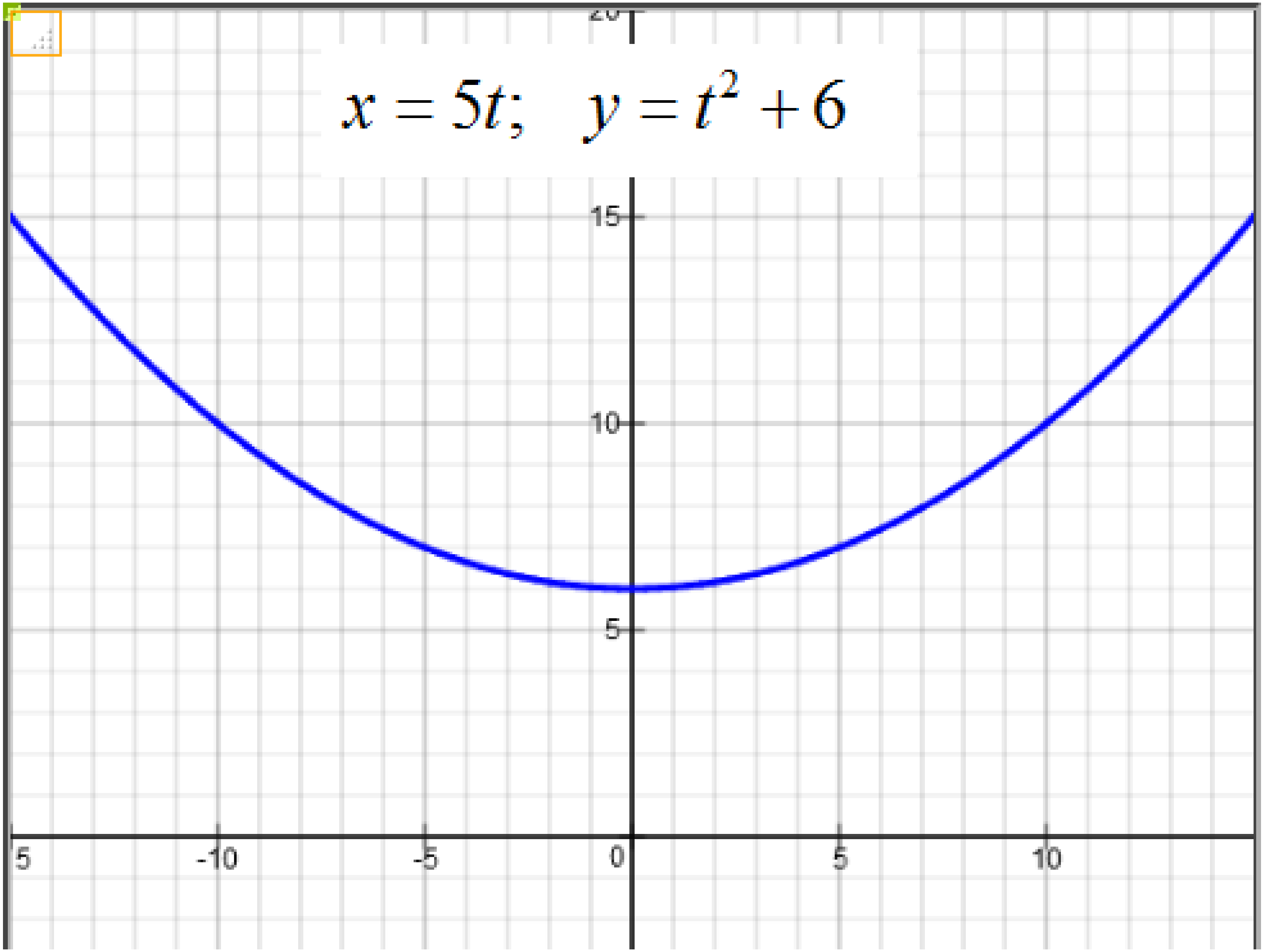
Parametric Equations: $x = f(t),\ \ y = g(t)$

Example: $x = 5t;\quad y = t^2 + 6$

Table of Values for Parametric Equations

t	$X(t) = 5t$	$Y(t) = t^2 + 6$
-4	-20	22
-3	-15	15
-2	-10	10
-1	-5	7
0	0	6
1	5	7
2	10	10
3	15	15
4	20	22


$x = 5t; \quad y = t^2 + 6$
15
10
5
5
-10
-5
0
5
10

Parametric Form of the Derivative:

a) $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

b) $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt}$$

Example 1:

Parametric Equations: $x = 4t^2, \quad y = 4 - 3t$

Find $y' = \dfrac{dy}{dx}$ = Derivative of y with respect to x.

We can find $\dfrac{dy}{dx}$ by using the Direct Method or Chain Rule Method.

Direct Method:

$$x = 4t^2 \Rightarrow t^2 = \frac{x}{4} \Rightarrow \sqrt{t^2} = \pm\sqrt{\frac{x}{4}} \Rightarrow t = \pm\frac{\sqrt{x}}{2}$$

$$y = 4 - 3t \Rightarrow y = 4 - 3\left(\pm\frac{\sqrt{x}}{2}\right) \Rightarrow y = 4 \pm \frac{3\sqrt{x}}{2} \Rightarrow y = 4 \pm \frac{3}{2}x^{1/2}$$

$$y' = \pm\frac{3}{2}\left(\frac{1}{2}x^{-1/2}\right) = \pm\frac{3}{4}x^{-1/2} = \pm\frac{3}{4x^{1/2}} = \pm\frac{3}{4\sqrt{x}}$$

Parametric Equations: $x = 4t^2, \quad y = 4 - 3t$

Chain Rule Method:

Note: $x = 4t^2 \Rightarrow t^2 = \dfrac{x}{4} \Rightarrow \sqrt{t^2} = \pm\sqrt{\dfrac{x}{4}} \Rightarrow t = \pm\dfrac{\sqrt{x}}{2}$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3}{8t} = \frac{-3}{8\left(\pm\dfrac{\sqrt{x}}{2}\right)} = \frac{-3}{\pm 4\sqrt{x}} = \pm\frac{3}{4\sqrt{x}}$$

Parametric Equations: $x = 4t^2, \quad y = 4 - 3t$

We can also find y'' by usng the Chain Rule Method:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-3}{8t} = \frac{-3}{8\left(\pm\frac{\sqrt{x}}{2}\right)} = \frac{-3}{\pm 4\sqrt{x}} = \pm\frac{3}{4\sqrt{x}}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\frac{d}{dt}\left[\frac{-3}{8t}\right]}{8t} = \frac{\frac{d}{dt}\left[-\frac{3}{8}t^{-1}\right]}{8t} = \frac{\frac{3}{8}t^{-2}}{8t} = \frac{3}{64t^3}$$

$$= \frac{3}{64\left(\pm\frac{3}{4\sqrt{x}}\right)^3}$$

Example 2:

Parametric Equations: $x = 4\sqrt{t}, \quad y = 2t$

Find $y' = \dfrac{dy}{dx}$

$Note: x = 4\sqrt{t} = 4t^{1/2}$

Find $y' = \dfrac{dy}{dx}$ by using the Direct Method.

$$x = 4\sqrt{t} \Rightarrow (x)^2 = \left(4\sqrt{t}\right)^2 \Rightarrow x^2 = 16t \Rightarrow t = \frac{x^2}{16}$$

$$y = 2t \Rightarrow y = 2\left(\frac{x^2}{16}\right) \Rightarrow y = \frac{1}{8}x^2$$

$$y' = \frac{1}{8}(2x) = \frac{1}{4}x$$

Example 2:

Parametric Equations: $x = 4\sqrt{t}, \quad y = 2t$

Find $y' = \dfrac{dy}{dx}$ and y'' by using the Chain Rule Method.

$$x = 4\sqrt{t} \Rightarrow (x)^2 = \left(4\sqrt{t}\right)^2 \Rightarrow x^2 = 16t \Rightarrow t = \frac{x^2}{16}$$

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2}{2t^{-1/2}} = \frac{1}{t^{-1/2}} = t^{1/2} = \left(\frac{x^2}{16}\right)^{1/2} = \frac{x}{4}$$

$$y'' = \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{dx/dt} = \frac{\frac{d}{dt}\left[t^{1/2}\right]}{2t^{-1/2}} = \frac{\frac{1}{2}t^{-1/2}}{2t^{-1/2}} = \frac{\frac{1}{2}}{2} = \frac{1}{4}$$

Example 3:

Parametric Equations: $x = 2t, \quad y = 5 - t^2$

Find the equation of the tangent line passing through (2,4).

At the point $(2,4)$, $x = 2$ and $y = 4$. To find t at this point,

we can use $x = 2t$.

$$x = 2t \quad \Rightarrow \quad 2 = 2t \quad \Rightarrow \quad t = 1$$

To find the slope of the tangent line, we need to find y'.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2t}{2} = -t$$

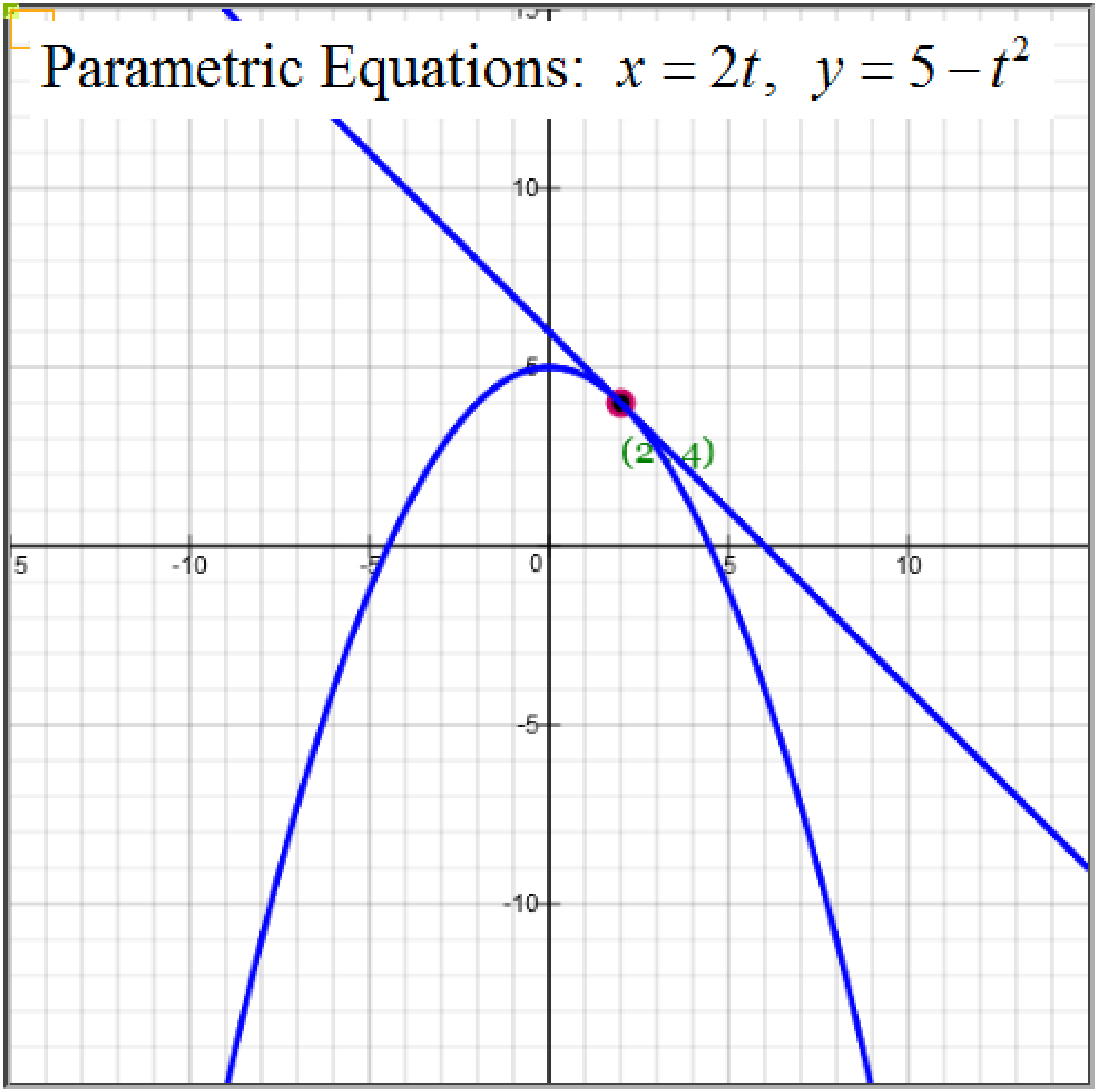
Example 3: (con't)

Slope of tangent line = $y' = \dfrac{dy}{dx} = -t = -(1) = -1$

Equation of tangent line at $(2,4)$:

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -1(x - 2)$$



Parametric Equations: $x = 2t$, $y = 5 - t^2$
10
5
-5
-10
0
-10
10
(2, 4)

Example 4:

Parametric Equations: $x = 2\cos t, \quad y = 2\sin t \qquad 0 \le t \le 2\pi$

Find the equation of the tangent line at the point $(0,2)$.

At the point $(0,2)$, we can find t as follows:

$$x = 2\cos t \quad \Leftrightarrow \quad 0 = 2\cos t \quad \Leftrightarrow \quad \cos t = 0 \quad \Leftrightarrow \quad t = \pi / 2$$

To find the slope of the tangent line, we need to find y'.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2\cos t}{-2\sin t} = -\frac{\cos t}{\sin t}$$

Example 4 (con't):

Slope of tangent line = $\dfrac{dy}{dx} = -\dfrac{\cos t}{\sin t} = -\dfrac{\cos(\pi/2)}{\sin(\pi/2)} = \dfrac{0}{1} = 0$

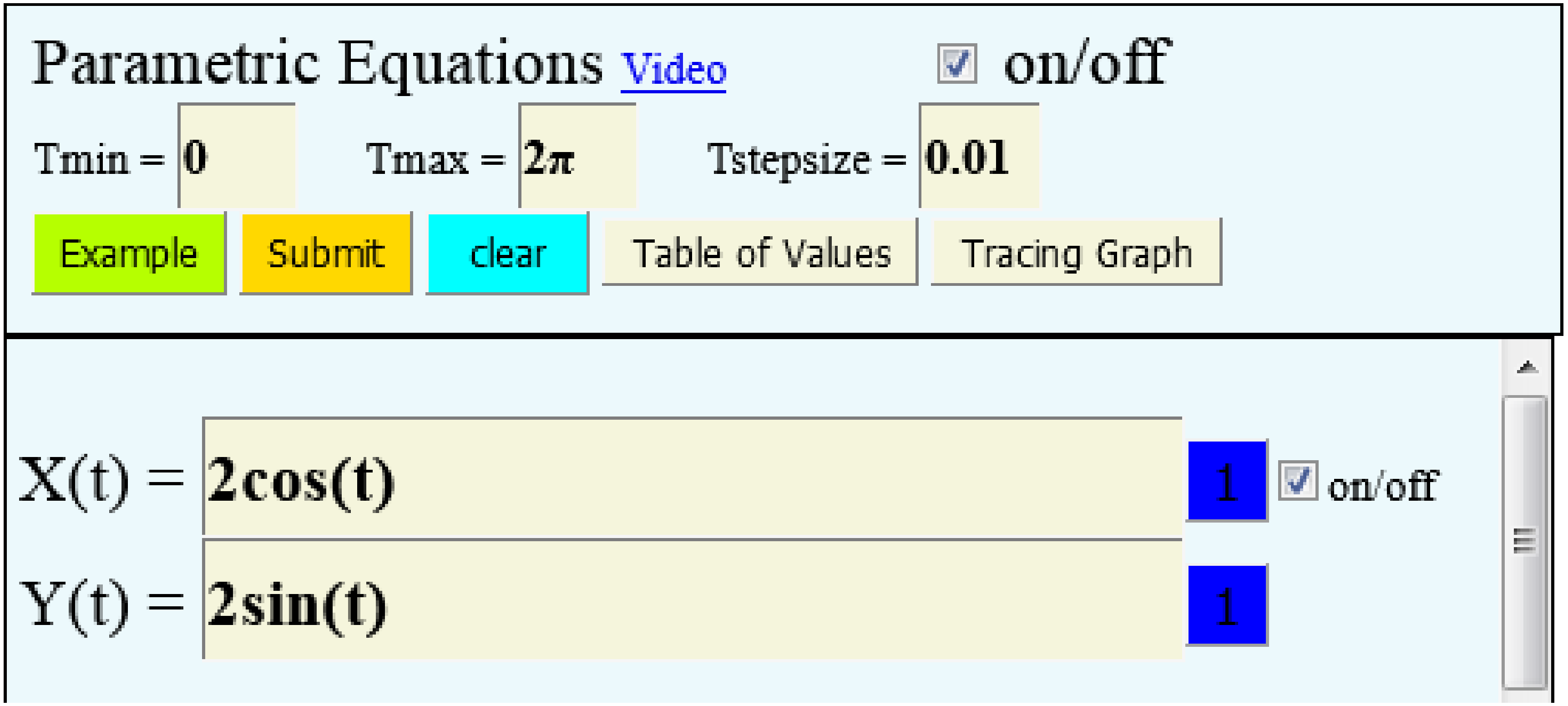
Equation of tangent line at $(0,2)$:

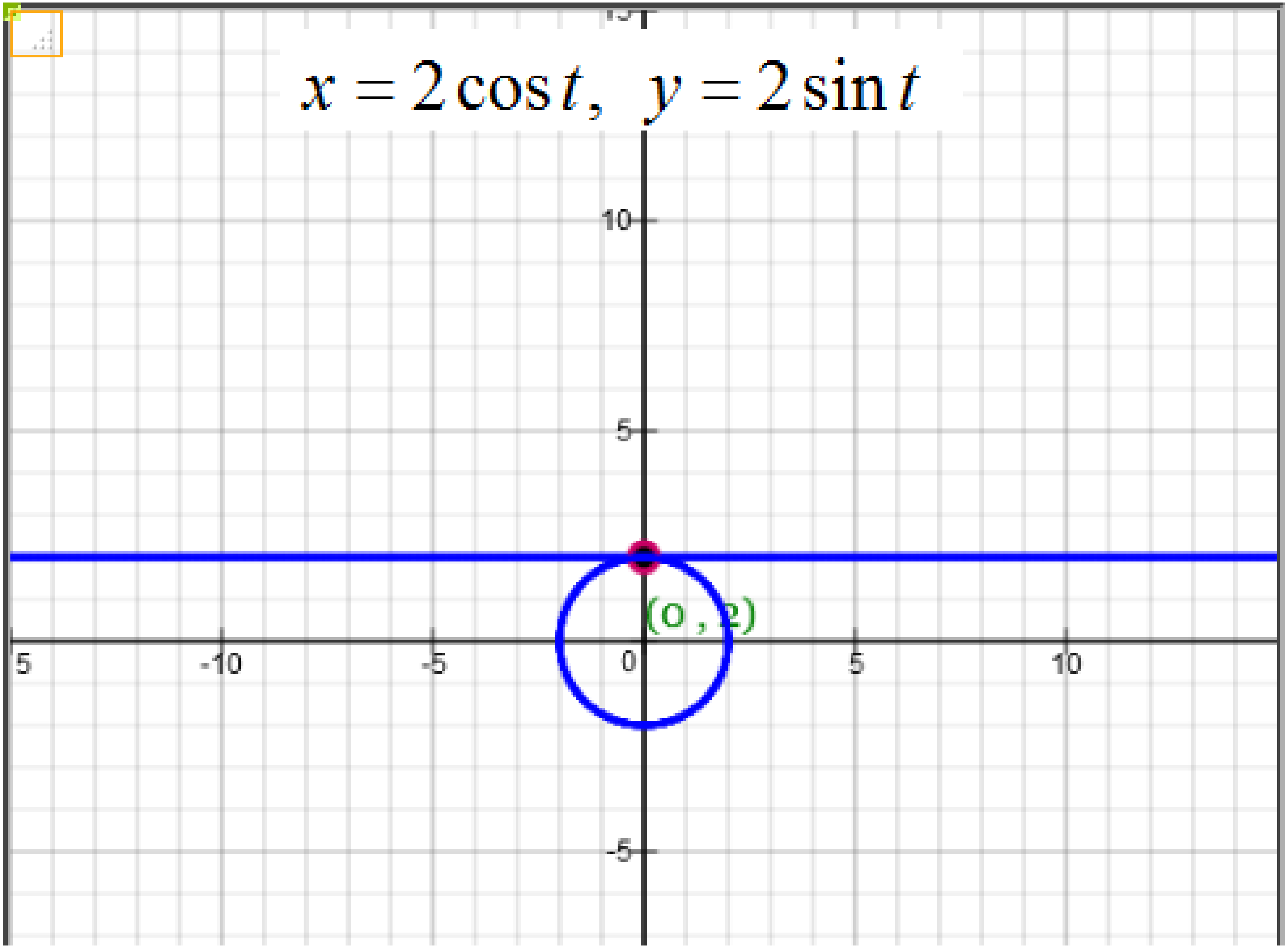
$$y - y_1 = m(x - x_1)$$

$$y - 2 = 0(x - 0)$$

$$y - 2 = 0$$

Calculator Input:





$x = 2\cos t, \quad y = 2\sin t$
10
5
-5
-10
0
(0 , 2)

Example 5:

Parametric Equations: $x = 4t + t^2, \quad y = 2t^{3/2}$

Find equation of the tangent line at $(5,2)$.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^{1/2}}{4+2t}$$

At the point $(5,2)$:

$$y = 2t^{3/2} \quad \Leftrightarrow \quad 2 = 2t^{3/2} \quad \Leftrightarrow \quad t^{3/2} = 1 \quad \Leftrightarrow \quad t = 1$$

Example 5 (con't):

Parametric Equations: $x = 4t + t^2, \quad y = 2t^{3/2}$

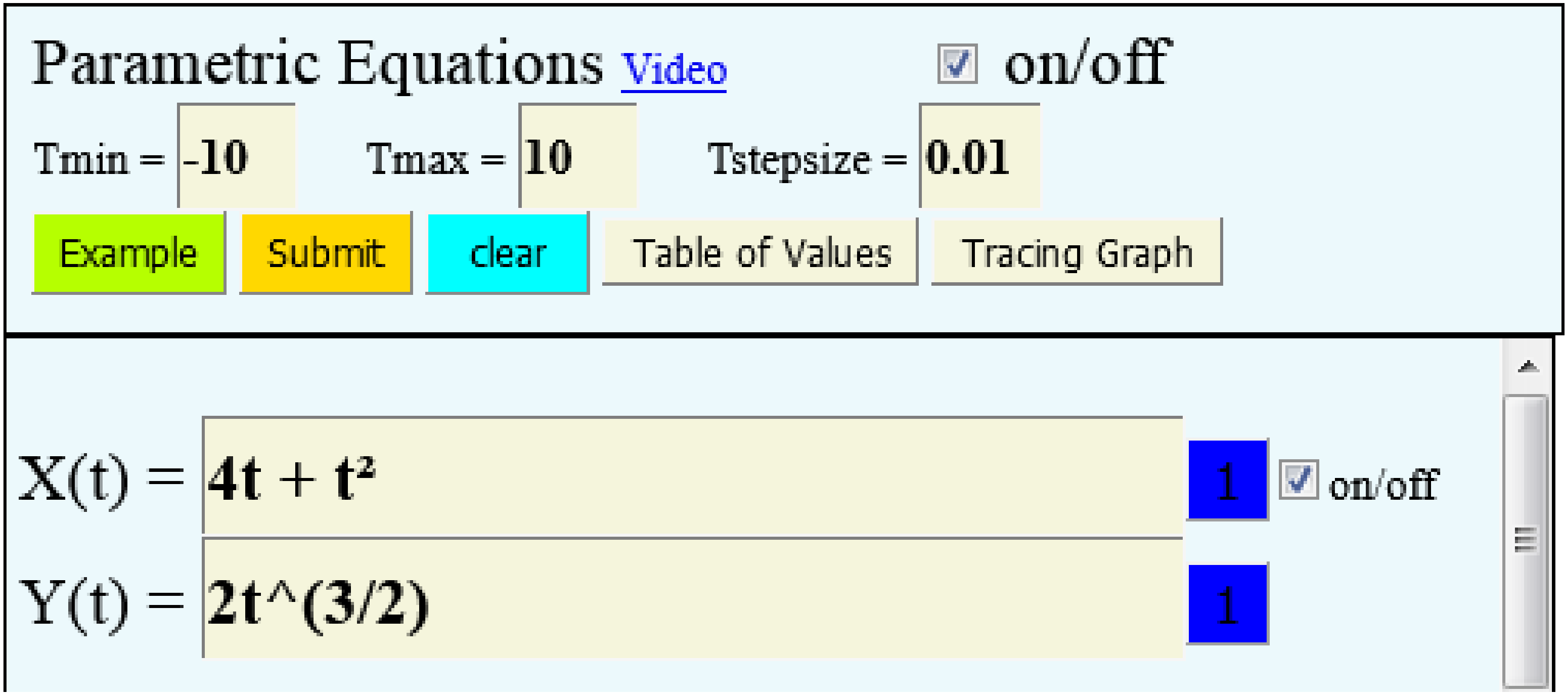
Slope of tangent line $= \dfrac{dy}{dx} = \dfrac{3t^{1/2}}{4+2t} = \dfrac{3(1)^{1/2}}{4+2(1)} = \dfrac{3}{6} = \dfrac{1}{2}$

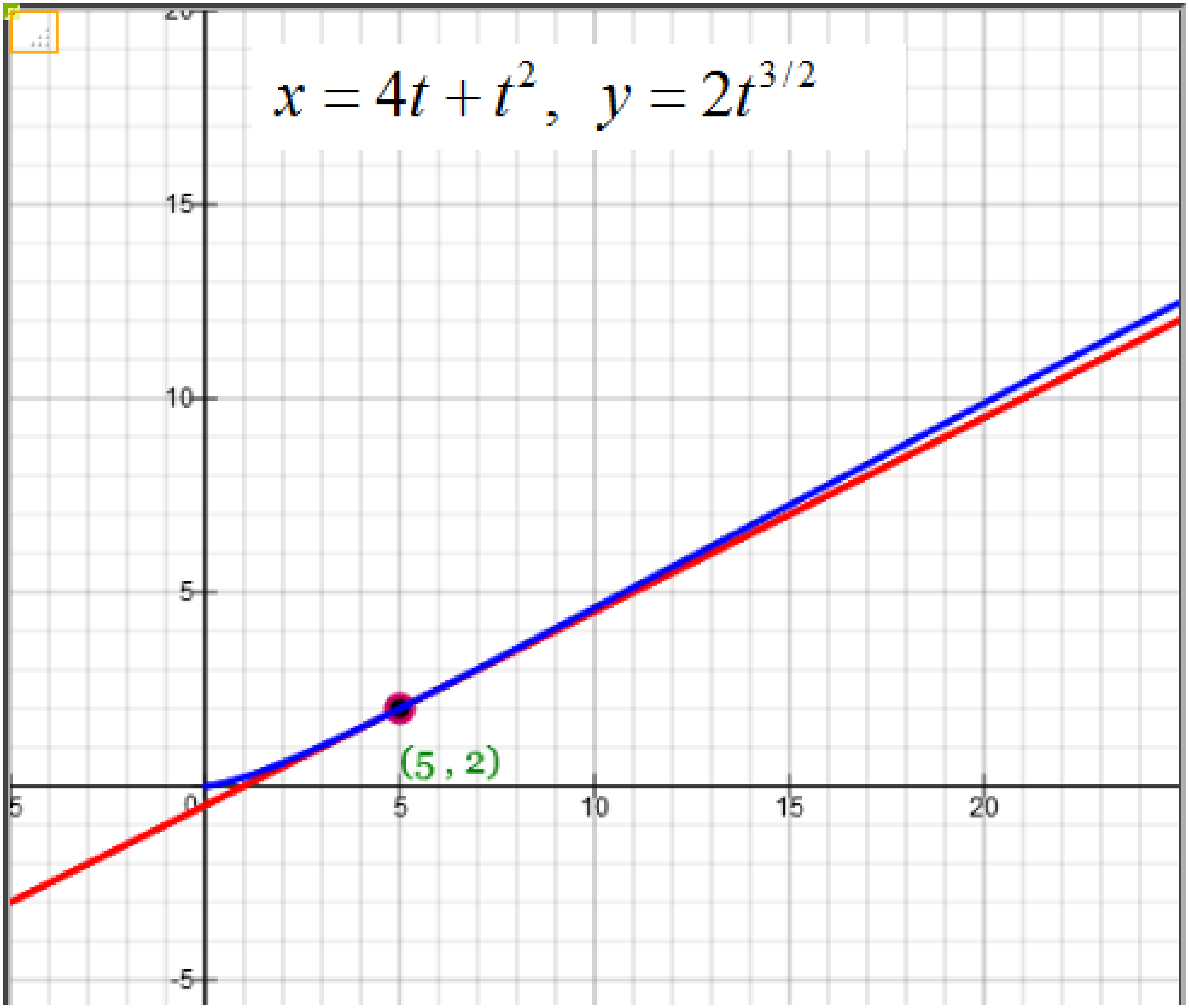
Equation of tangent line at $(5,2)$:

$$y - y_1 = m(x - x_1)$$

$$y - 2 = \frac{1}{2}(x - 5)$$

Calculator Input:





$x = 4t + t^2, \ \ y = 2t^{3/2}$
(5 , 2)
15
10
5
0
-5
5
10
15
20

Example 6:

Parametric Equations: $x = 6t + 2, \quad y = 3t \qquad 0 \le t \le 2$

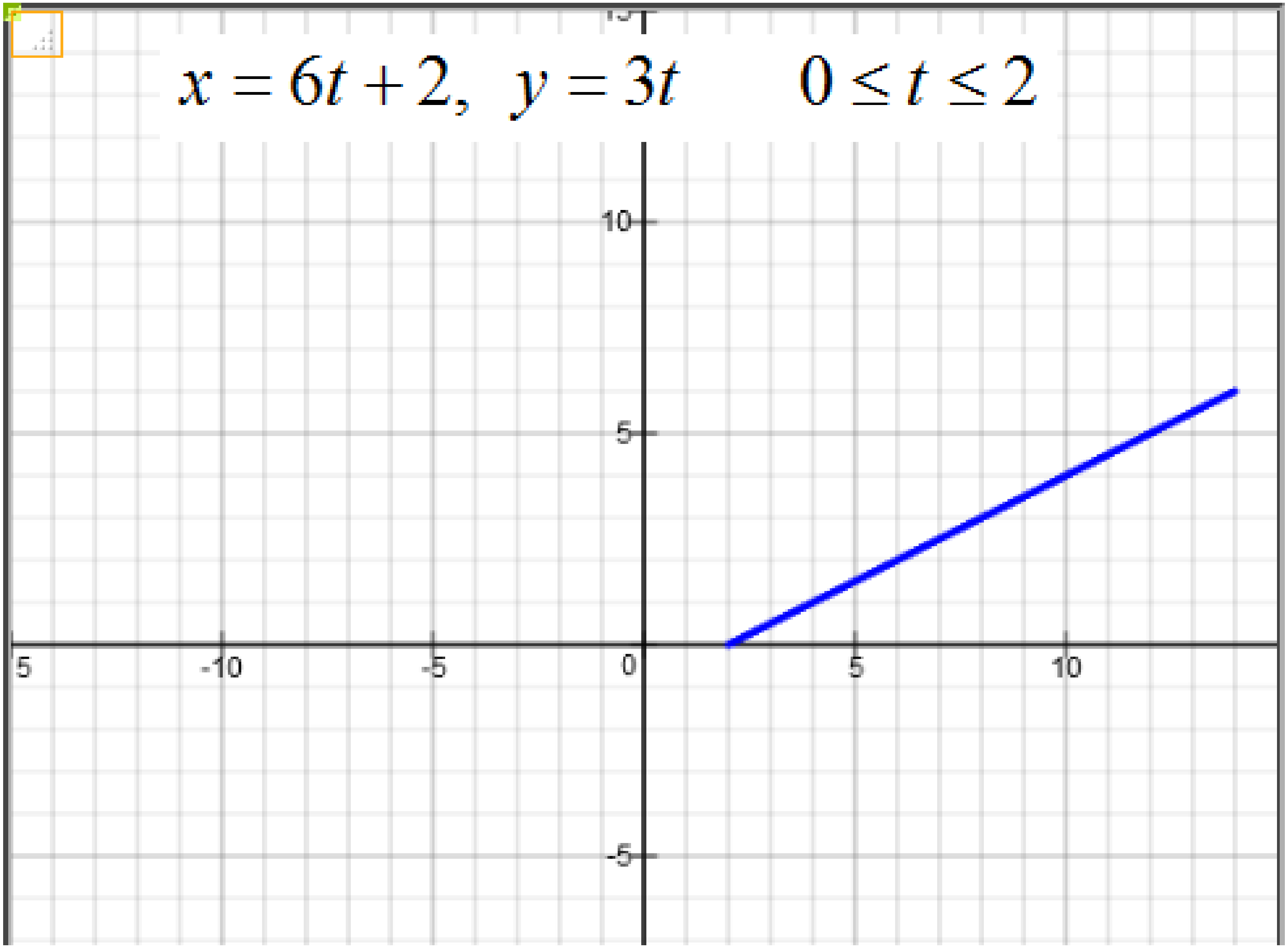
$$s = \text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt :$$

$$\frac{dx}{dt} = 6 \qquad\qquad \frac{dy}{dt} = 3$$

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(6)^2 + (3)^2}\, dt = \sqrt{45}t\Big|_0^2$$

$$= 2\sqrt{45} - 0 = 2\sqrt{45}$$

$$x = 6t + 2, \quad y = 3t \qquad 0 \le t \le 2$$



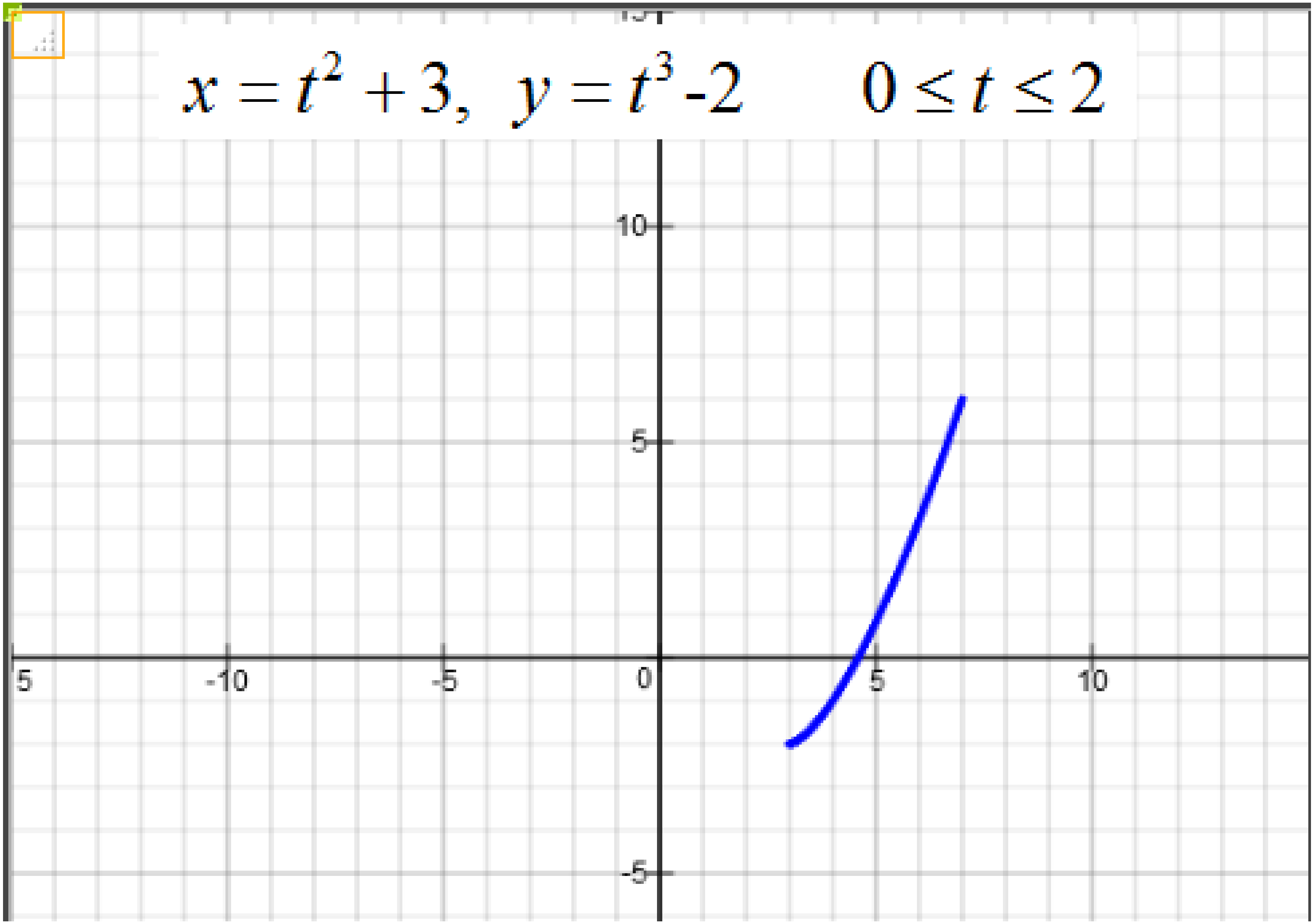
Example 7:

Parametric Equations: $x = t^2 + 3, \quad y = t^3 - 2 \qquad 0 \le t \le 2$

$$s = \text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt:$$

$$\frac{dx}{dt} = 2t \qquad\qquad \frac{dx}{dt} = 3t^2$$

$$s = \int_0^2 \sqrt{(2t)^2 + \left(3t^2\right)^2}\, dt = \int_0^2 \sqrt{4t^2 + 9t^4}\, dt = 9.0734$$


$x = t^2 + 3,\ \ y = t^3 - 2 \qquad 0 \le t \le 2$
10
5
-5
5
-10
-5
0
5
10

Example 8:

Parametric Equations: $x = 4\cos t + 1, \quad y = 3\sin t \qquad 0 \le t \le \pi/4$

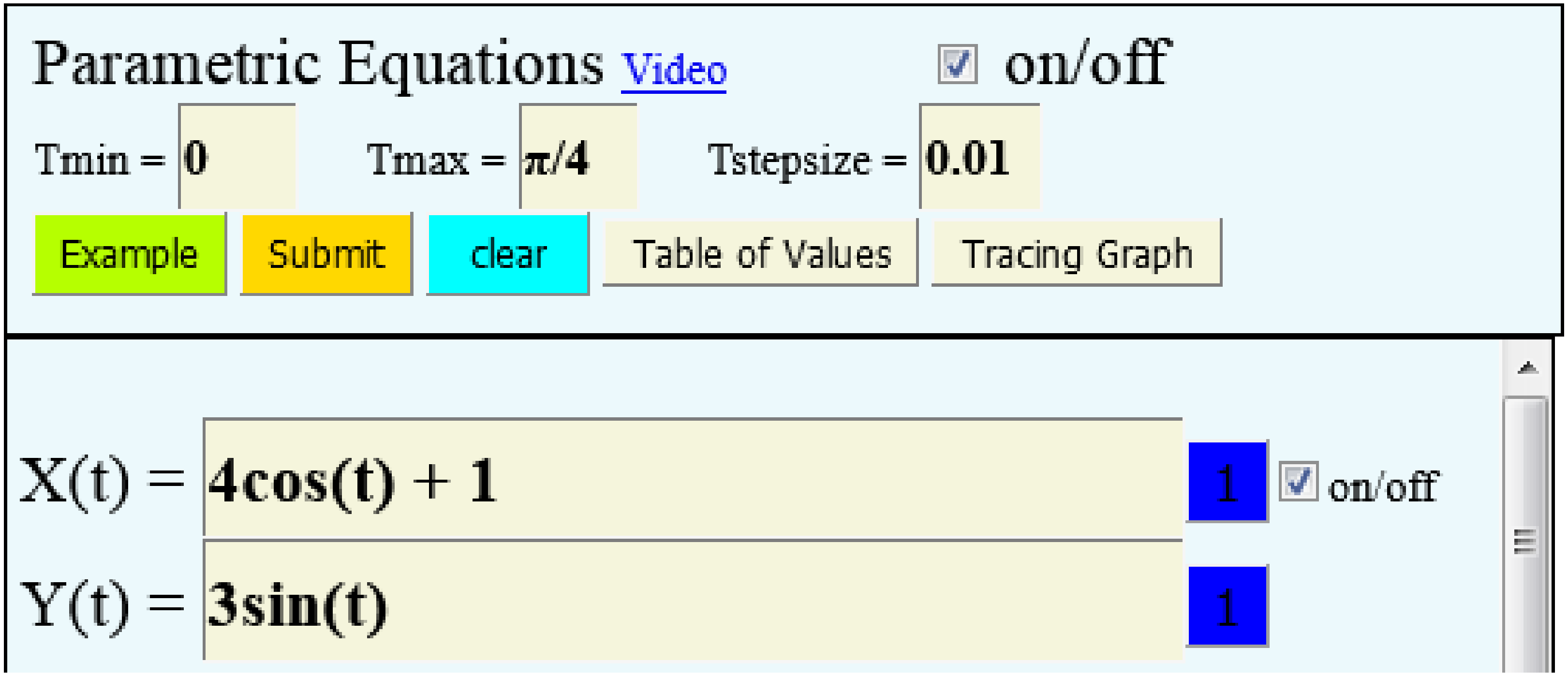
$$s = \text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt :$$

$$\frac{dx}{dt} = 4(-\sin t) = -4\sin t; \qquad \frac{dy}{dt} = 3\cos t$$

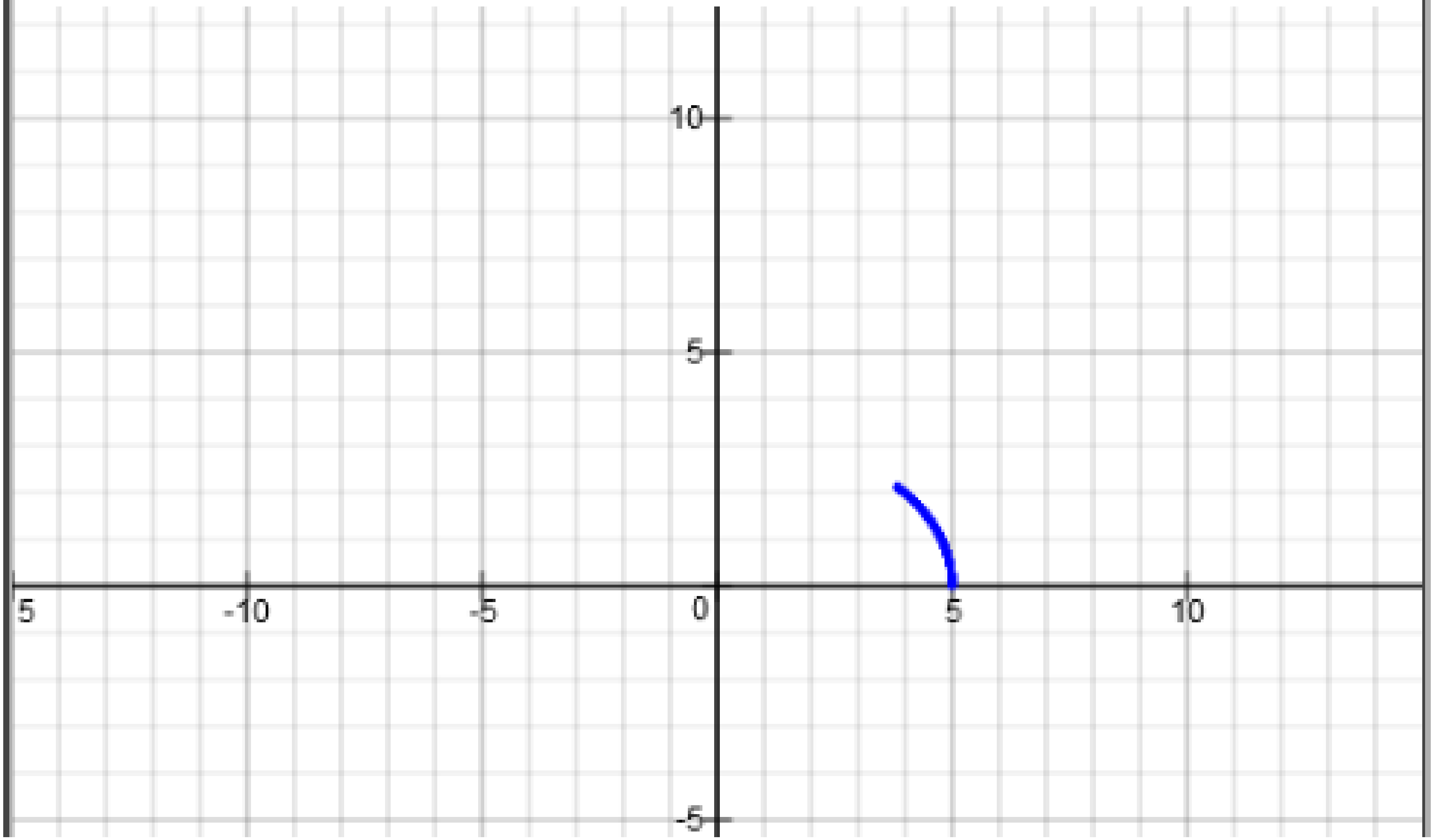
$$s = \int_0^{\pi/4} \sqrt{(-4\sin t)^2 + (3\cos t)^2}\, dt = \int_0^{\pi/4} \sqrt{16(\sin t)^2 + 9(\cos t)^2}\, dt$$

$$= 2.51378$$

Calculator Input:



$$x = 4\cos t + 1, \quad y = 3\sin t \qquad 0 \le t \le \pi/4$$



Find $\int_0^{\pi/4} \sqrt{16(\sin t)^2 + 9(\cos t)^2}\, dt$ by using Trapezoidal Rule.

$$\int_0^{\pi/4} \sqrt{16(\sin t)^2 + 9(\cos t)^2}\, dt = 2.51378$$

Input for finding definite integral

Trapezoidal Rule

Input f(x) = $\sqrt{(\ 16(\sin(x))^2 + 9(\cos(x))^2\ )}$ Find Area

Input value of a = **0**

Input value of b = **π/4**

Input value of n = number of subintervals = **5000**

Example 9:

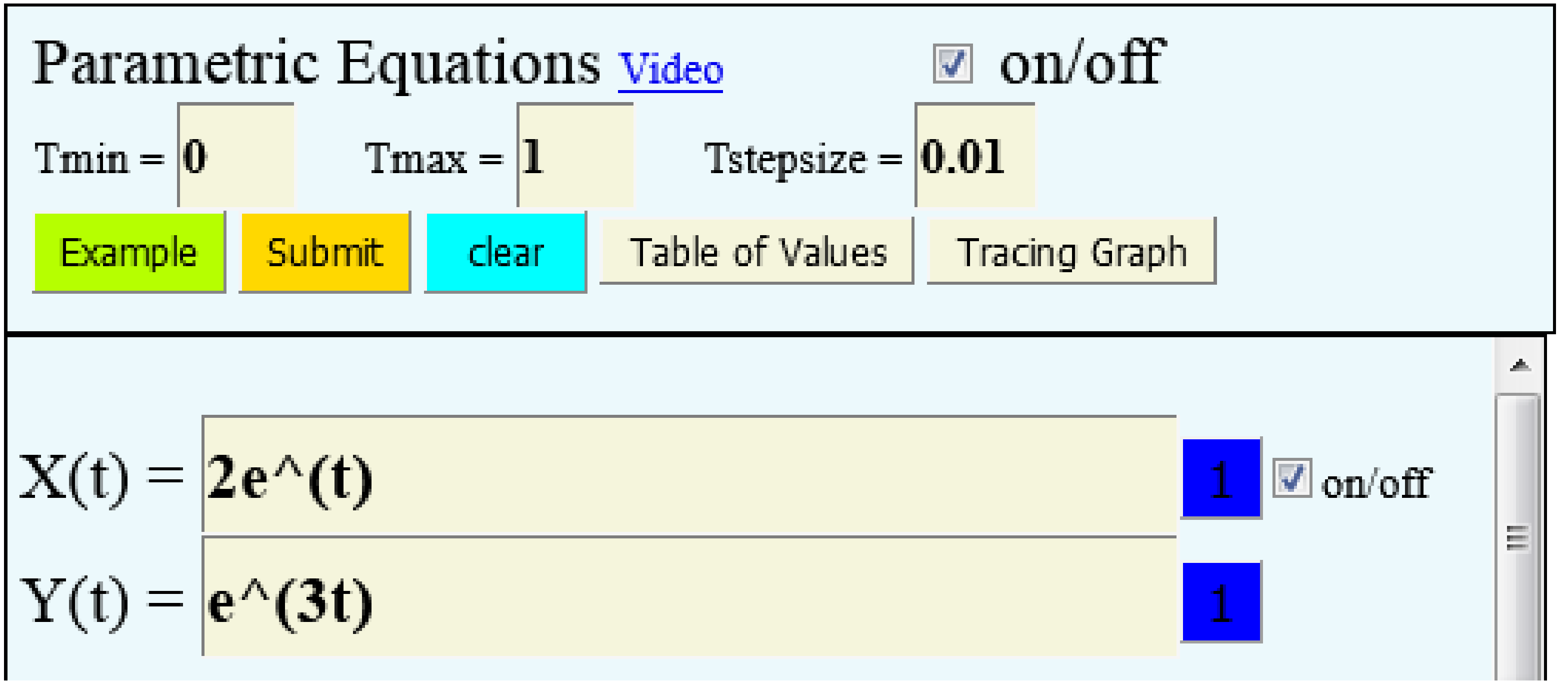
Parametric Equations: $x = 2e^t, \quad y = e^{3t} \qquad 0 \le t \le 1$

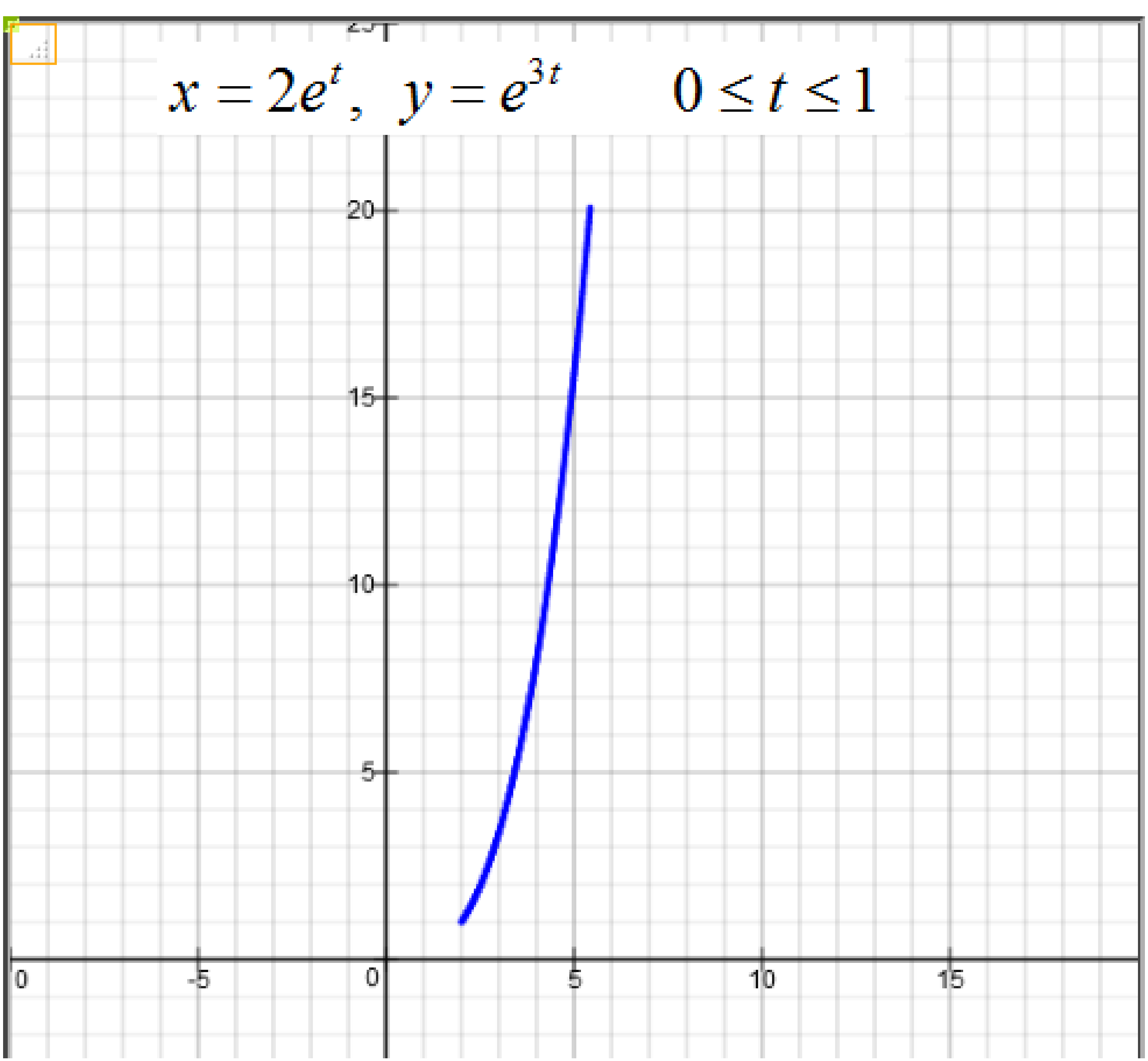
$$s = \text{Arc Length} = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt :$$

$$\frac{dx}{dt} = 2e^t \qquad \frac{dy}{dt} = e^{3t}(3) = 3e^{3t}$$

$$s = \int_0^1 \sqrt{\left(2e^t\right)^2 + \left(3e^{3t}\right)^2}\, dt = \int_0^1 \sqrt{4e^{2t} + 9e^{6t}}\, dt = 19.4937$$

Calculator Input




$x = 2e^t, \ y = e^{3t} \qquad 0 \le t \le 1$
20
15
10
5
0
-5
0
5
10
15

Find $\int_0^1 \sqrt{4e^{2t}+9e^{6t}}\,dt$ by using Trapezoidal Rule.

$$\int_0^1 \sqrt{4e^{2t}+9e^{6t}}\,dt = 19.4937$$

Input for finding definite integral

Trapezoidal Rule

Input f(x) = √(4e^(2x) +9e^(6x))

Find Area

Input value of a = **0**

Input value of b = **1**

Input value of n = number of subintervals = **5000**