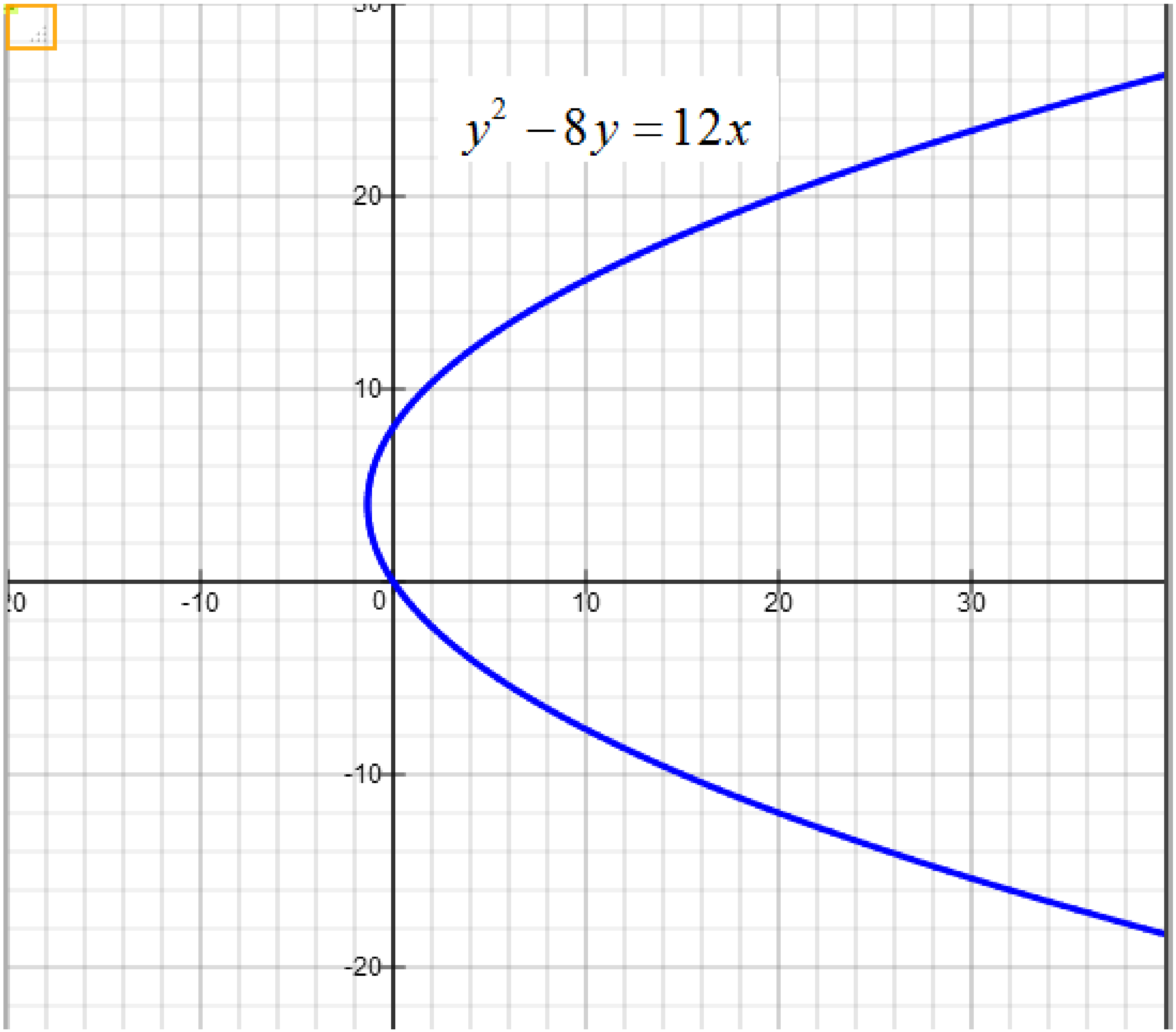
$\left(y-4\right)^2 = 4 \cdot 3(x + 4/3)$ Parabola opens to the right

$$p = 3$$

a) Find the vertex: $\left(h, k\right) = \left(-4/3,\ 4\right)$

b) Find the focus: $\left(h + p, k\right) = \left(-4/3+3,\ 4\right)$

c) Find the directrix: $x = h - p \quad \Rightarrow \quad x = -4/3 - 3$



$y^2 - 8y = 12x$
30
20
10
-10
-20
20
-10
0
10
20
30

Find vertex, focus, and directrix for $x^2 + 4x + 4y - 10 = 0$.

$x^2 + 4x = -4y + 10$ Rearrange equation

Note: half of 4 = 2; $(2)^2 = 4$; add 4 to both sides of equation

$x^2 + 4x + 4 = -4y + 10 + 4$

$(x+2)(x+2) = -4y + 14$

$(x+2)^2 = -4(y - 14/4)$

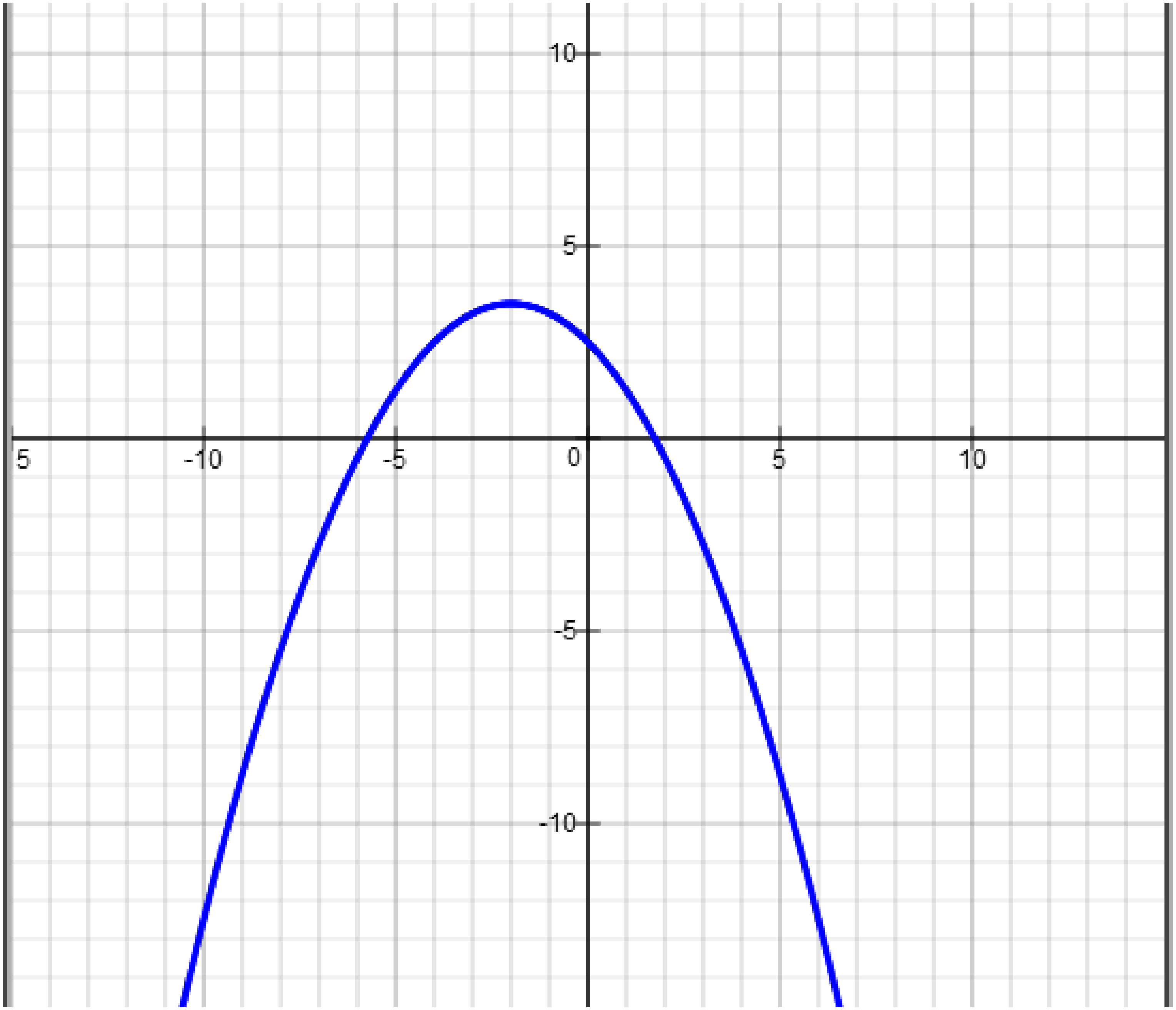
$(x+2)^2 = -4 \cdot 1(y - 7/2)$ Parabola opens down

$p = 1$

a) Find the vertex: $(h, k) = ($-2, 7/2$)$

b) Find the focus: $(h, k - p) = ($-2, 7/2-1$)$

c) Find the directrix: $y = k + p \Rightarrow y = 7/2+1$


10
5
-5
-10
5
-10
-5
0
5
10

Find an equation of the parabola that has the following vertex and focus: vertex: (4, 7) ; focus: (1, 7)

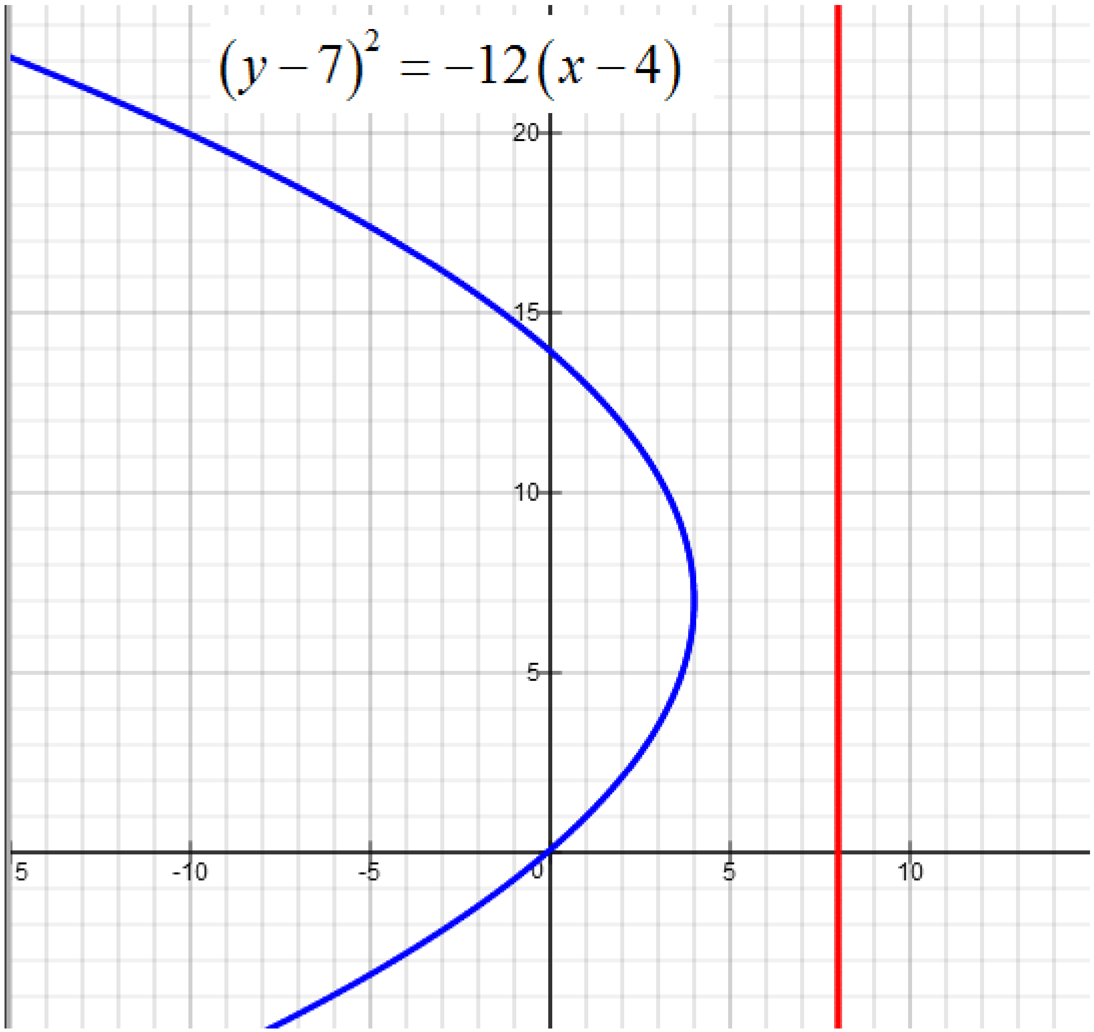
Note: Parabola opens left.

$p = 3 =$ distance from vertex to focus

Equation of parabola: $(y-k)^2 = -4p(x-h)$

$$(y-7)^2 = -4 \cdot 3(x-4)$$

$$(y-7)^2 = -12(x-4)$$


$(y-7)^2=-12(x-4)$
20
15
10
5
5
-10
-5
0
5
10

Find an equation of the parabola that has the following vertex and directrix: vertex: (5, 3) ; directrix: $x = 1$

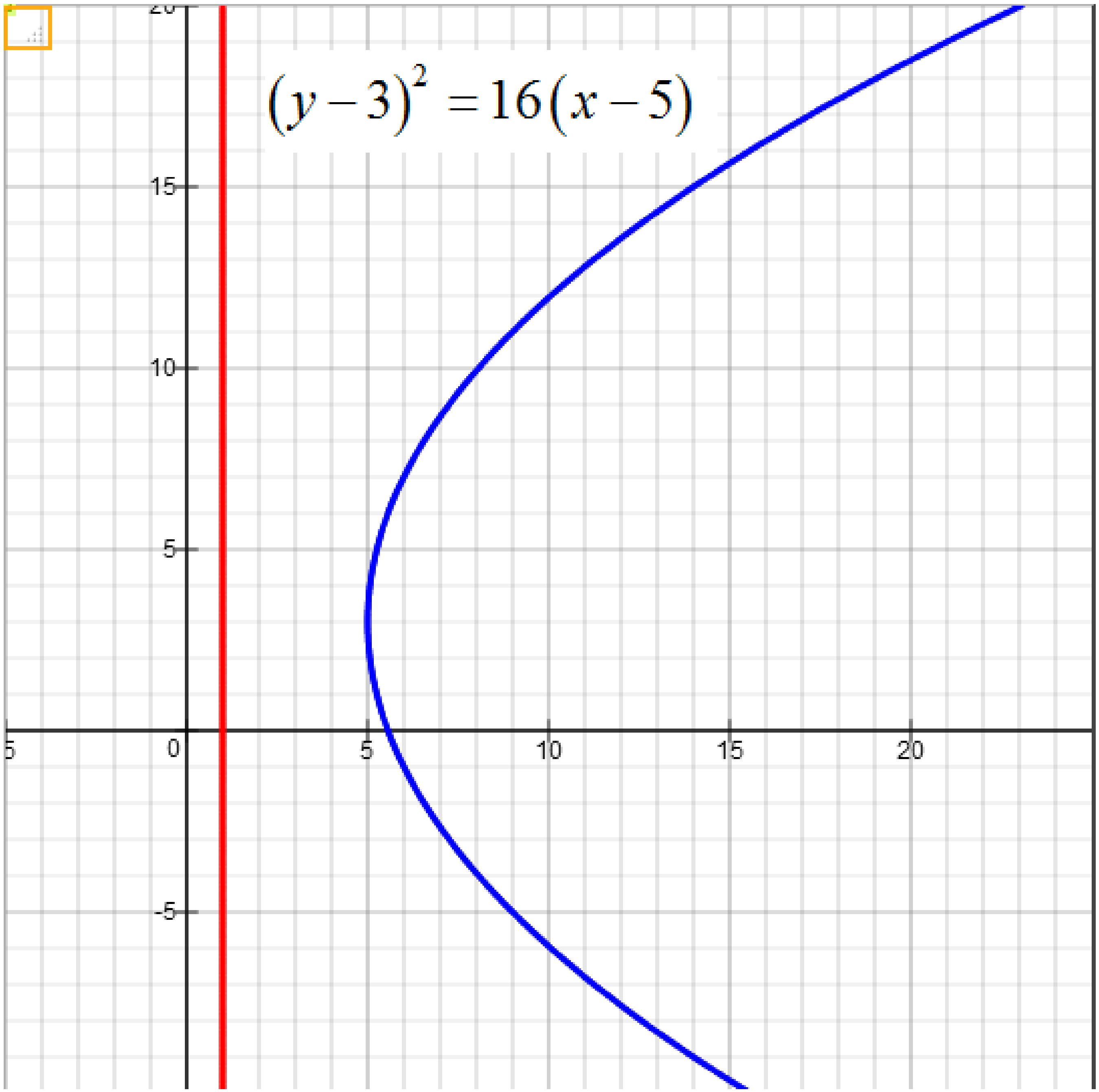
Note: Parabola opens to the right.

$p = 4 =$ distance from vertex to directrix

Equation of parabola: $(y-k)^2 = 4p(x-h)$

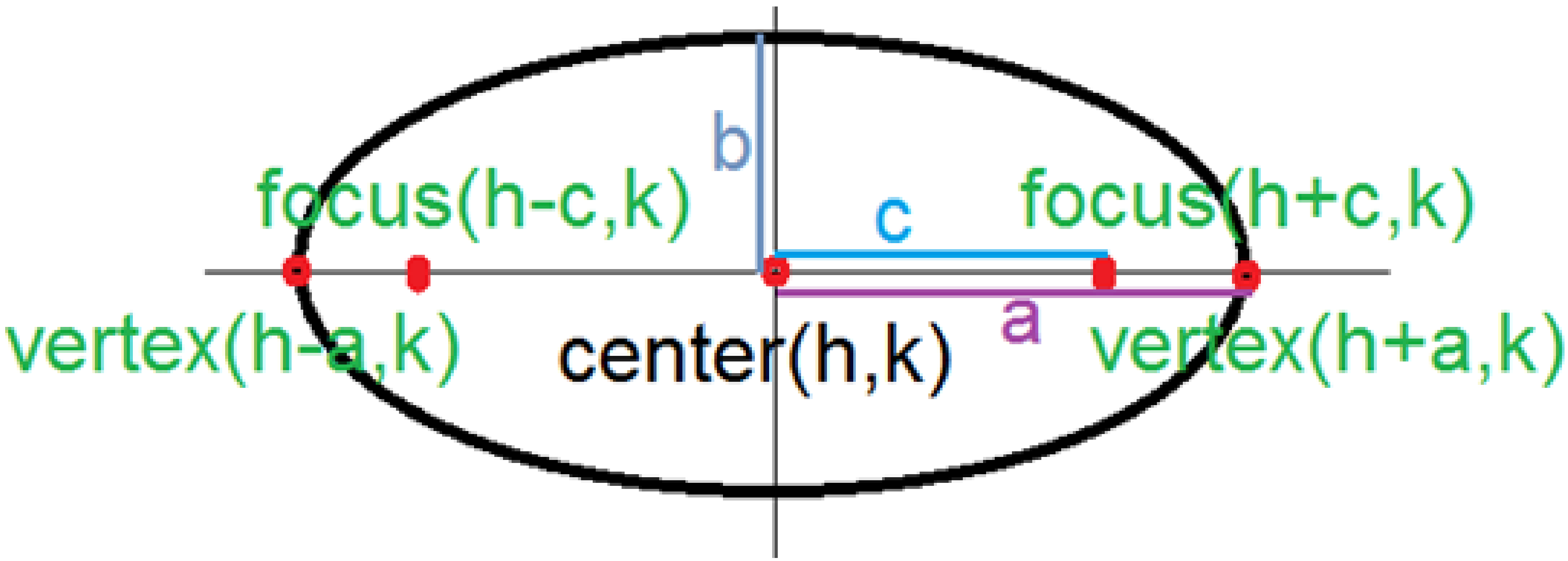
$$(y-3)^2 = 4 \cdot 4(x-5)$$

$$(y-3)^2 = 16(x-5)$$


$(y-3)^2 = 16(x-5)$
20
15
10
5
0
-5
5
10
15
20

$$a^2 = b^2 + c^2$$

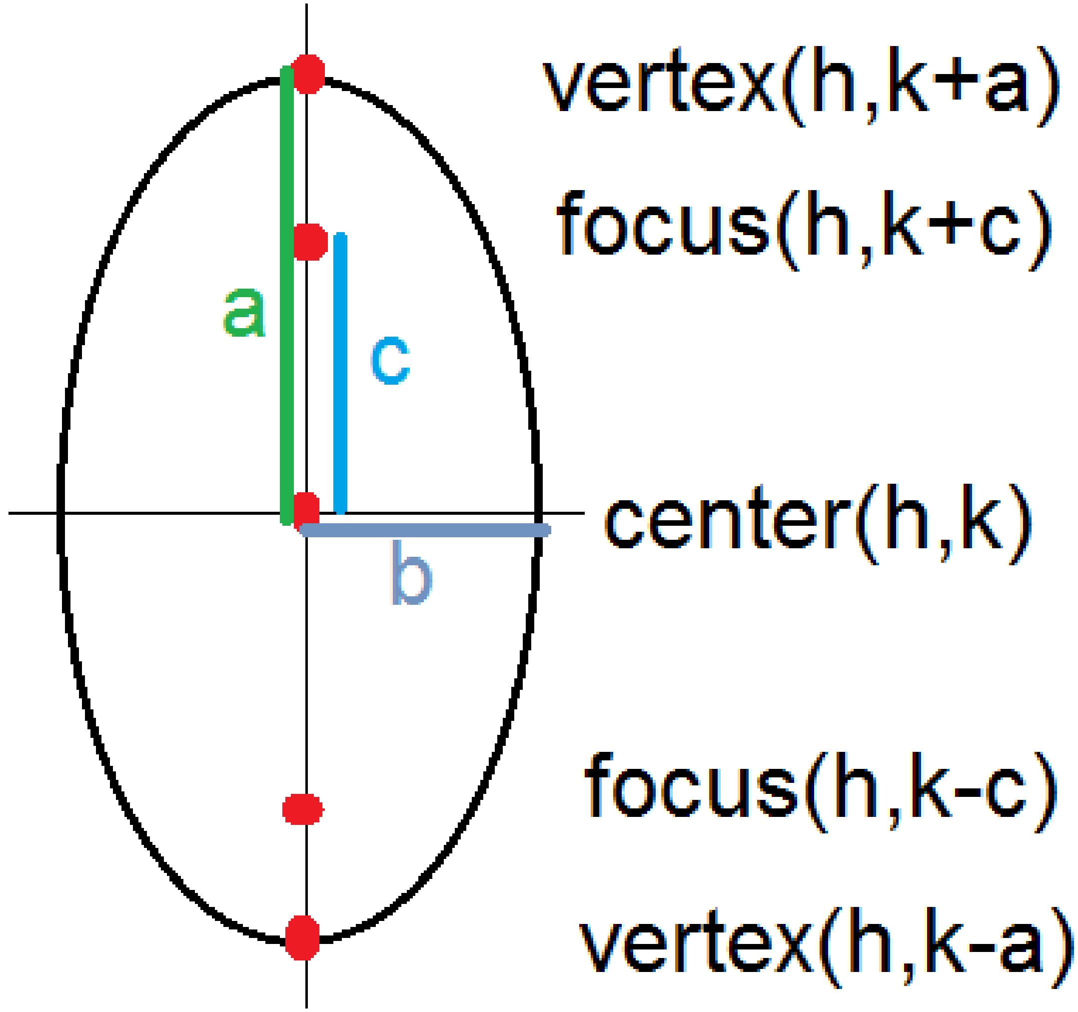
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



Ellipse Elongated Horizontally

$$a^2 = b^2 + c^2$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$$



Ellipse Elongated Vertically

Ellipse Equations

$$a^2 = b^2 + c^2$$

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1\,; \quad a > b \qquad \text{Elongated Horizontally}$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1; \quad a > b \qquad \text{Elongated Vertically}$$

$$\frac{x^2}{9}+\frac{(y-4)^2}{25}=1$$

$$a^2=25; \quad a=5; \qquad b^2=9; \quad b=3$$

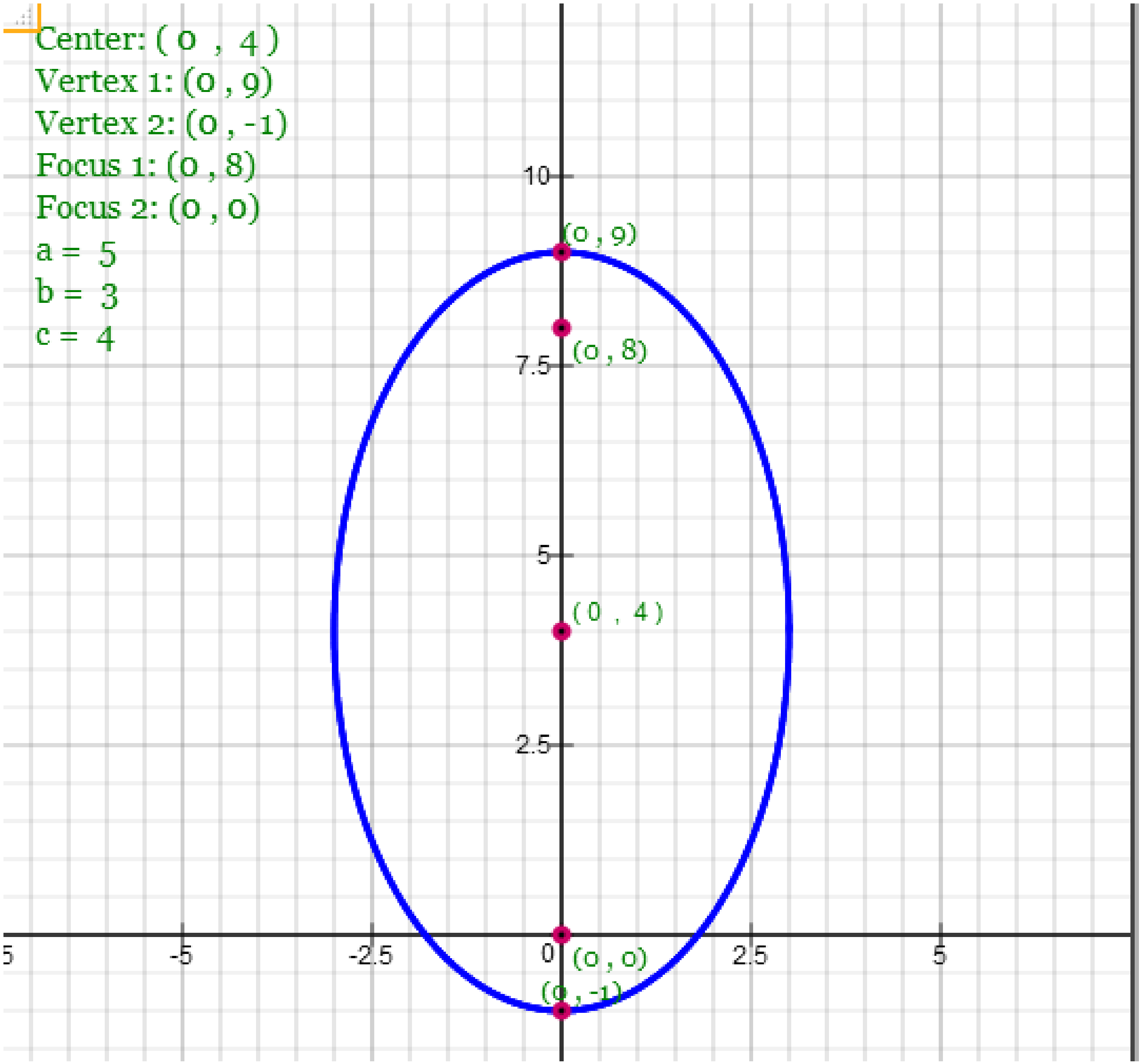
$$a^2=b^2+c^2 \quad \Rightarrow \quad 25=9+c^2 \quad \Rightarrow \quad c^2=16 \quad \Rightarrow \quad c=4$$

Note: a^2 is under $(y-4)^2$; ellipse elongates vertically

a) Find the center of the ellipse: $(h,k)=(0,4)$

b) Find the vertices: $(h,k\pm a)=(0,4\pm 5)$

c) Find the foci: $(h,k\pm c)=(0,4\pm 4)$


Center: (0 , 4)
Vertex 1: (0 , 9)
Vertex 2: (0 , -1)
Focus 1: (0 , 8)
Focus 2: (0 , 0)
a = 5
b = 3
c = 4
(0 , 9)
(0 , 8)
(0 , 4)
(0 , 0)
(0 , -1)
10
7.5
5
2.5
-5
-2.5
0
2.5
5

$$\frac{(x+2)^2}{25}+\frac{(y-3)^2}{12}=1$$

$a^2=25;\quad a=5;\quad b^2=12;\quad b=\sqrt{12}$

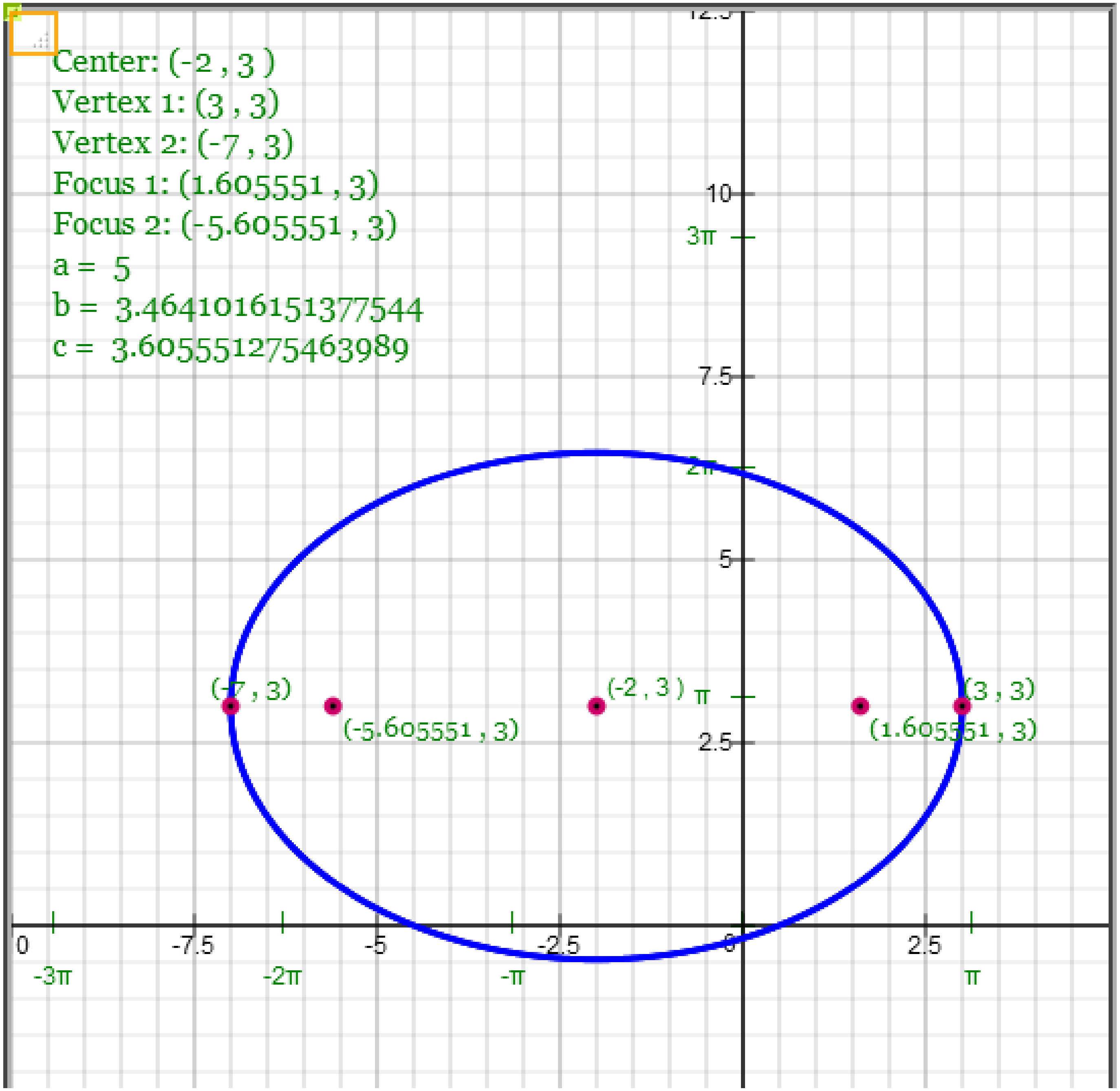
$a^2=b^2+c^2 \quad\Rightarrow\quad 25=12+c^2 \quad\Rightarrow\quad c^2=13 \quad\Rightarrow\quad c=\sqrt{13}$

Note: a^2 is under $(x+2)^2$; ellipse elongates horizontally

a) Find the center of the ellipse: $(h,k)=(-2,3)$

b) Find the vertices: $(h\pm a,k)=(-2\pm 5,3)$

c) Find the foci: $(h\pm c,k)=\left(-2\pm\sqrt{13},3\right)$


Center: (-2 , 3)
Vertex 1: (3 , 3)
Vertex 2: (-7 , 3)
Focus 1: (1.605551 , 3)
Focus 2: (-5.605551 , 3)
a = 5
b = 3.4641016151377544
c = 3.605551275463989
(-7 , 3)
(-5.605551 , 3)
(-2 , 3)
(3 , 3)
(1.605551 , 3)
10
3π
7.5
2π
5
π
2.5
0
-7.5
-5
-2.5
2.5
-3π
-2π
-π
π

Ellipse in General Form: $4x^2 + 9y^2 + 16x - 18y + 5 = 0$.

Find center, vertices, and foci.

$$4x^2 + 9y^2 + 16x - 18y + 5 = 0$$

$$\left(4x^2 + 16x\right) + \left(9y^2 - 18y\right) = -5$$

$$4\left(x^2 + 4x\right) + 9\left(y^2 - 2y\right) = -5$$

Half of 4 = 2; $\left(2\right)^2 = 4$; Half of -2 = -1; $\left(-1\right)^2 = 1$

$$4\left(x^2 + 4x + 4\right) + 9\left(y^2 - 2y + 1\right) = -5 + 4\left(4\right) + 9\left(1\right)$$

$$4\left(x + 2\right)\left(x + 2\right) + 9\left(y - 1\right)\left(y - 1\right) = 20$$

$$4\left(x + 2\right)^2 + 9\left(y - 1\right)^2 = 20$$

$$\frac{4\left(x + 2\right)^2}{20} + \frac{9\left(y - 1\right)^2}{20} = 1$$

Ellipse in General Form: $4x^2 + 9y^2 + 16x - 18y + 5 = 0$

$$\frac{(x+2)^2}{20/4} + \frac{(y-1)^2}{20/9} = 1$$

$$\frac{(x+2)^2}{5} + \frac{(y-1)^2}{20/9} = 1$$

$$a^2 = 5;\quad a = \sqrt{5};\quad b^2 = 20/9;\quad b = \sqrt{20/9}$$

$$a^2 = b^2 + c^2 \quad \Rightarrow\ 5 = 20/9 + c^2$$

$$\Rightarrow\ c^2 = 25/9 \quad \Rightarrow\ c = \sqrt{25/9} = 5/3$$

Note: a^2 is under $(x+2)^2$; ellipse elongates horizontally

a) Find the center of the ellipse: $(h,k) = (-2,1)$

b) Find the vertices: $(h \pm a, k) = (-2 \pm 5, 1)$

c) Find the foci: $(h \pm c, k) = (-2 \pm 5/3, 1)$

Find an equation of the ellipse that has the following:

Center:(0, 0) Vertex: (0, 5) ; Focus(0,3)

Note: Ellipse is elongated vertically.

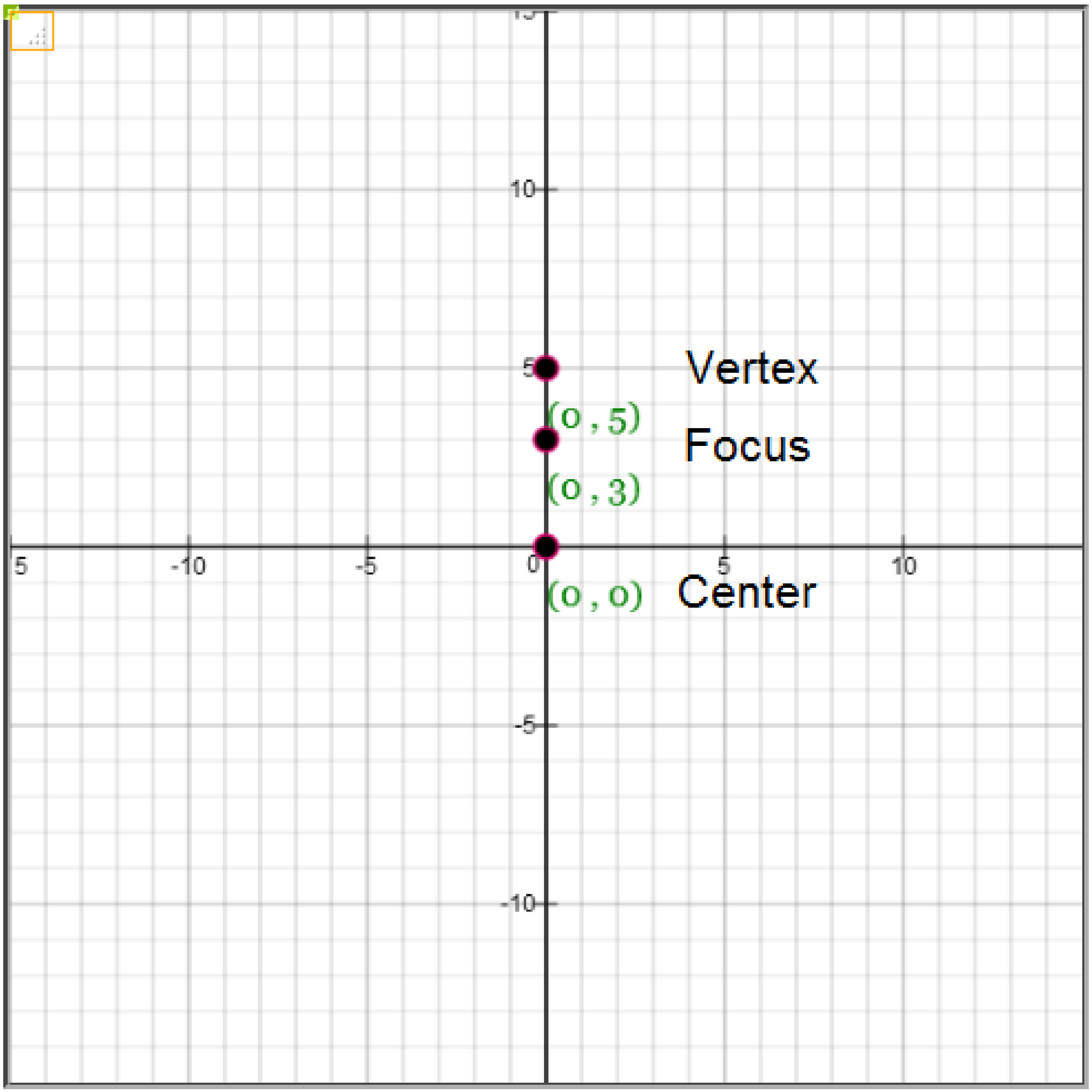
$a = 5 =$ distance from center to vertex

$c = 3 =$ distance from center to focus

$$a^2 = b^2 + c^2 \quad \Leftrightarrow \quad 25 = b^2 + 9 \quad \Leftrightarrow \quad b^2 = 16 \quad \Leftrightarrow \quad b = 4$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1; \quad a > b \quad \text{Elongated Vertically}$$

$$\frac{(x-0)^2}{16} + \frac{(y-0)^2}{25} = 1; \quad a > b \quad \text{Elongated Vertically}$$


Vertex
(0 , 5)
Focus
(0 , 3)
Center
(0 , 0)
10
5
-5
-10
5
-10
-5
0
5
10

Find an equation of the ellipse that has the following:

Vertices: (0, 10) and (0, 2); Eccentricity: $\dfrac{3}{5}$

Note: Ellipse is elongated vertically.

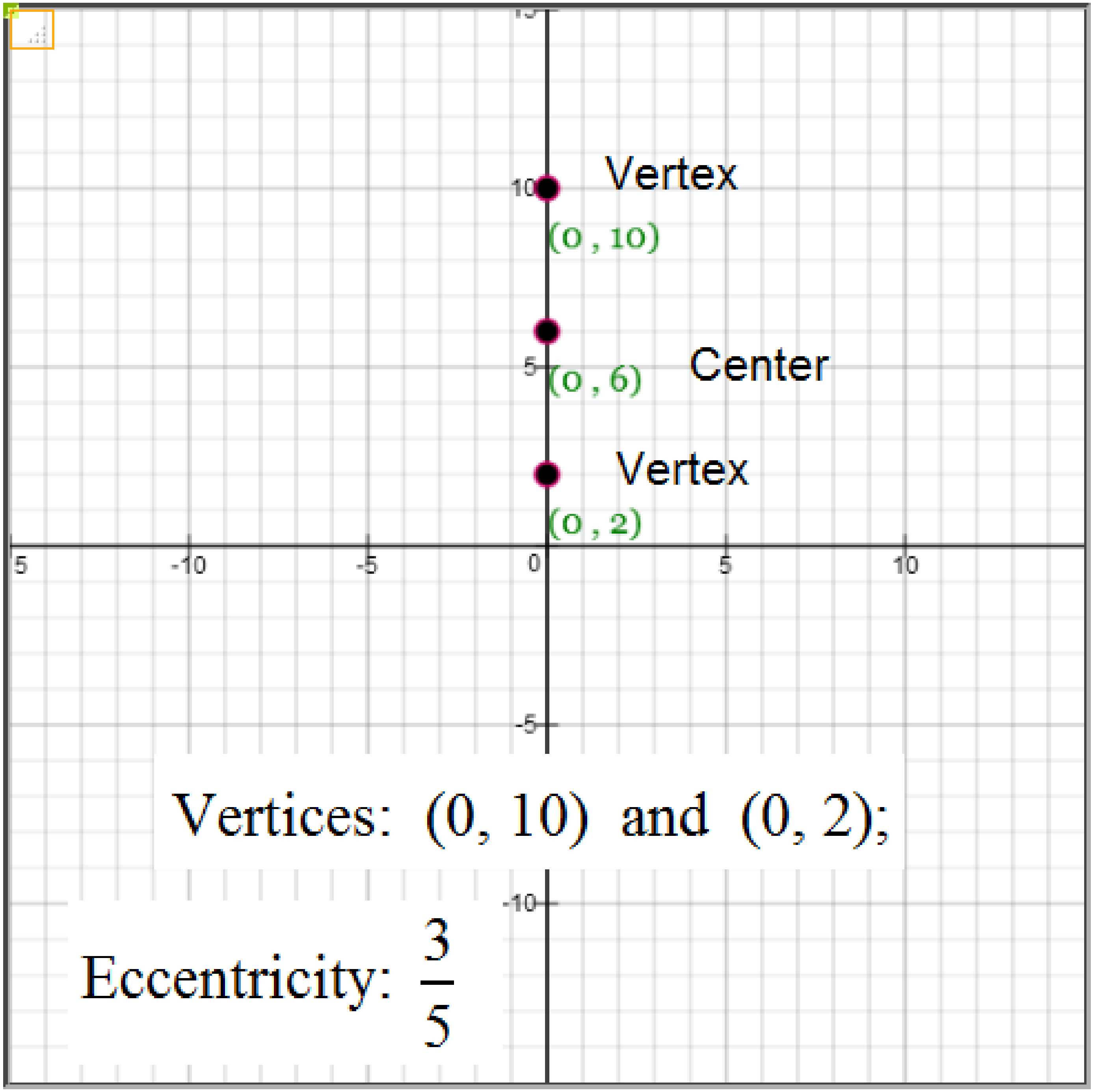
Center: (0, 6); $a = 4$ = distance from center to vertex

Eccentricity: $\dfrac{3}{5} = \dfrac{c}{a} \quad \Leftrightarrow \quad \dfrac{3}{5} = \dfrac{c}{4} \quad \Leftrightarrow \quad 5c = 12 \quad \Leftrightarrow \quad c = 12/5$

$$a^2 = b^2 + c^2 \quad \Leftrightarrow \quad 4^2 = b^2 + (12/5)^2 \quad \Leftrightarrow \quad b^2 = 256/25$$

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1; \quad a > b \quad \text{Elongated Vertically}$$

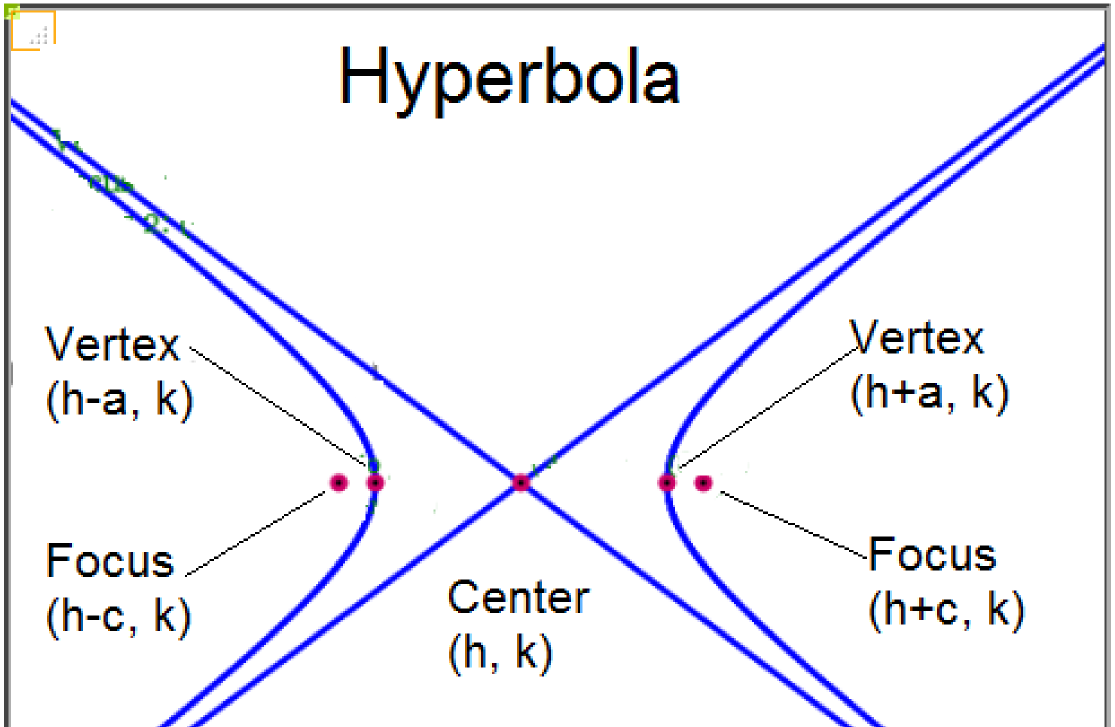
$$\frac{(x-0)^2}{256/25} + \frac{(y-6)^2}{16} = 1; \quad a > b \quad \text{Elongated Vertically}$$



Vertices: $(0, 10)$ and $(0, 2)$;

Eccentricity: $\dfrac{3}{5}$

Hyperbola with Branches Opening Left and Right



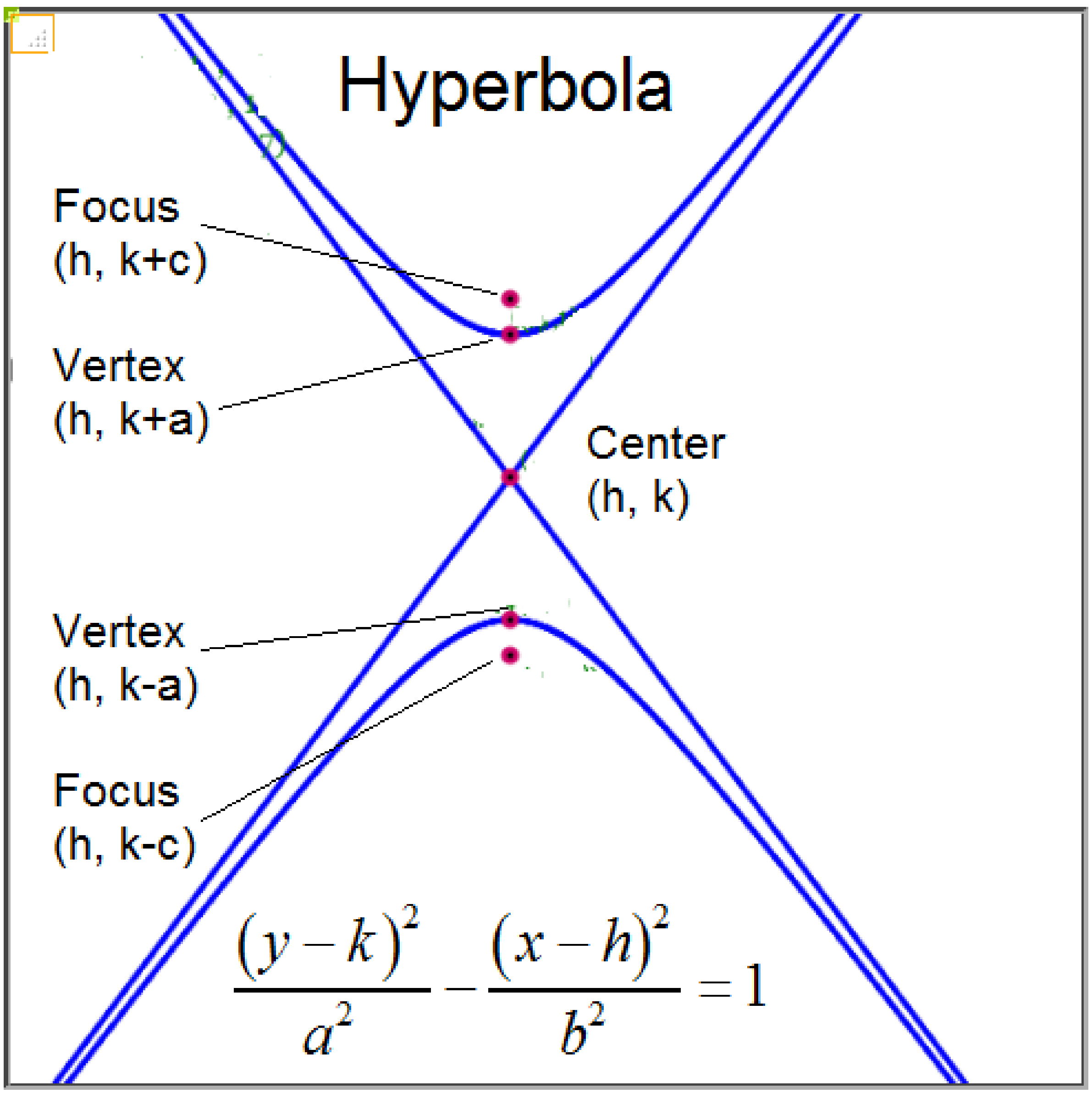
$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

$a =$ distance from center to vertex

$c =$ distance from center to focus

$c^2 = a^2 + b^2$

Hyperbola with Branches Opening Up and Down



Find center, vertices, and foci for $\dfrac{x^2}{16} - \dfrac{(y-4)^2}{25} = 1.$

$Note:$ Hyperbola has branches opening left and right.

$a^2 = 16;\ a = 4; \qquad b^2 = 25;\ b = 5$

$c^2 = a^2 + b^2 \iff c^2 = 16 + 25 \iff c^2 = 41 \iff c = \sqrt{41}$

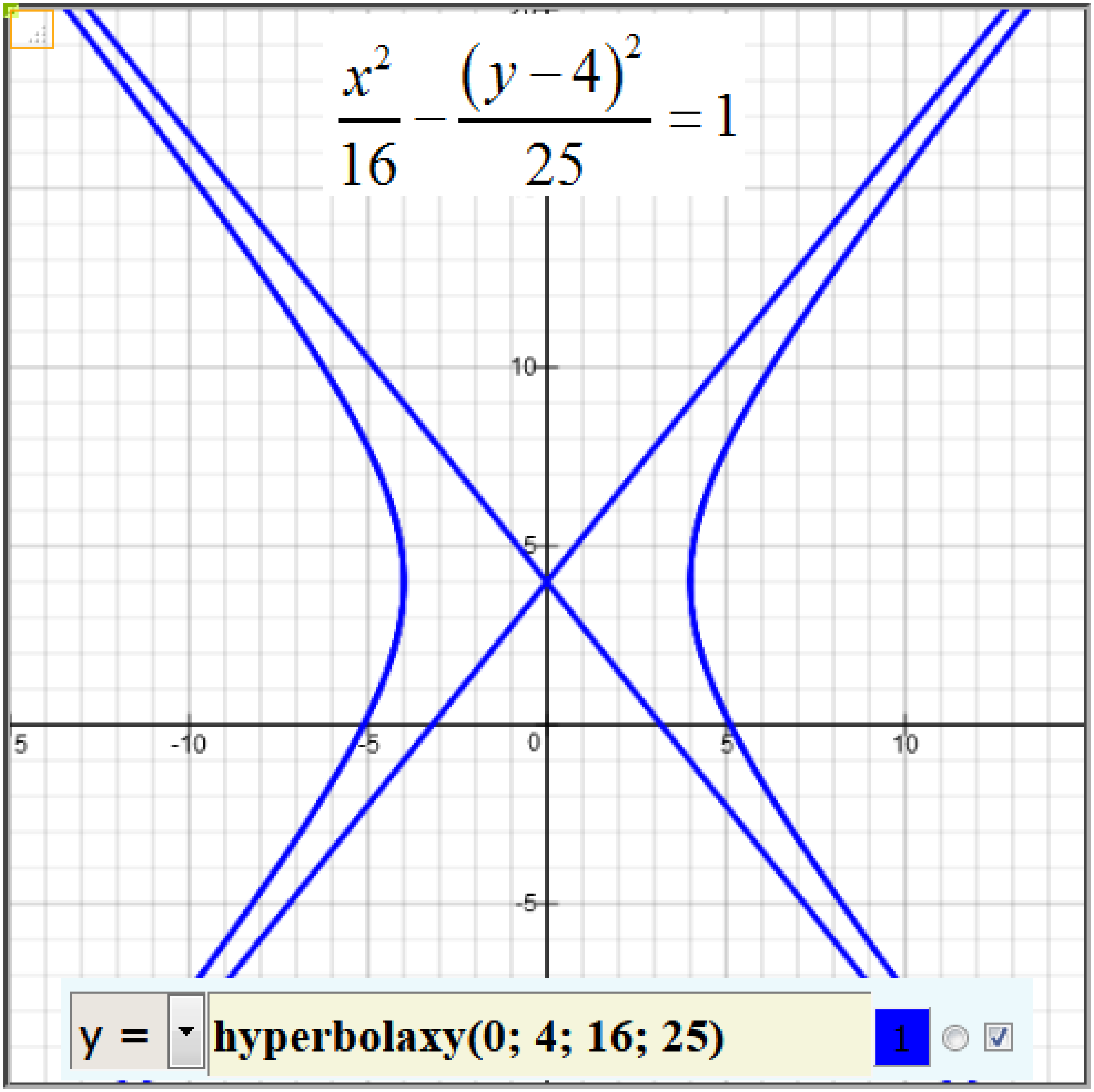
a) Find the center of the hyperbola: $(h,k) = (0,4)$

b) Find the vertices: $(h \pm a, k) = (0 \pm 4, 4)$

c) Find the foci: $(h \pm c, k) = \left(0 \pm \sqrt{41}, 4\right)$

d) Equations of asymptotes:

$$y = \pm \frac{b}{a}(x-h) + k \iff y = \pm \frac{5}{4}(x-0) + 4$$


$\frac{x^2}{16} - \frac{(y-4)^2}{25} = 1$
10
5
-5
-10
0
10
y =
hyperbolaxy(0; 4; 16; 25)
1

Find center, vertices, and foci for $\frac{(y+2)^2}{9} - \frac{(x-1)^2}{25} - = 1$

Note: Hypobola has branche opening up and down.

$a^2 = 9; \quad a = 3; \quad b^2 = 25; \quad b = 5$

$c^2 = a^2 + b^2 \quad \Leftrightarrow \quad c^2 = 9 + 25 \quad \Leftrightarrow \quad c^2 = 34 \quad \Leftrightarrow \quad c = \sqrt{34}$

a) Find the center of the hyperbola: $(h, k) = (1, -2)$

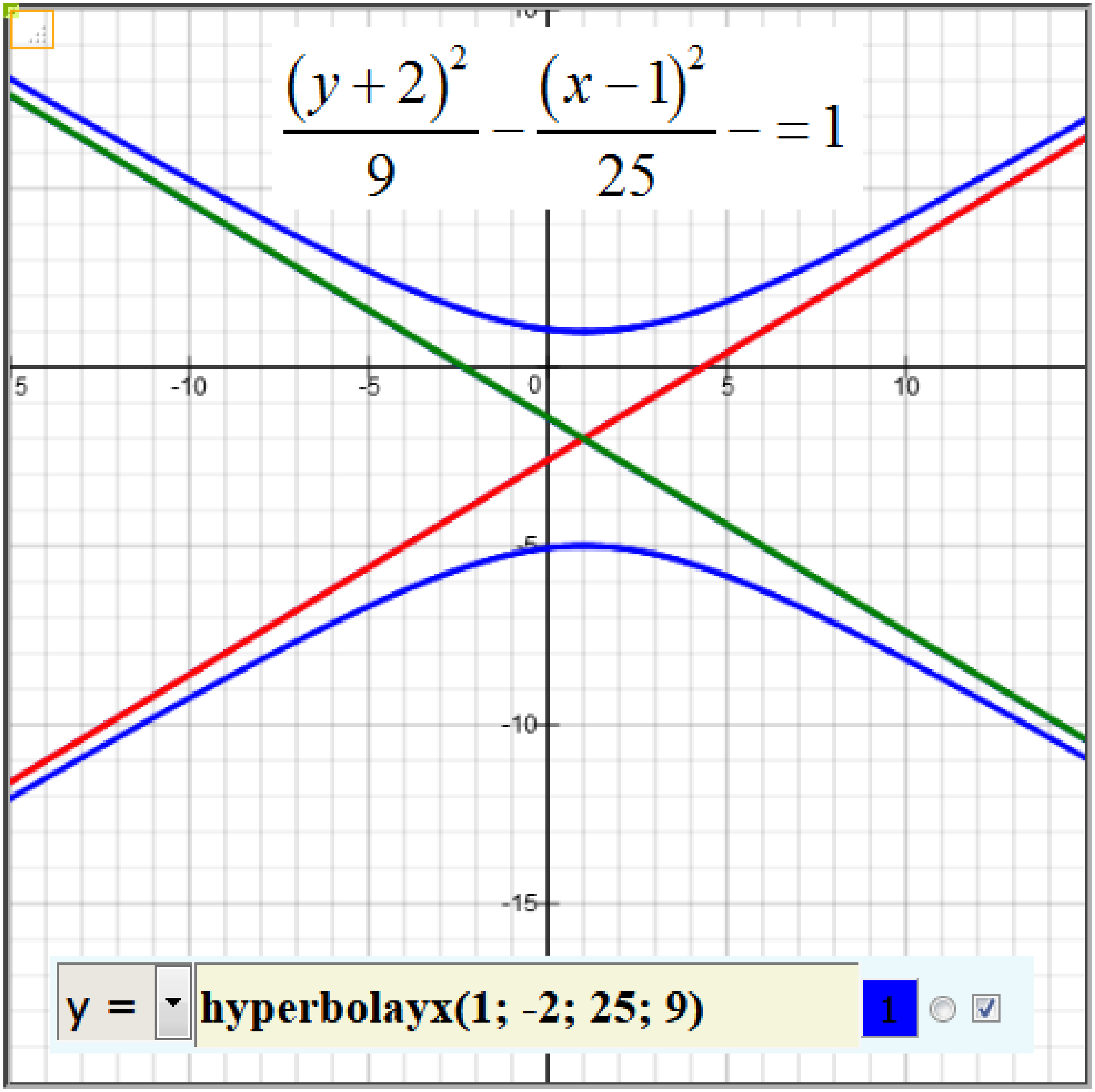
b) Find the vertices: $(h, k \pm a) = (1, -2 \pm 3)$

c) Find the foci: $(h, k \pm c) = \left(1, -2 \pm \sqrt{34}\right)$

d) Equations of Asymptotes:

$$y = \pm \frac{a}{b}(x - h) + k \Leftrightarrow y = \pm \frac{3}{5}(x - 1) + -2$$

$$\Leftrightarrow y = \pm \frac{3}{5}(x - 1) - 2$$



$$\frac{(y+2)^2}{9} - \frac{(x-1)^2}{25} = 1$$
-15
-10
-5
0
5
10
-10
-15
y =
hyperbolayx(1; -2; 25; 9)
1

Equation of hyperbola: $36x^2 - 72x + 36 - 16y^2 - 96y - 144 = 576$

Find center, vertices, and foci.

$$36x^2 - 72x + 36 - 16y^2 - 96y - 144 = 576$$

$$\left(36x^2 - 72x\right) + \left(-16y^2 - 96y\right) = 576 - 36 + 144$$

$$36\left(x^2 - 2x\right) - 16\left(y^2 + 6y\right) = 684$$

Half of $-2 = -1$; $(-1)^2 = 1$; Half of 6 = 3; $(3)^2 = 9$

$$36\left(x^2 - 2x + 1\right) - 16\left(y^2 + 6y + 9\right) = 684 + 36(1) + -16(9)$$

$$36(x-1)(x-1) - 16(y+3)(y+3) = 576$$

$$36(x-1)^2 - 16(y+3)^2 = 576$$

$$\frac{36(x-1)^2}{576} - \frac{16(y+3)^2}{576} = 1 \quad \Leftrightarrow \quad \frac{(x-1)^2}{16} - \frac{(y+3)^2}{36} = 1$$

$$\frac{36(x-1)^2}{576} - \frac{16(y+3)^2}{576} = 1$$

$$\frac{(x-1)^2}{16} - \frac{(y+3)^2}{36} = 1$$

Note: Hyperbola has branches opening left and right.

$a^2 = 16;\ a = 4; \qquad b^2 = 36;\ b = 6$

$c^2 = a^2 + b^2 \Leftrightarrow c^2 = 16 + 36 \Leftrightarrow c^2 = 52 \Leftrightarrow c = \sqrt{52}$

a) Find the center of the hyperbola: $(h,k) = (1,-3)$

b) Find the vertices: $(h \pm a, k) = (1 \pm 4, -3)$

c) Find the foci: $(h \pm c, k) = \left(1 \pm \sqrt{52}, -3\right)$

d) Equations of asymptotes:

$$y = \pm\frac{b}{a}(x-h) + k \Leftrightarrow y = \pm\frac{6}{4}(x-1) + \text{-}3$$

$$\Leftrightarrow y = \pm\frac{3}{2}(x-1) - 3$$

Find an equation of the hyperbola that has the following:

Center:(0, 0) Vertex: (4, 0) ; Focus(6, 0)

Note: Hyperbola has branches opening left and right.

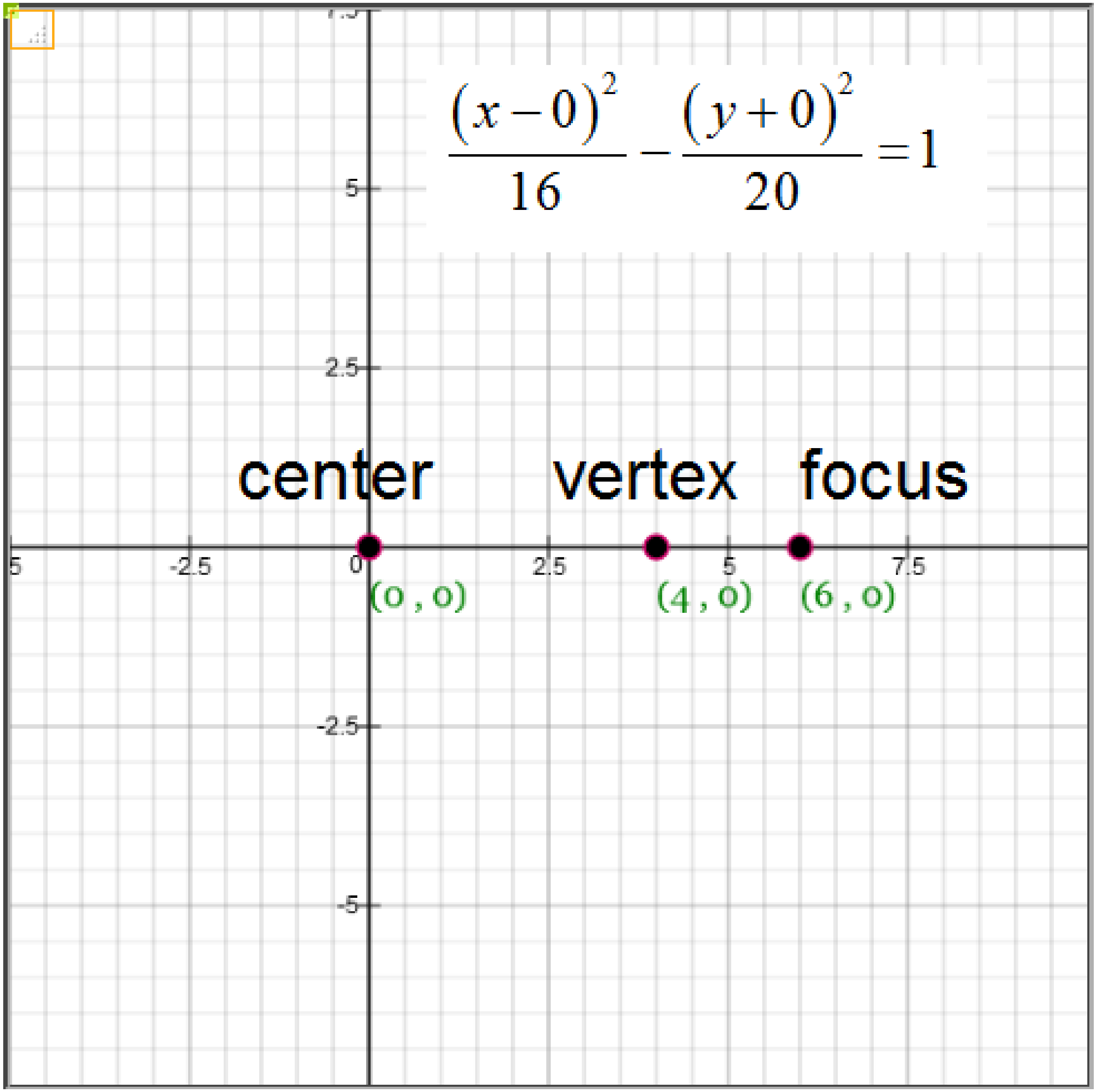
$a = 4 =$ distance from center to vertex

$c = 6 =$ distance from center to focus

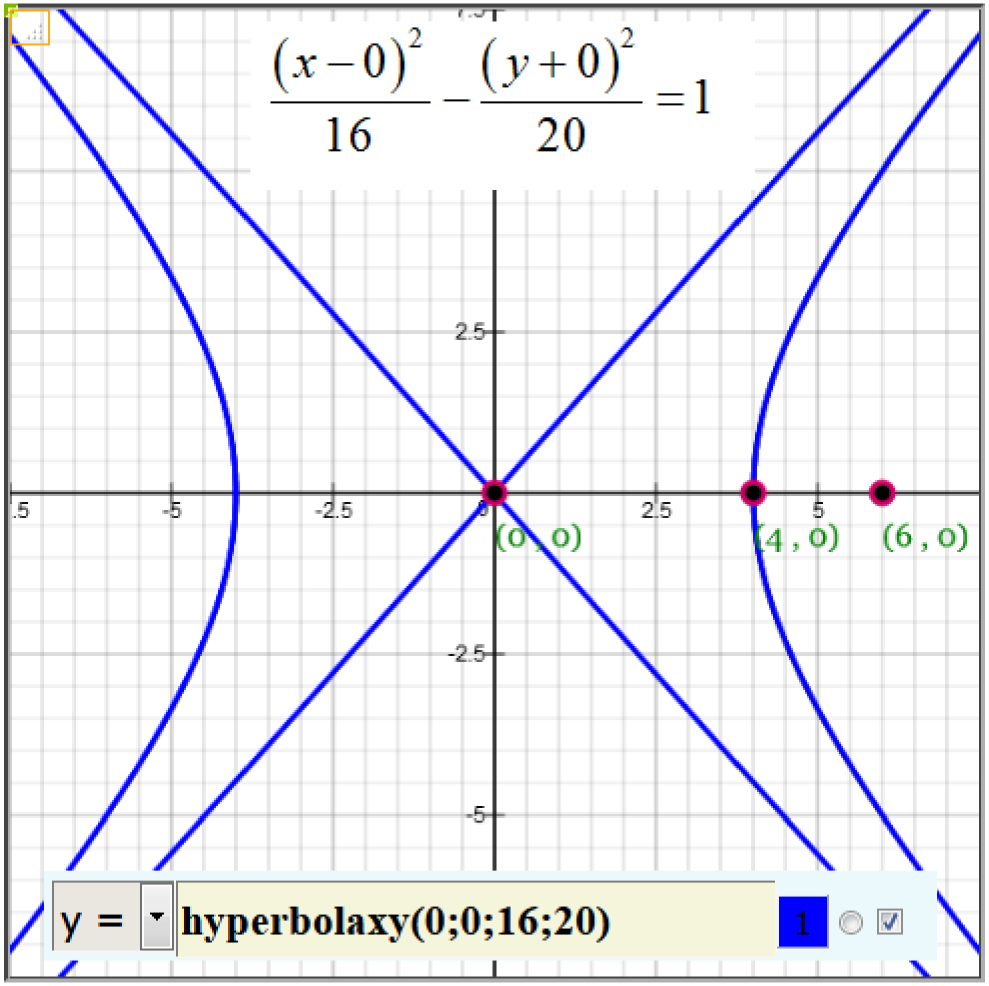
$c^2 = a^2 + b^2 \quad \Leftrightarrow \quad 36 = 16 + b^2 \quad \Leftrightarrow \quad b^2 = 20$

Equation of hyperbola: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$

$$\frac{(x-0)^2}{16} - \frac{(y+0)^2}{20} = 1$$



$$\frac{(x-0)^2}{16}-\frac{(y+0)^2}{20}=1$$
center
vertex
focus
(0 , 0)
(4 , 0)
(6 , 0)
7.5
5
2.5
0
-2.5
-5
2.5
5
7.5



$$\frac{(x-0)^2}{16} - \frac{(y+0)^2}{20} = 1$$
2.5
-2.5
-5
-5
-2.5
2.5
5
(0 , 0)
(4 , 0)
(6 , 0)
y =
hyperbolaxy(0;0;16;20)
1

Find an equation of the hyperbola that has the following:

Vertices: (0, 10) and (0, 4); Foci: (0,12), (0,2)

Note: Hyperbola has branches opening up and down.

Center: (0, 7)

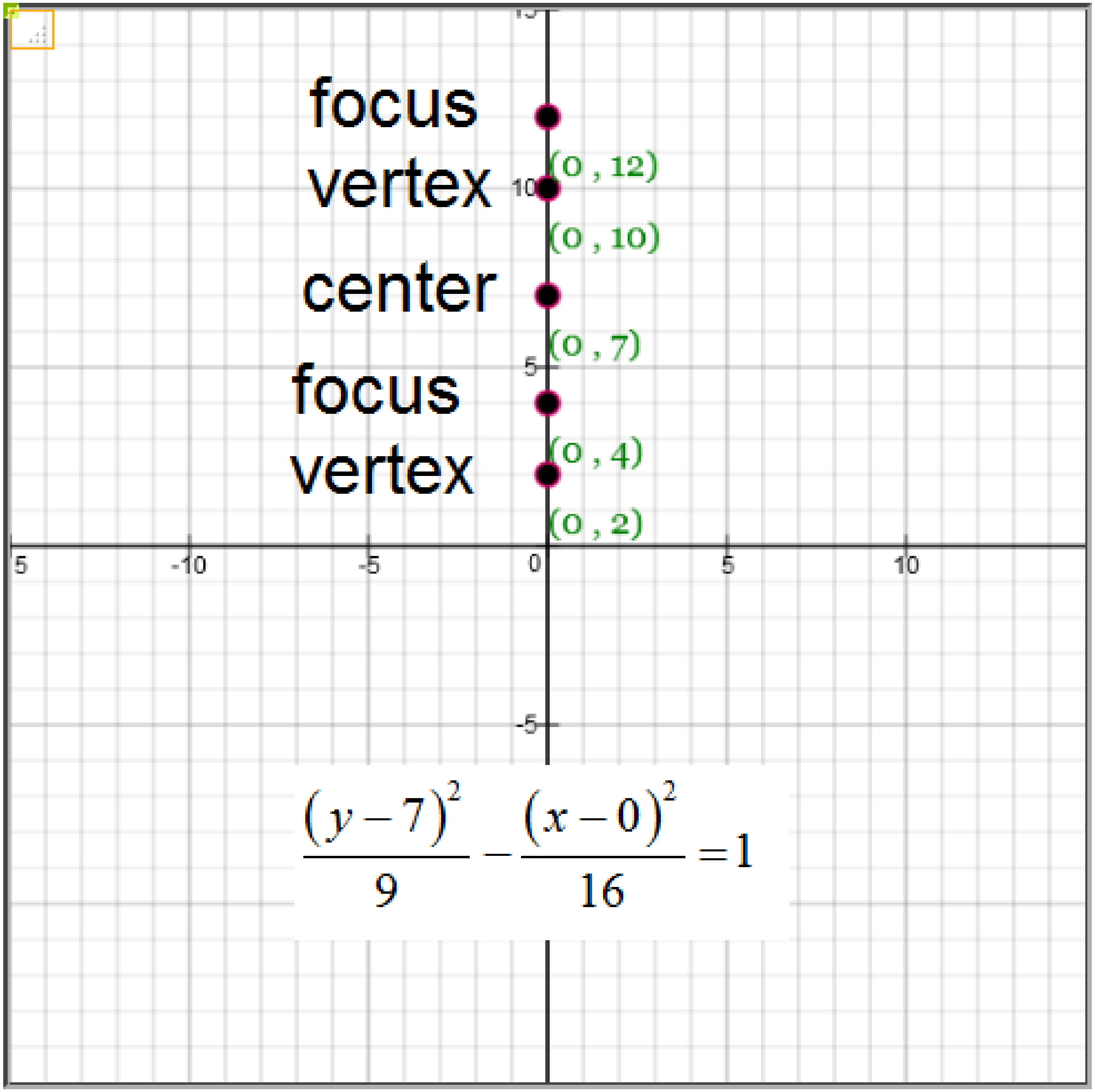
$a = 3 =$ distance from center to vertex

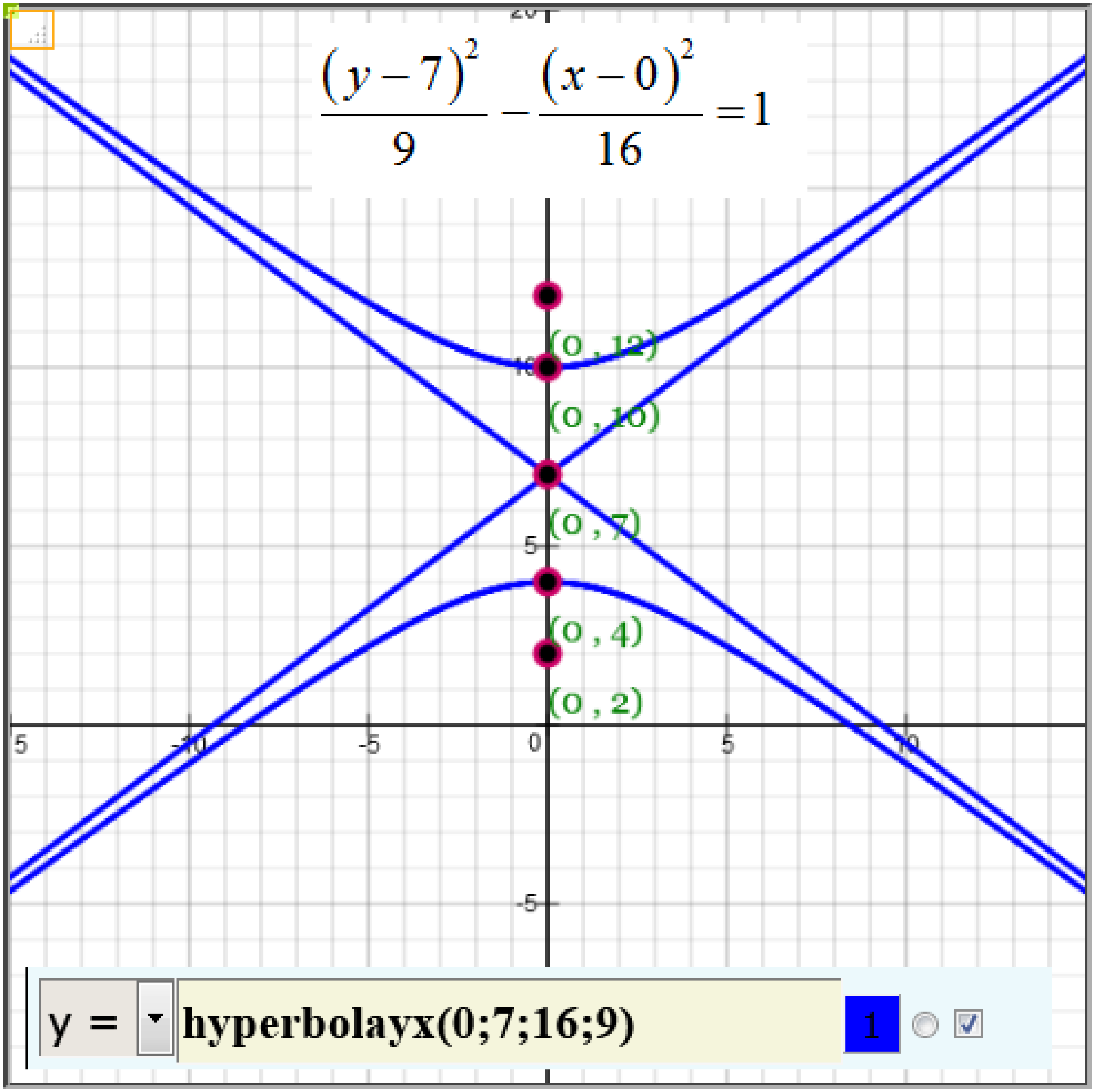
$c = 5 =$ distance from center to focus

$c^2 = a^2 + b^2 \quad \Leftrightarrow \quad 25 = 9 + b^2 \quad \Leftrightarrow \quad b^2 = 16$

Equation of hyperbola: $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$

$$\frac{(y-7)^2}{9} - \frac{(x-0)^2}{16} = 1$$


focus
vertex
center
focus
vertex
(0 , 12)
(0 , 10)
(0 , 7)
(0 , 4)
(0 , 2)
$\frac{(y-7)^2}{9}-\frac{(x-0)^2}{16}=1$



$$\frac{(y-7)^2}{9}-\frac{(x-0)^2}{16}=1$$
(0 , 12)
(0 , 10)
(0 , 7)
(0 , 4)
(0 , 2)
y = hyperbolayx(0;7;16;9)