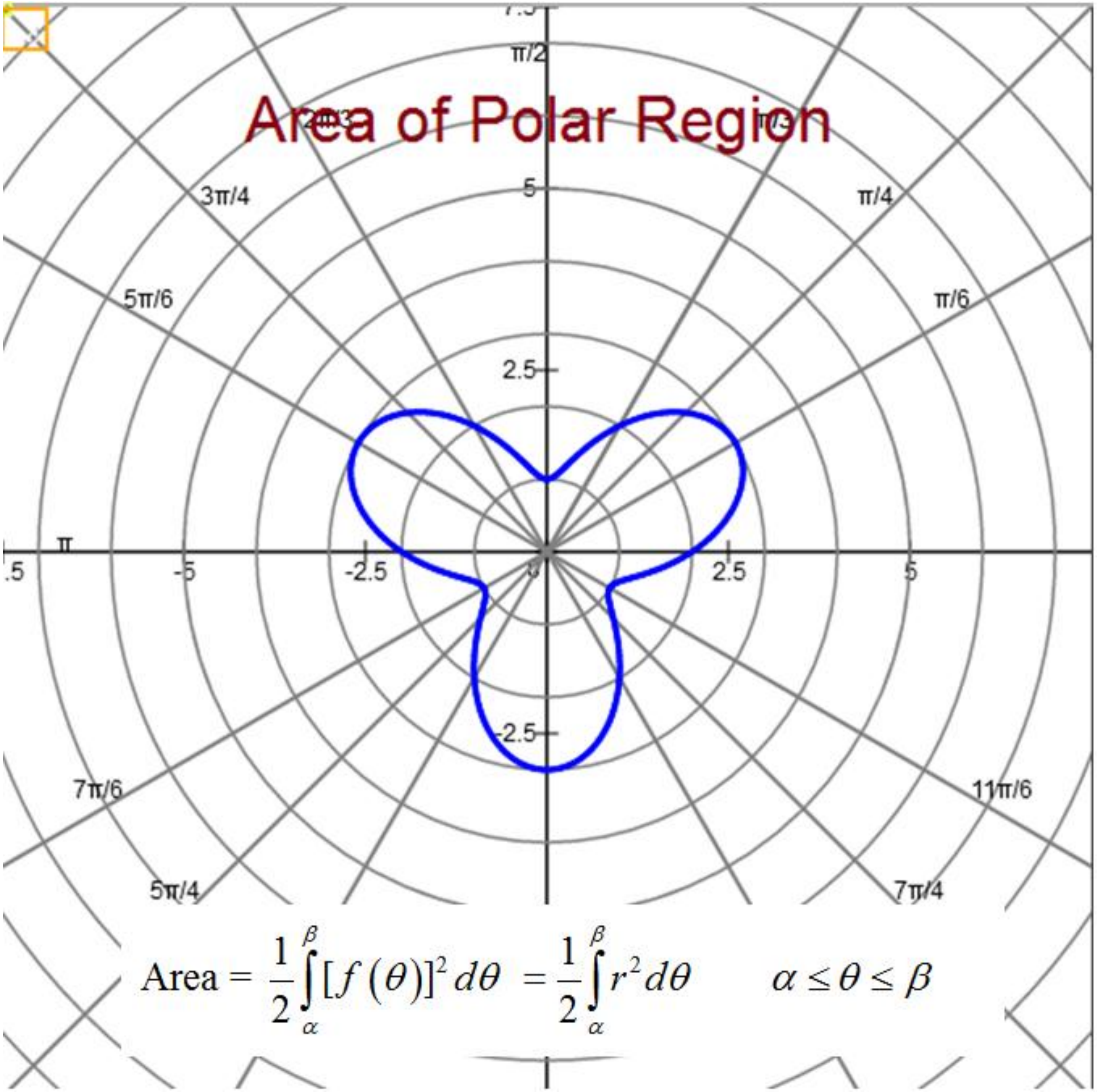
Area of Polar Region

$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta \qquad \alpha \le \theta \le \beta$$

Area of Polar Region



$$\text{Area} = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta \qquad \alpha \le \theta \le \beta$$

Example 1:

Find the area of the interior of $r = 4\sin\theta$.

First graph $r = 4\sin\theta$ with $0 \le \theta \le \pi$.

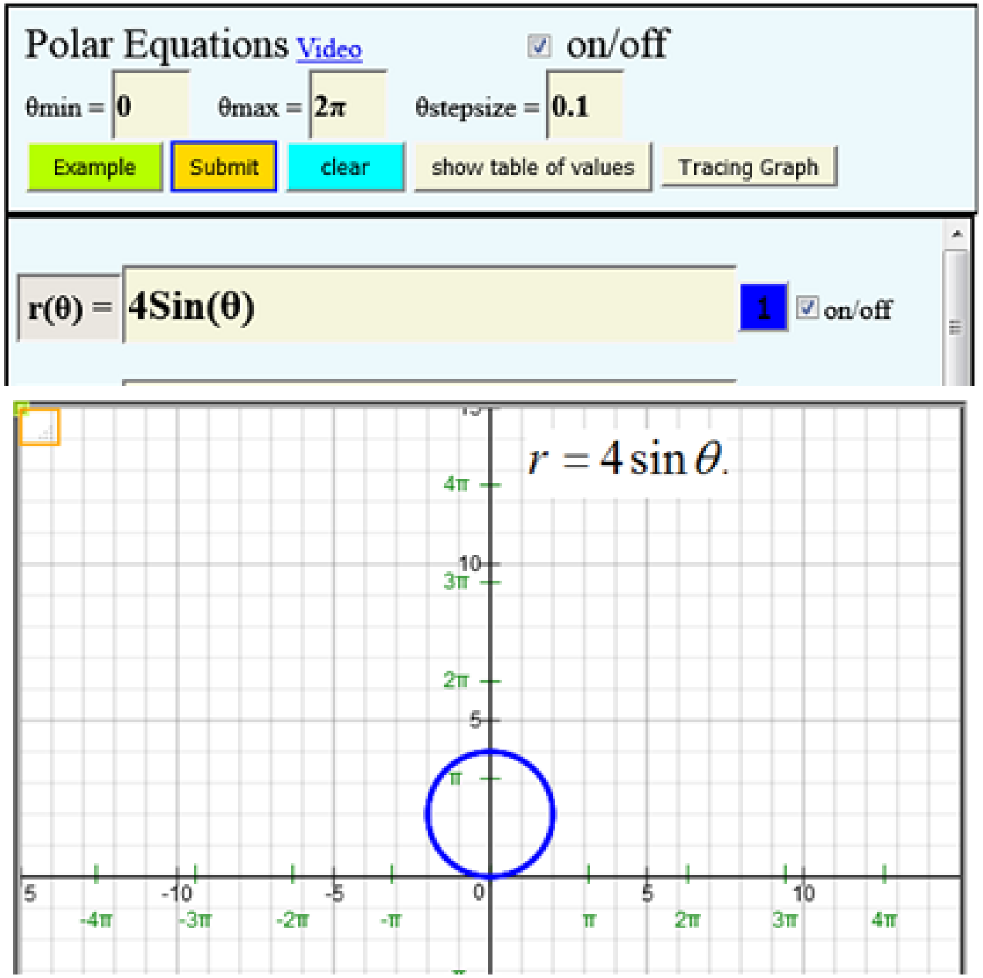
Hence, Area = $\frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

$$= \frac{1}{2}\int_{0}^{\pi} [4\sin\theta]^2 \, d\theta = 12.56637122$$

To evaluate definite integral, use numerical integration method like Trapezoid's Rule.

Example 1 (con't)

How to graph $r = 4\sin\theta$ with $0 \le \theta \le \pi$:



Example 2:

Find the area of one petal of $r = \sin 3\theta$.

First graph $r = \sin 3\theta$ with $0 \le \theta \le 2\pi$.

From graph (see below) we see that there are three petals.

Note that each petal starts and ends at the pole (origin); and at the pole, $r = 0$.

To find where one petal starts and ends, set $r = 0$.

Hence $r = \sin 3\theta$

$$0 = \sin 3\theta$$

To solve this equation, we note that sin(0) = 0 ; sin(π) = 0; sin(2π) = 0; and so on.

Hence, $3\theta = 0, \pi, 2\pi, 3\pi, \ldots$

$$\Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \ldots$$

Now graph $r = \sin 3\theta$ with $0 \le \theta \le \pi / 3$.

We can see from graph (see below) that we have one petal.

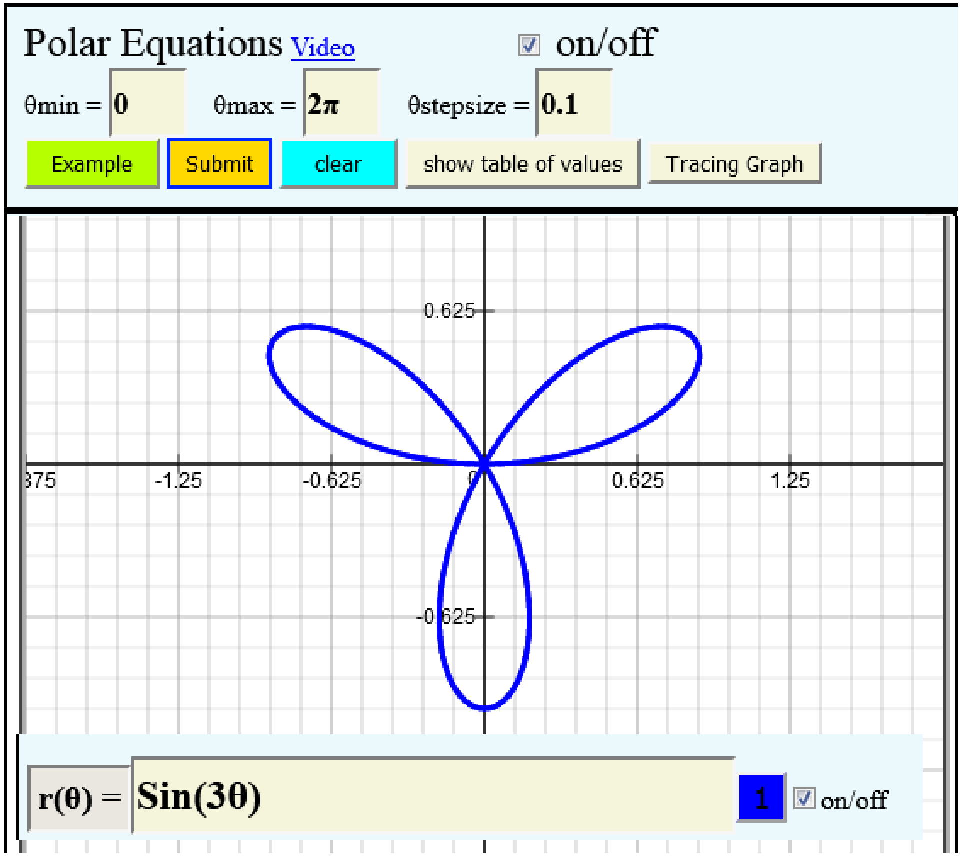
Hence, one petal starts at $\theta = 0$ and ends at $\theta = \frac{\pi}{3}$'

Area of one petal = $\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

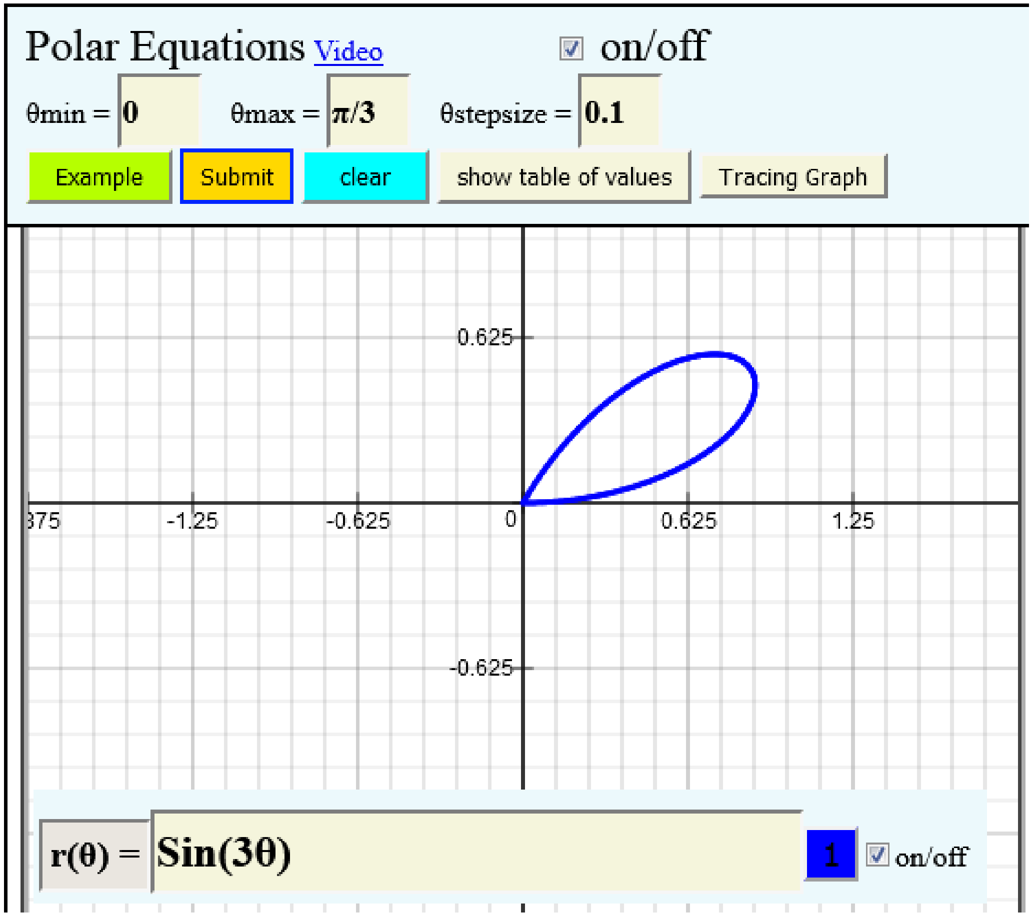
Area of one petal = $\frac{1}{2}\int_{0}^{\pi/3}[\sin 3\theta]^2 d\theta = 0.2617994$

To evaluate definite integral, use numerical integration method like Trapezoid's Rule.

Example 2 (con't)



Example 2 (con't)



Example 3:

Find the area of one petal of $r = 2\cos 5\theta$.

First graph $r = 2\cos 5\theta$ with $0 \le \theta \le 2\pi$.

From graph (see below) we can see that there are five petals.

Note that each petal starts and ends at the pole (origin); and at the pole, $r = 0$.

To find where one petal starts and ends, set $r = 0$.

set $r = 2\cos 5\theta$

$\Rightarrow \cos 5\theta = 0$

To solve this equation, note that $\cos(\pi/2) = 0;$ $\cos(3\pi/2) = 0;$ $\cos(5\pi/2) = 0;$ and so on.

Hence, $5\theta = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2,\ 9\pi/2...$

$$\Rightarrow \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}, ...$$

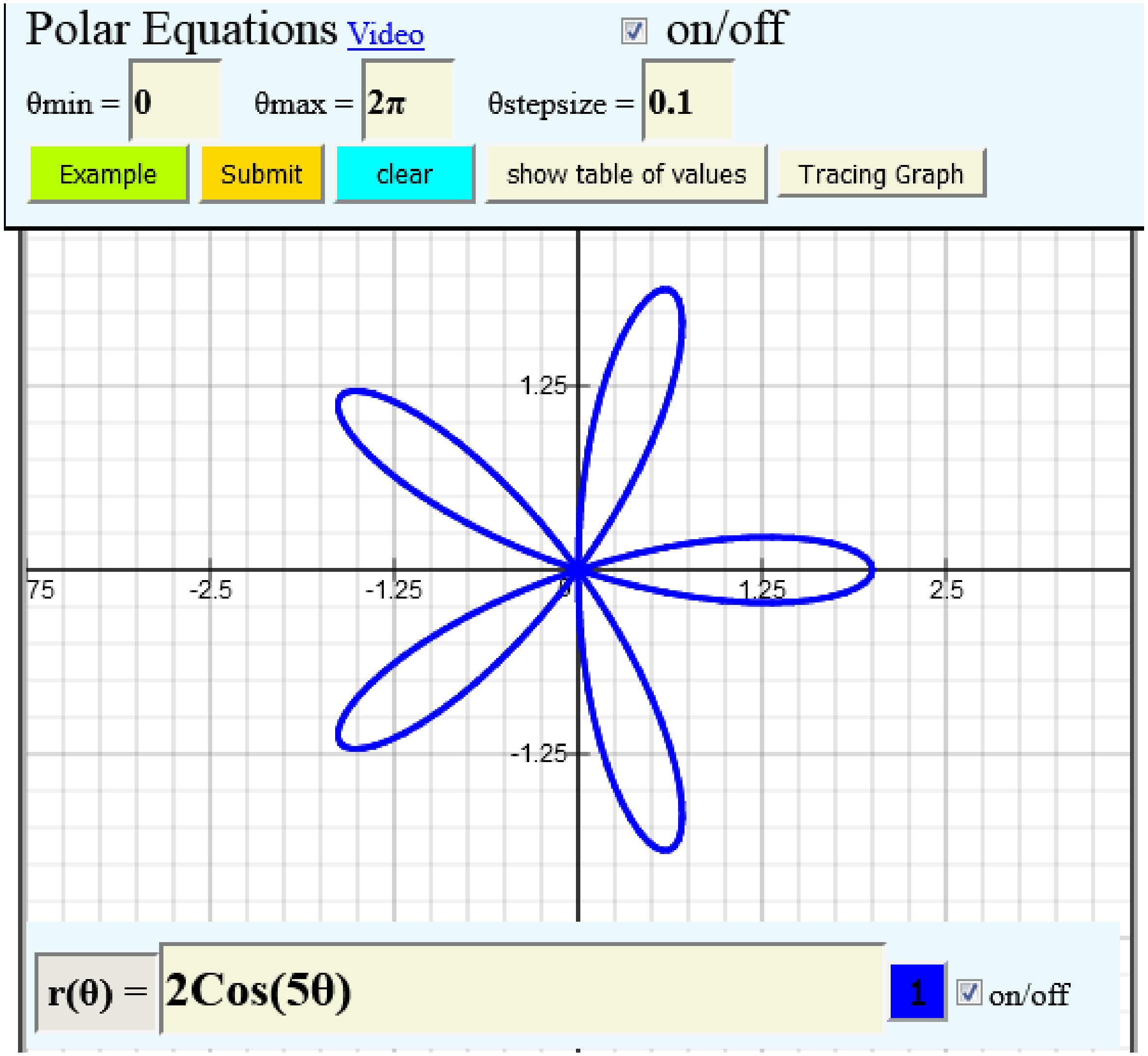
Now graph $r = 2\cos 5\theta$ with $\dfrac{\pi}{10} \le \theta \le \dfrac{3\pi}{10}$.

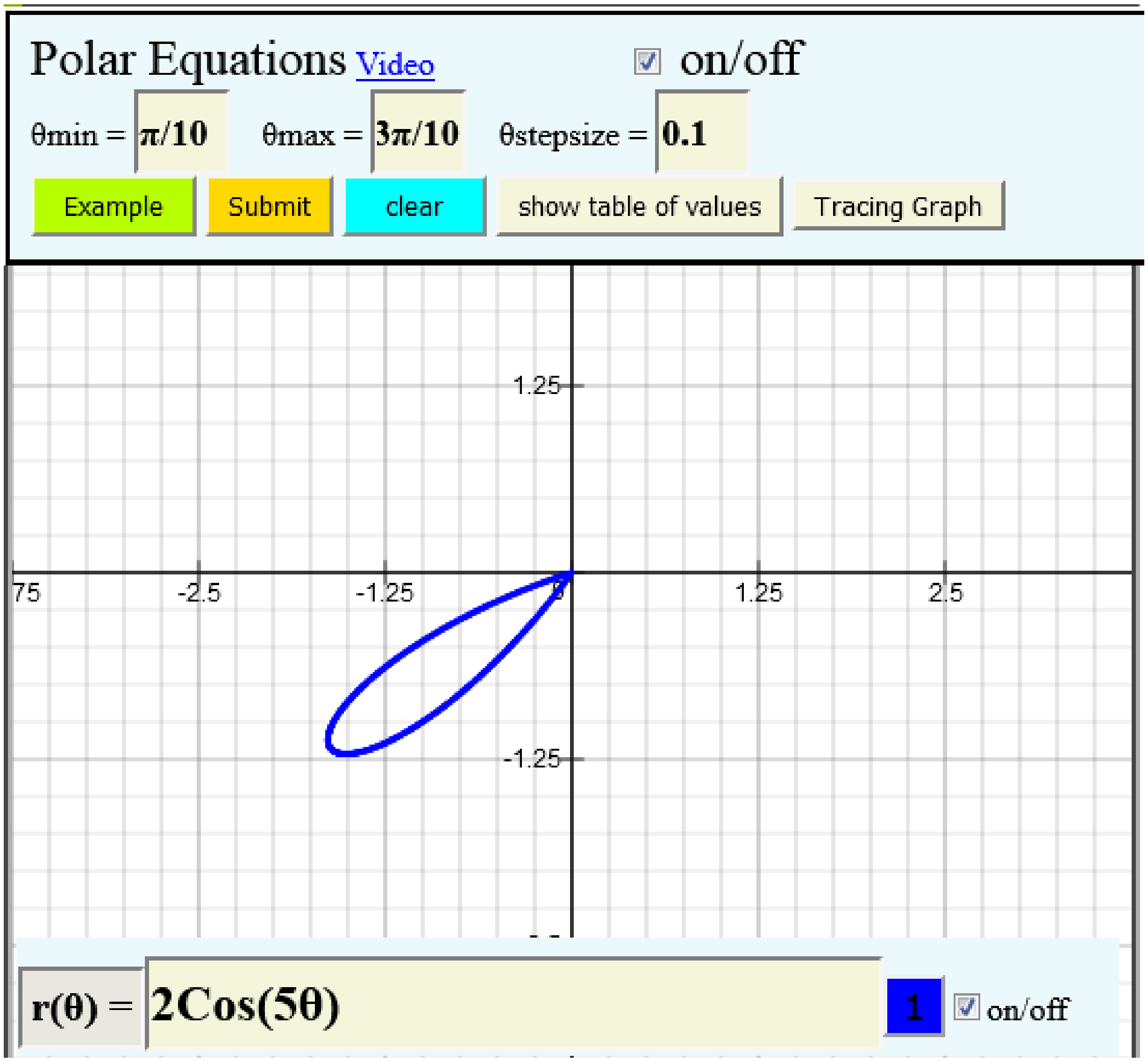
From graph (see below) we can see that we have one petal.

Therefore,

Area of one petal = $\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$:

Area of one petal = $\frac{1}{2}\int_{\pi/10}^{3\pi/10}[2\cos 5\theta]^2 d\theta = 0.628318$


Polar Equations Video
on/off
θmin = 0
θmax = 2π
θstepsize = 0.1
Example
Submit
clear
show table of values
Tracing Graph
1.25
-1.25
75
-2.5
-1.25
1.25
2.5
r(θ) = 2Cos(5θ)
1
on/off


Polar Equations Video
on/off
θmin = π/10
θmax = 3π/10
θstepsize = 0.1
Example
Submit
clear
show table of values
Tracing Graph
1.25
-2.5
-1.25
1.25
2.5
-1.25
r(θ) = 2Cos(5θ)
1
on/off

Example 4:

Find the area of the inner loop of $r = 1 + 2\sin\theta$

First graph $r = 1 + 2\sin\theta$ with $0 \le \theta \le 2\pi$.

Note that the inner loop starts and ends at the pole (origin); and at the pole, $r = 0$.

To find where the inner loop starts and ends, set $r = 0$.

Hence $r = 1 + 2\sin\theta$

$$\Leftrightarrow \quad 1 + 2\sin\theta = 0$$

$$\Leftrightarrow \quad \sin\theta = -1/2$$

To solve this equation, note that from Trigonometric Table of Values $\sin(7\pi/6) = -1/2$ and $\sin(11\pi/6) = -1/2$.

$$\Rightarrow \theta = 7\pi/6,\ 11\pi/6,\ 7\pi/6 + 2\pi = 19\pi/6,\ 11\pi/6 + 2\pi = 23\pi/6,\ \ldots$$

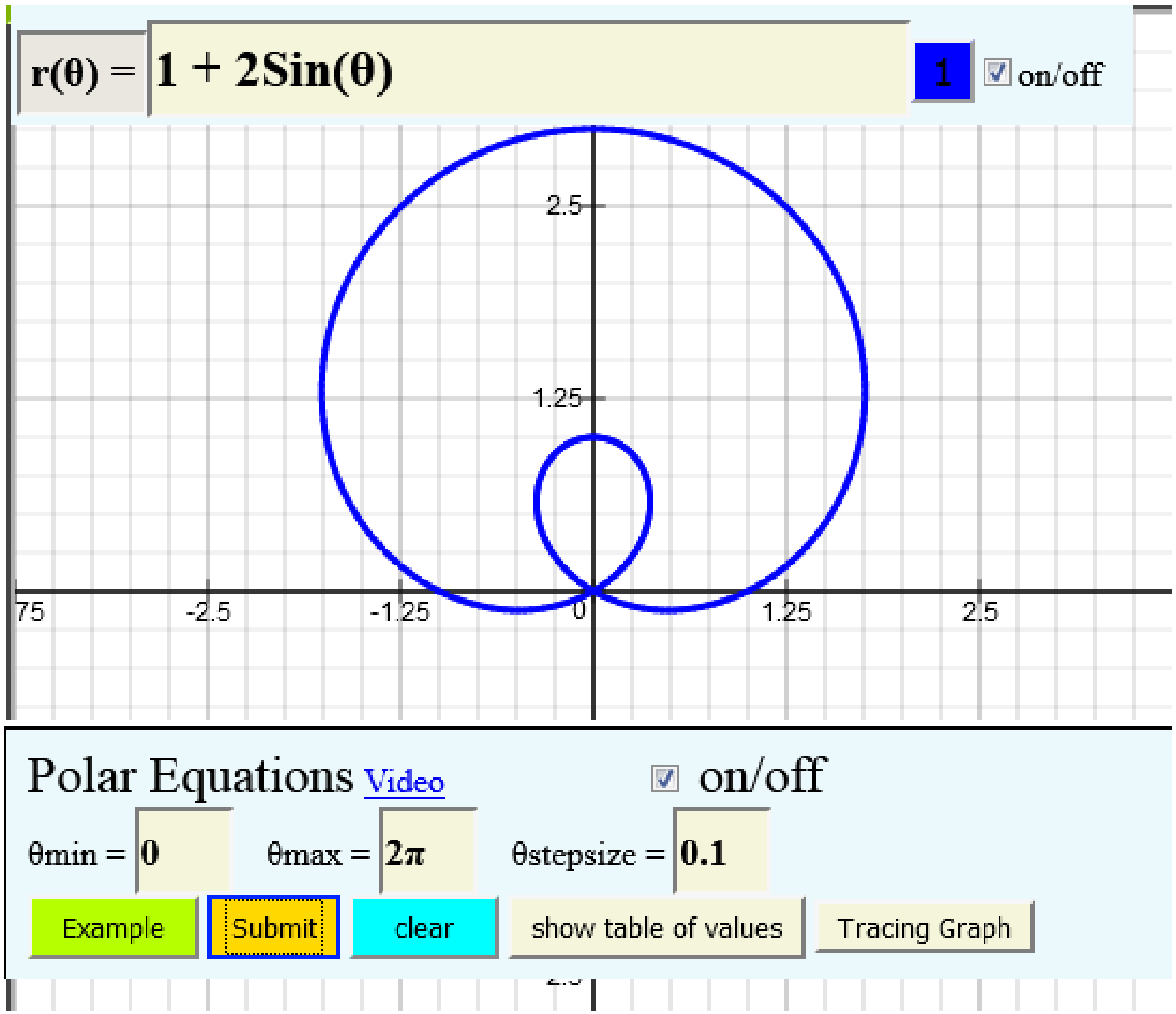
Now graph $r = 1 + 2\sin\theta$ with $7\pi/6 \le \theta \le 11\pi/6$.

We can see from graph (see below) that we have the inner loop.

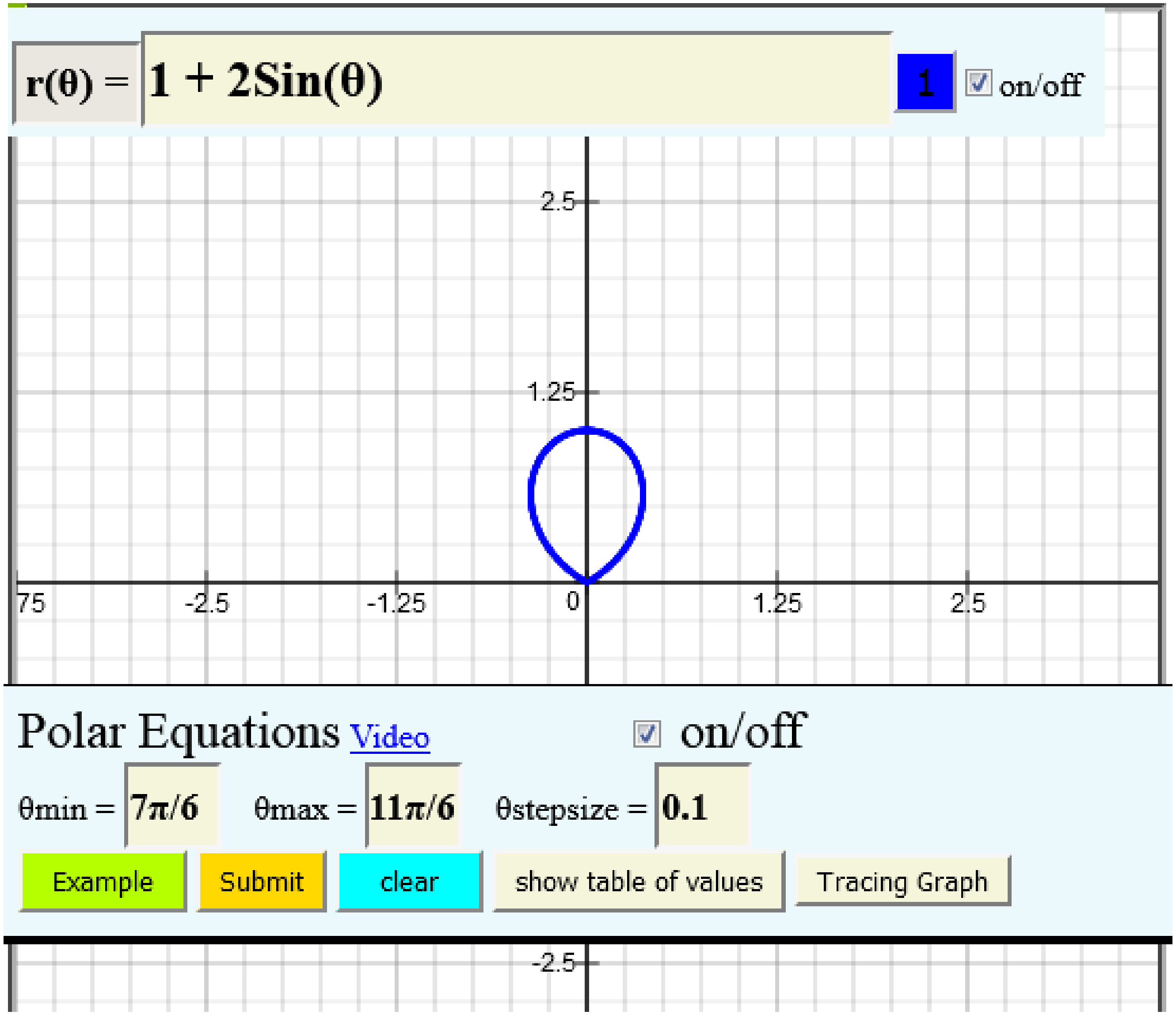
Therefore, the inner loop starts at $\theta = \dfrac{7\pi}{6}$ and ends at $\theta = \dfrac{11\pi}{6}$; and

Area of the inner loop= $\dfrac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 \, d\theta = \dfrac{1}{2}\int_{\alpha}^{\beta} r^2 \, d\theta$:

Area of inner loop = $\dfrac{1}{2}\int_{7\pi/6}^{11\pi/6} [1 + 2\sin\theta]^2 \, d\theta = 0.54351646$



r(θ) =
1 + 2Sin(θ)
1
on/off
2.5
1.25
75
-2.5
-1.25
0
1.25
2.5
Polar Equations Video
on/off
θmin = 0
θmax = 2π
θstepsize = 0.1
Example
Submit
clear
show table of values
Tracing Graph


r(θ) =
1 + 2Sin(θ)
1
on/off
2.5
1.25
75
-2.5
-1.25
0
1.25
2.5
Polar Equations
Video
on/off
θmin =
7π/6
θmax =
11π/6
θstepsize =
0.1
Example
Submit
clear
show table of values
Tracing Graph
-2.5

Arc Length of Polar Region

$$\text{Arc Length} = \int_{\alpha}^{\beta} \sqrt{\left[f(\theta) \right]^2 + \left[f'(\theta) \right]^2} \, d\theta$$

$$= \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta} \right)^2} \, d\theta$$

Example 5:

Find the length of the arc for $r = 4\sin\theta \qquad 0 \le \theta \le \pi$

a) Graph $r = 2\cos 5\theta \qquad 0 \le \theta \le \pi$ (see below).

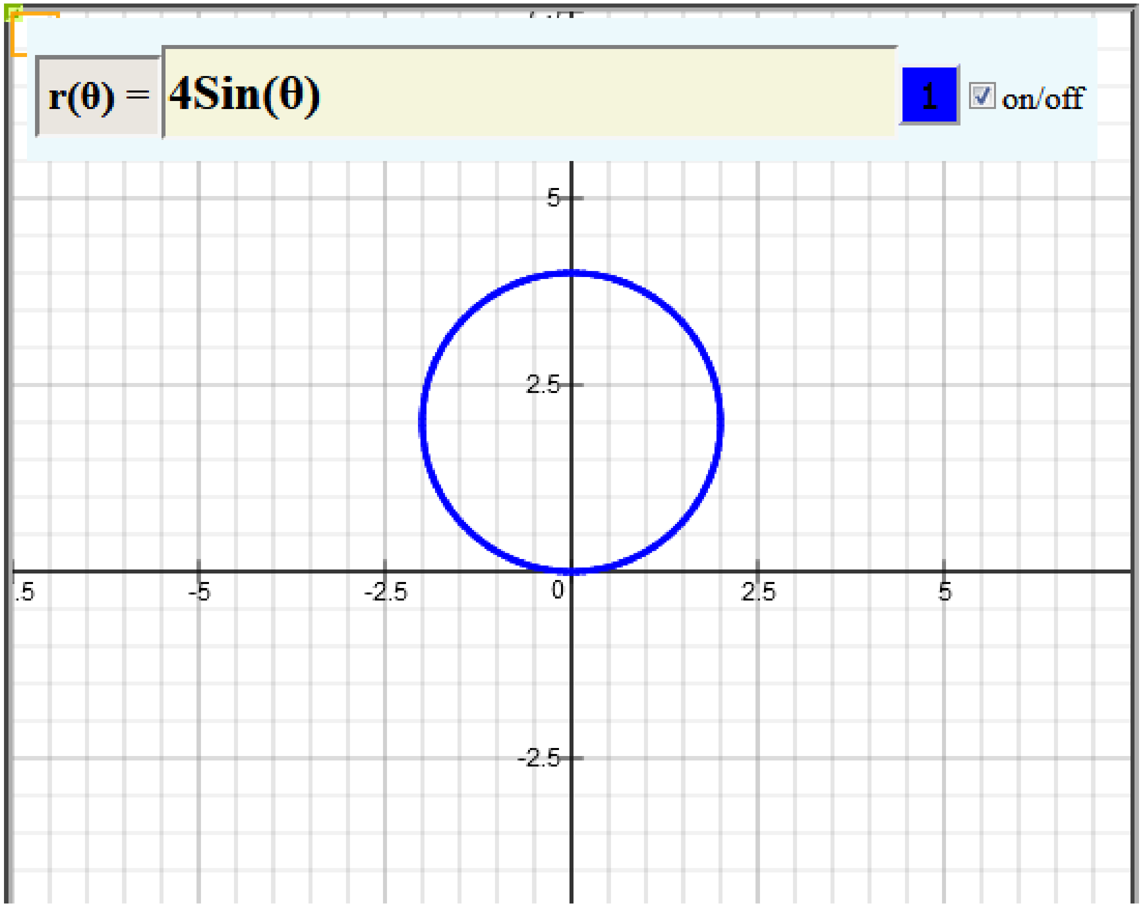
b) Find Arc Length = $\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$:

$$\frac{dr}{d\theta} = 4\cos\theta$$

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\pi} \sqrt{[4\sin\theta]^2 + [4\cos\theta]^2}\, d\theta$$

$$= \int_0^{\pi} \sqrt{16[\sin\theta]^2 + 16[\cos\theta]^2}\, d\theta = \int_0^{\pi} \sqrt{16\left([\sin\theta]^2 + [\cos\theta]^2\right)}\, d\theta$$

$$= \int_0^{\pi} \sqrt{16(1)}\, d\theta = \int_0^{\pi} 4\, d\theta = 4\theta\Big|_0^{\pi} = 4\pi$$


r(θ) =
4Sin(θ)
1
on/off
5
2.5
-2.5
.5
-5
-2.5
0
2.5
5

Example 6:

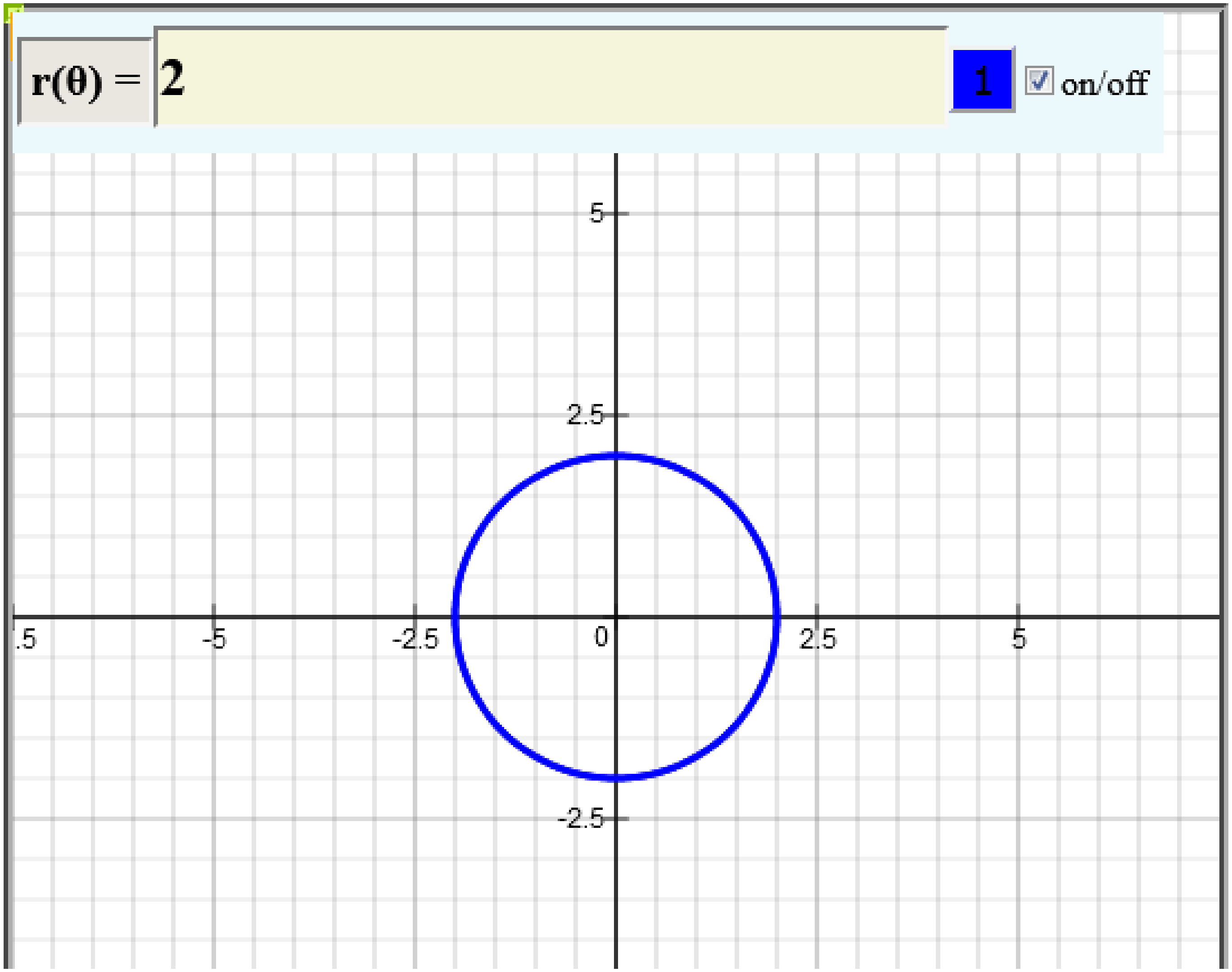
Find the length of the arc for $r = 2 \qquad 0 \le \theta \le 2\pi$

a) Graph $r = 2 \qquad 0 \le \theta \le 2\pi \qquad$ (see below).

b) Find Arc Length = $\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$:

$$\frac{dr}{d\theta} = 0$$

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{2\pi} \sqrt{2^2 + 0^2}\, d\theta = \int_{0}^{2\pi} 2\, d\theta \;=\; 2\theta\Big|_0^{2\pi} = 4\pi$$


r(θ) =
2
1
on/off
5
2.5
-2.5
.5
-5
-2.5
0
2.5
5

Example 7:

Find the length of the arc for $r = 2 + 3\sin\theta \qquad 0 \le \theta \le \pi/4$

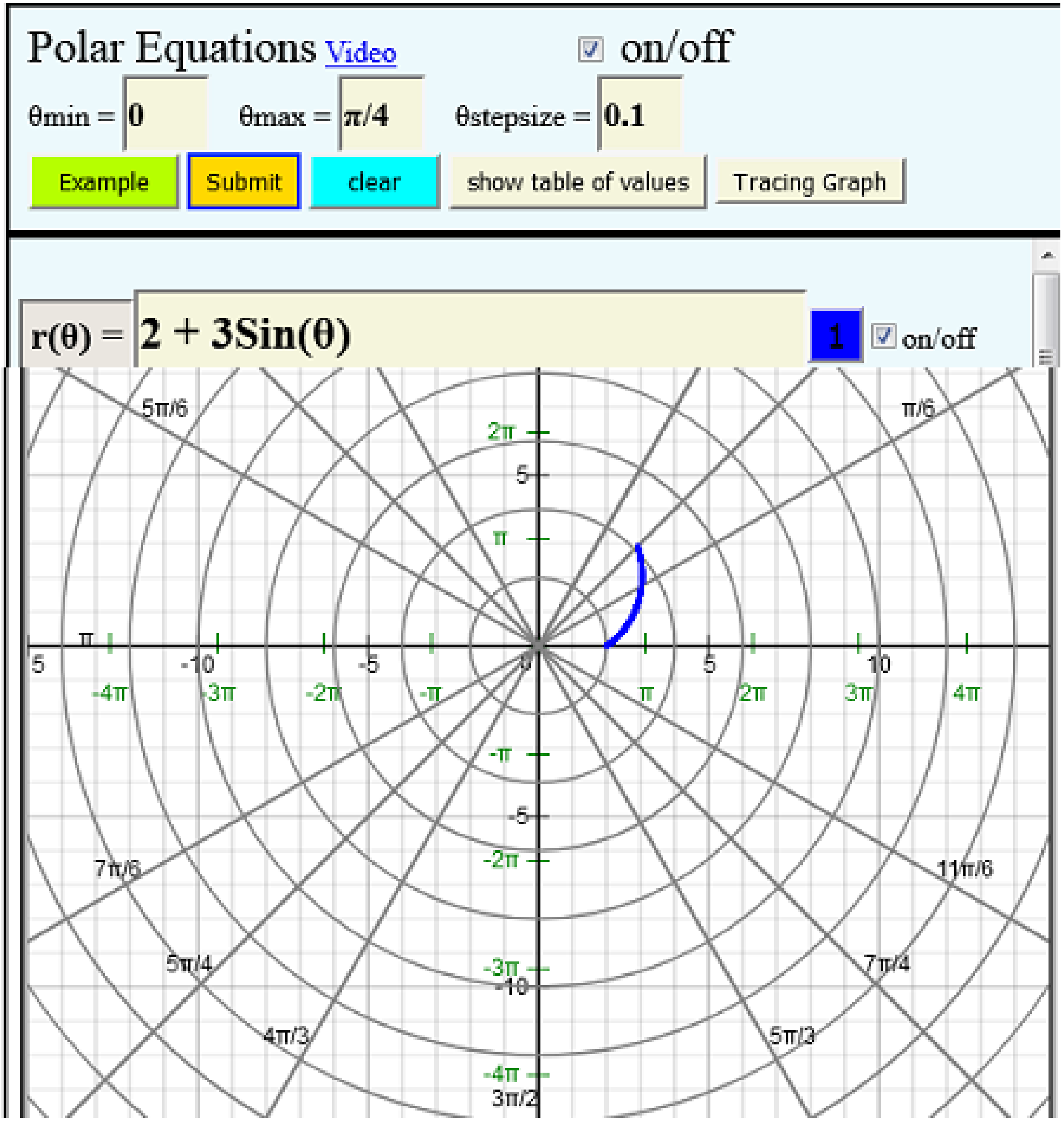
a) Graph $r = 2 + 3\sin\theta \qquad 0 \le \theta \le \pi/4$ (see below).

b) Find Arc Length = $\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$:

$$\frac{dr}{d\theta} = 3\cos\theta$$

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{\pi/4} \sqrt{(2+3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta = 3.274805725$$

To evaluate definite integral, we can use numerical integration method like Trapezoid's Rule.


Polar Equations Video
on/off
θmin = 0
θmax = π/4
θstepsize = 0.1
Example
Submit
clear
show table of values
Tracing Graph
r(θ) = 2 + 3Sin(θ)
1
on/off
5π/6
π/6
7π/6
11π/6
5π/4
7π/4
4π/3
5π/3
3π/2