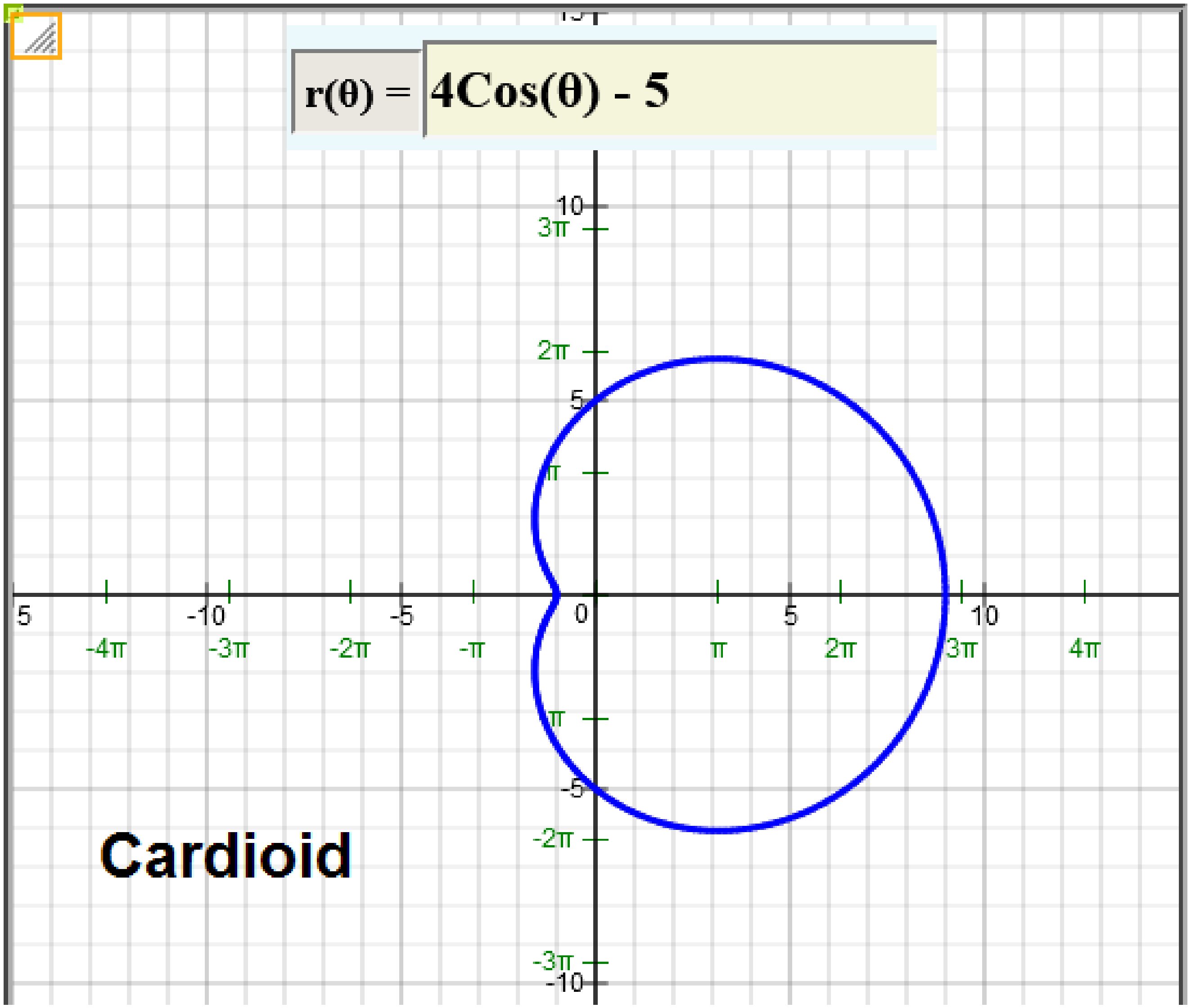
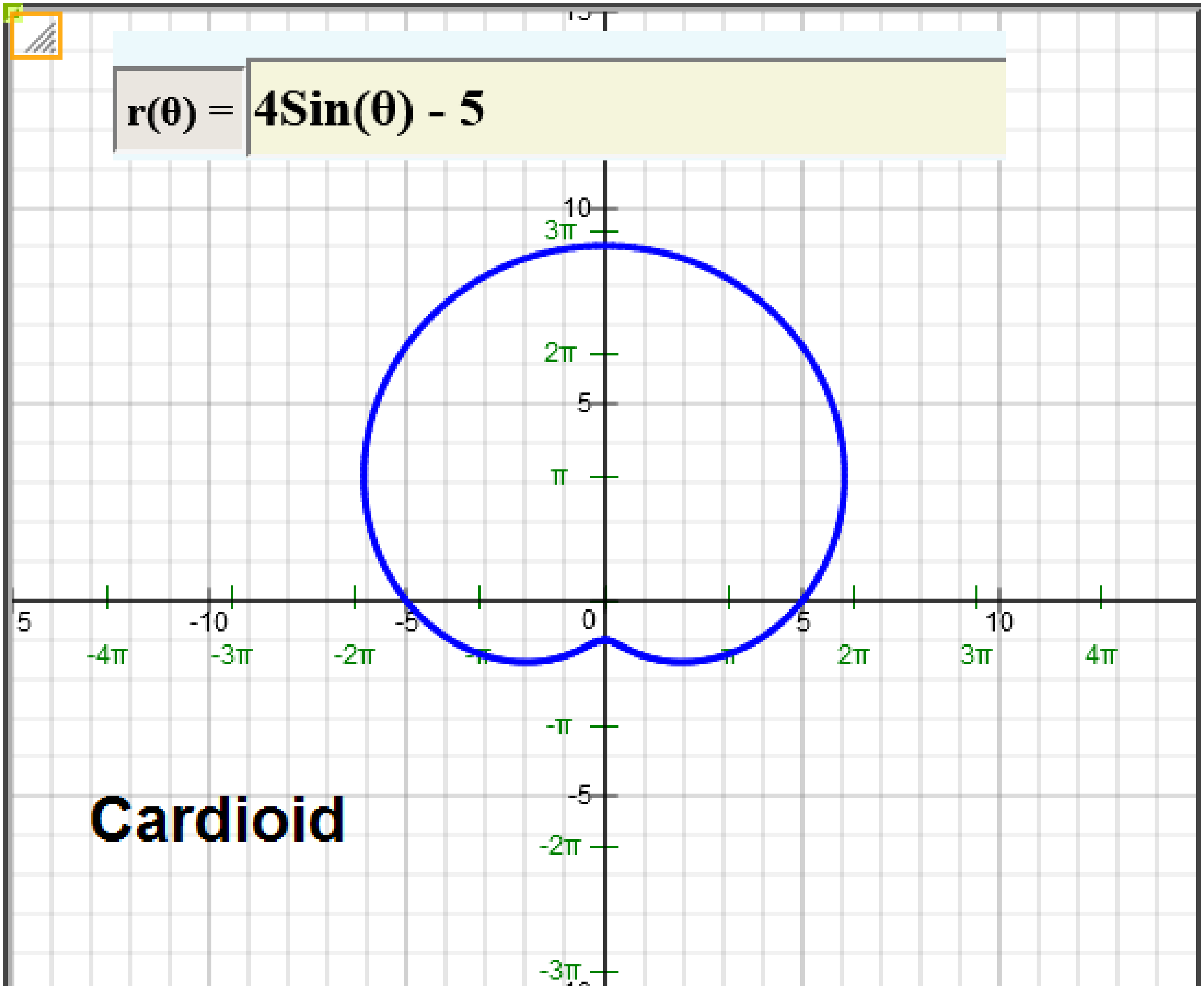
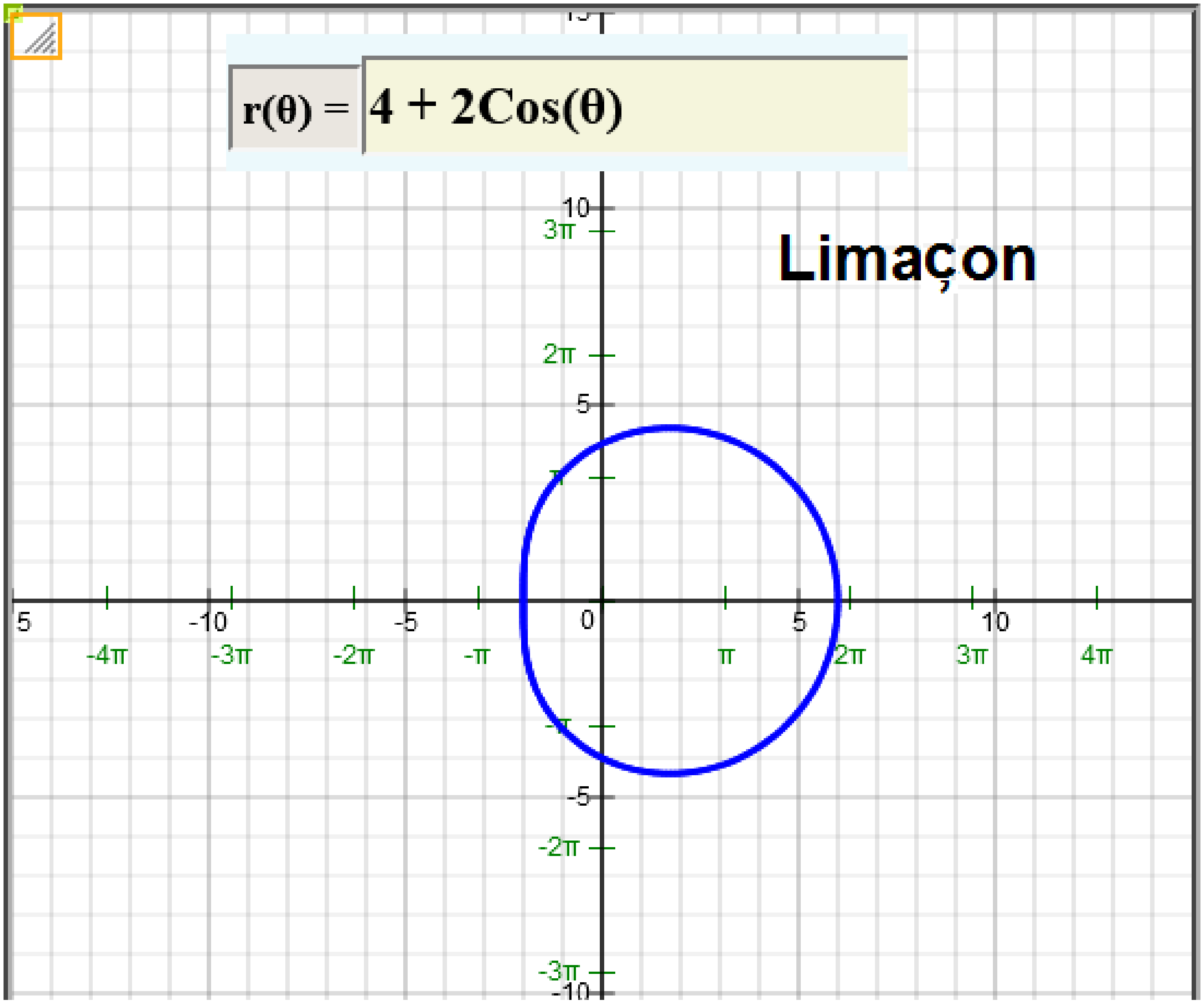
Calculus II Area of Polar Region

Examples of Polar Region

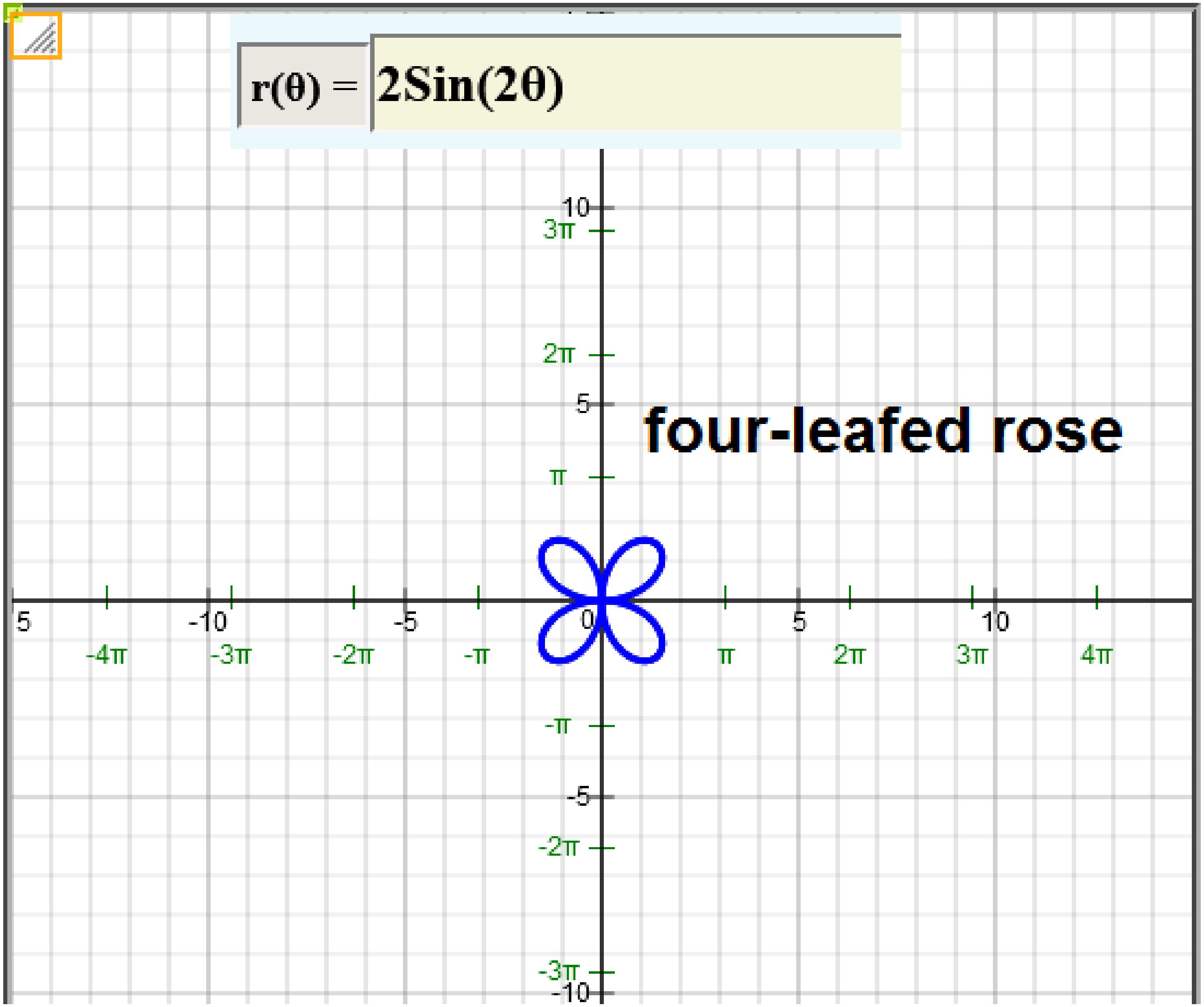




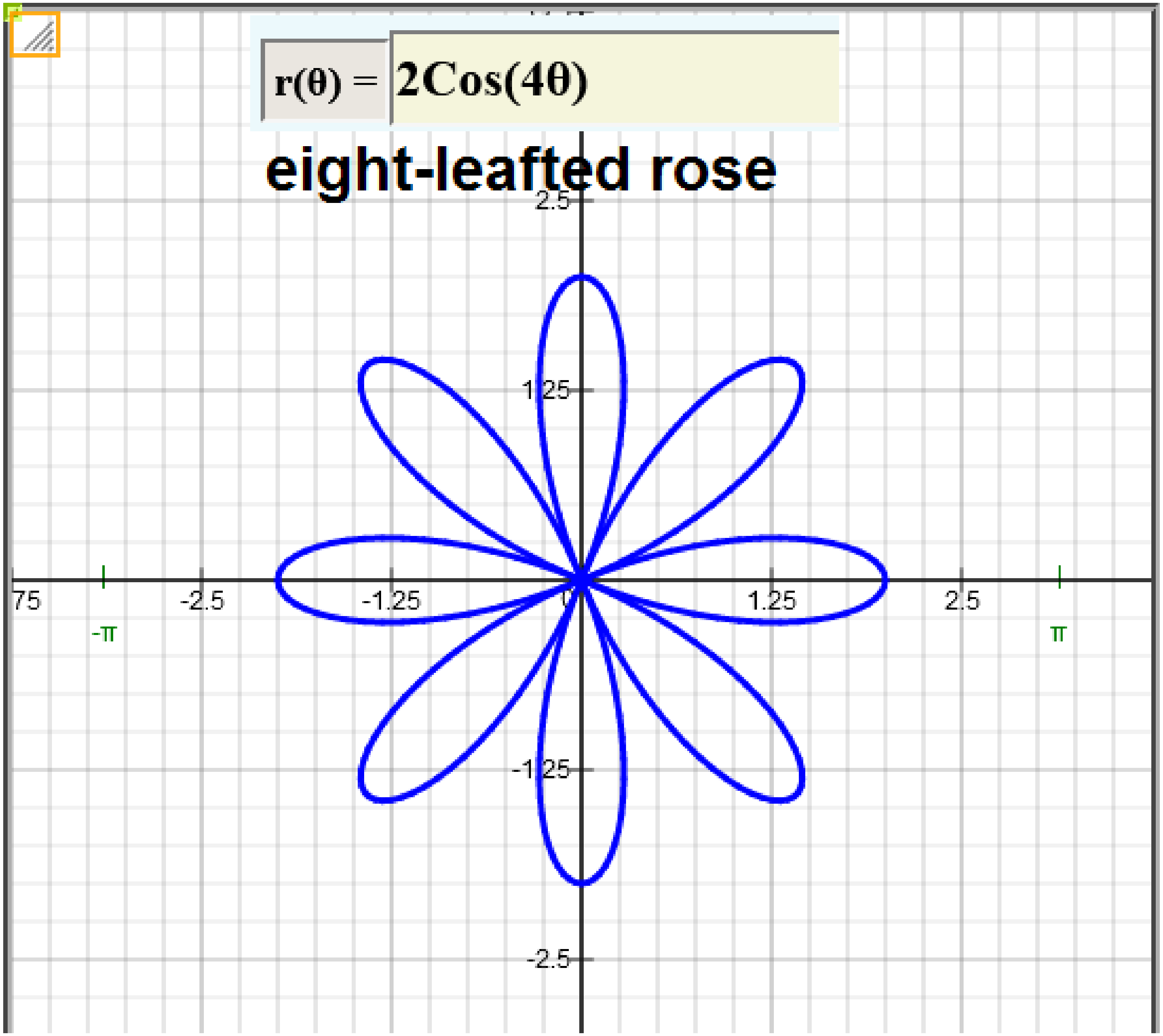
r(θ) = 4Sin(θ) - 5
Cardioid



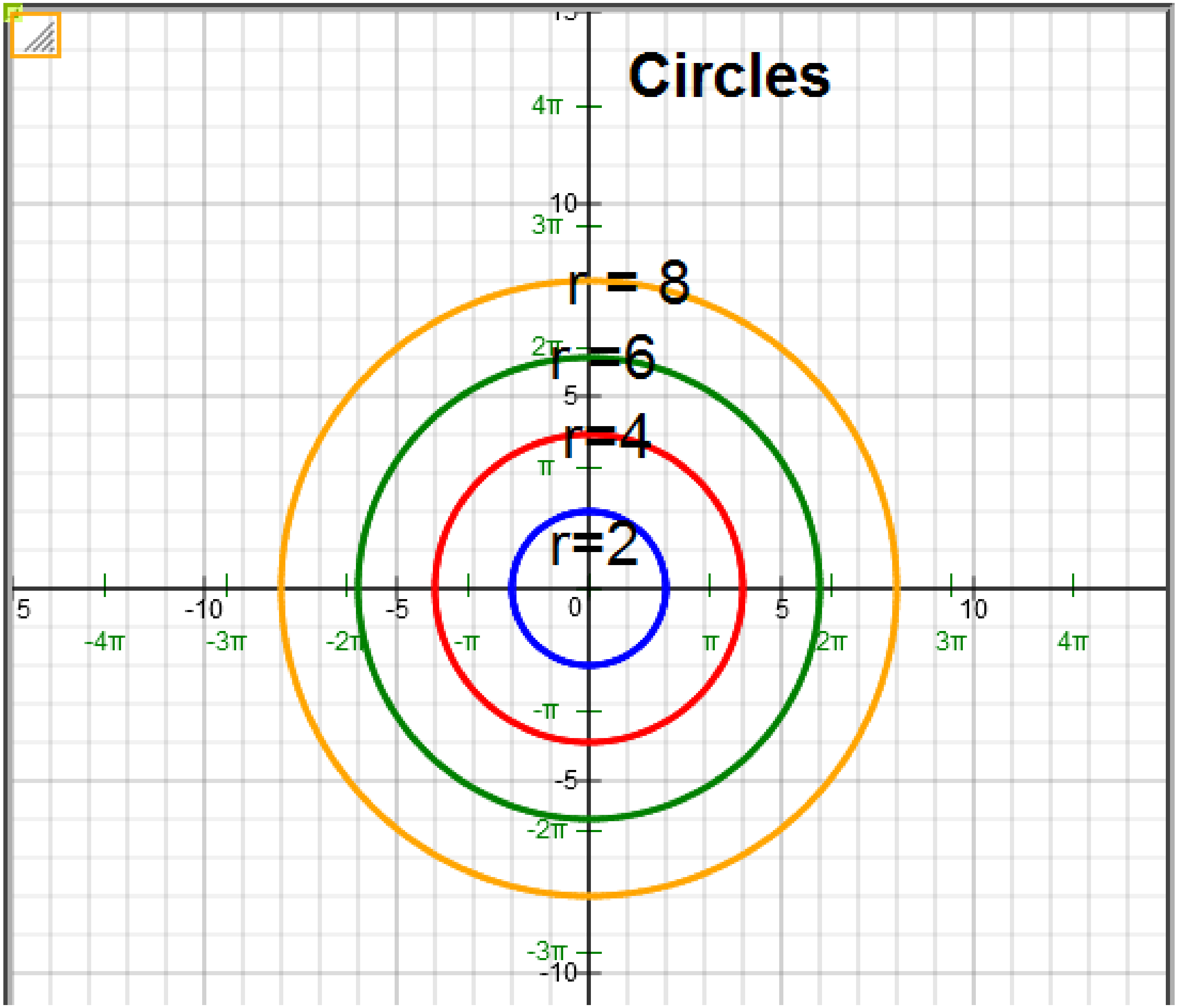
r(θ) = 4 + 2Cos(θ)
Limaçon

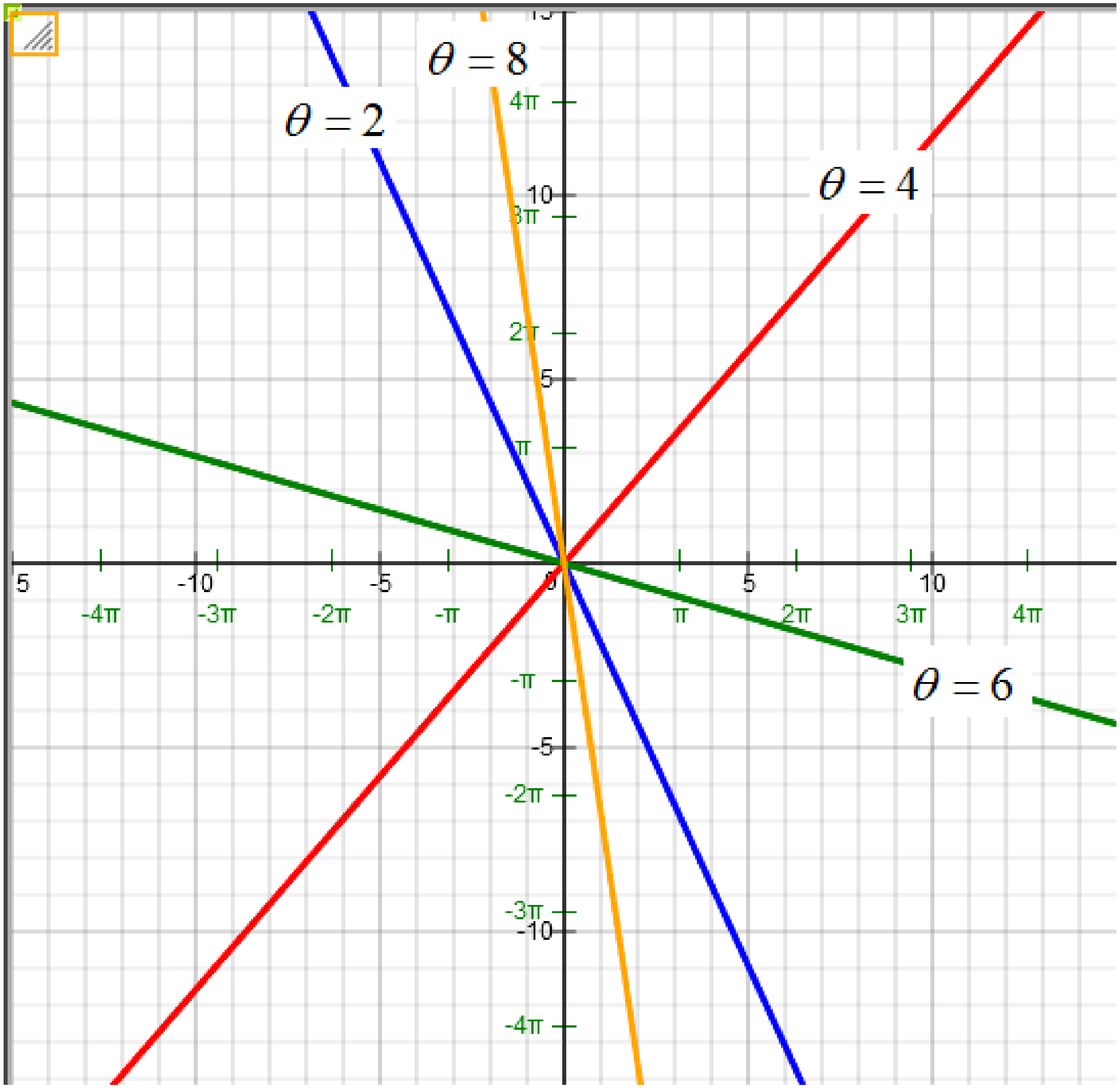


r(θ) = 2Sin(2θ)
four-leafed rose
10
3π
2π
5
π
-π
-5
-2π
-3π
-10
5
-10
-5
0
5
10
-4π
-3π
-2π
-π
π
2π
3π
4π



r(θ) = 2Cos(4θ)
eight-leafted rose
2.5
1.25
-1.25
-2.5
75
-2.5
-1.25
1.25
2.5
-π
π


Circles
r = 8
r=6
r=4
r=2
4π
10
3π
2π
5
π
0
-π
-5
-2π
-3π
-10
5
-10
-5
5
10
-4π
-3π
-2π
-π
π
2π
3π
4π



Note:

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan(\theta) = \frac{y}{x}$$

$$(x)\tan(\theta) = \frac{y}{x}(x)$$

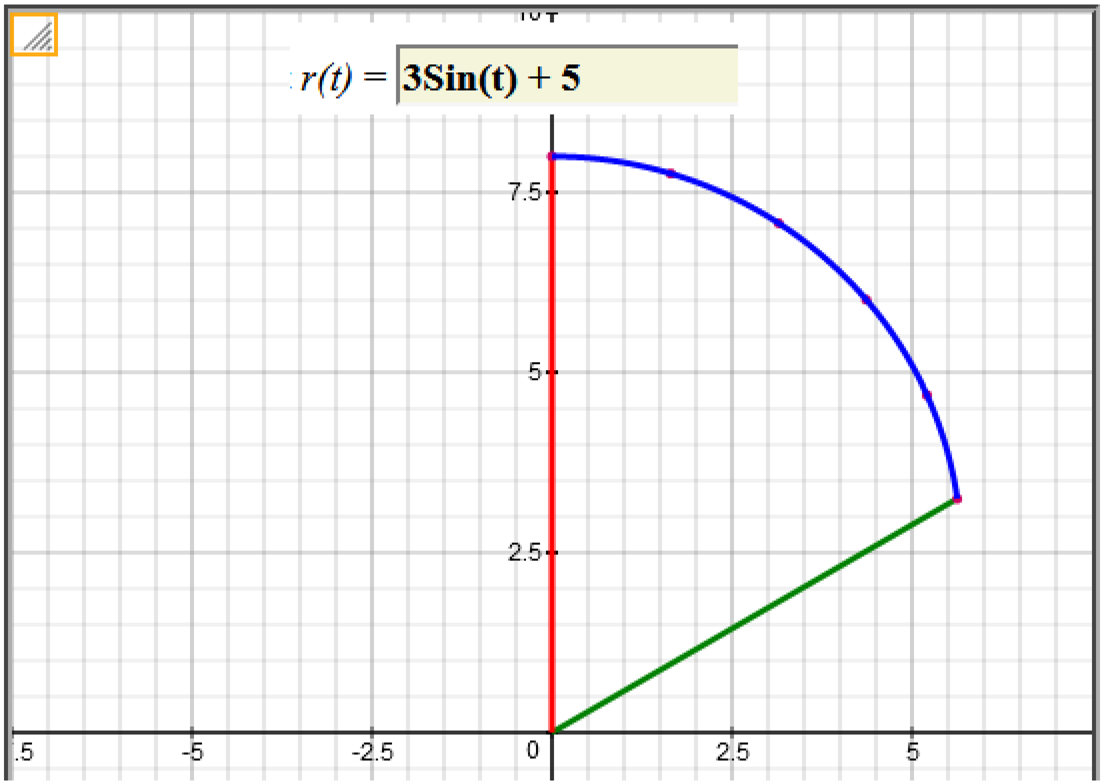
$$y = x \cdot \tan(\theta)$$

Area of Polar Region

Claim: $\text{Area} = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta} r^2 d\theta \qquad \alpha \le \theta \le \beta$

Find the area of the following polar region

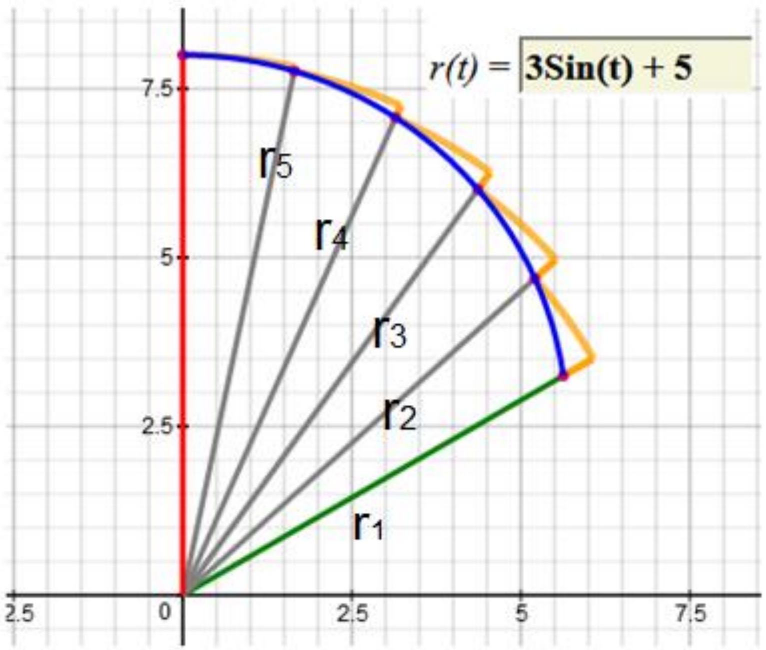
$$r = 3\sin(\theta) + 5 \quad \text{with} \quad \pi/6 \le \theta \le \pi/2$$



We will divide the polar region into 5 sectors.

Let r_i = radius of ith sector

$\Delta\theta_i$ = central angle of ith sector



sector 1: $\text{Area} = \frac{1}{2}r_1^2\Delta\theta_1$

sector 2: $\text{Area} = \frac{1}{2}r_2^2\Delta\theta_2$

sector 3: $\text{Area} = \frac{1}{2}r_3^2\Delta\theta_3$

sector 4: $\text{Area} = \frac{1}{2}r_4^2\Delta\theta_4$

sector 5: $\text{Area} = \frac{1}{2}r_5^2\Delta\theta_5$

$$\text{Area of Polar Region} \approx \frac{1}{2}r_1^2\Delta\theta_1 + \frac{1}{2}r_2^2\Delta\theta_2 + \frac{1}{2}r_3^2\Delta\theta_3 + \frac{1}{2}r_4^2\Delta\theta_4 + \frac{1}{2}r_5^2\Delta\theta_5$$

Suppose we partition the polar region for $r = 3\sin(t) + 5$ into n sectors, then

$$\text{Area of Polar Region } \approx \frac{1}{2}r_1^2\Delta\theta_1 + \frac{1}{2}r_2^2\Delta\theta_2 + \frac{1}{2}r_3^2\Delta\theta_3 + \cdots + \frac{1}{2}r_{n-1}^2\Delta\theta_{n-1} + \frac{1}{2}r_n^2\Delta\theta_n$$

$$\text{Area of Polar Region } \approx \sum_{i=1}^{n}\frac{1}{2}r_i^2\Delta\theta_i$$

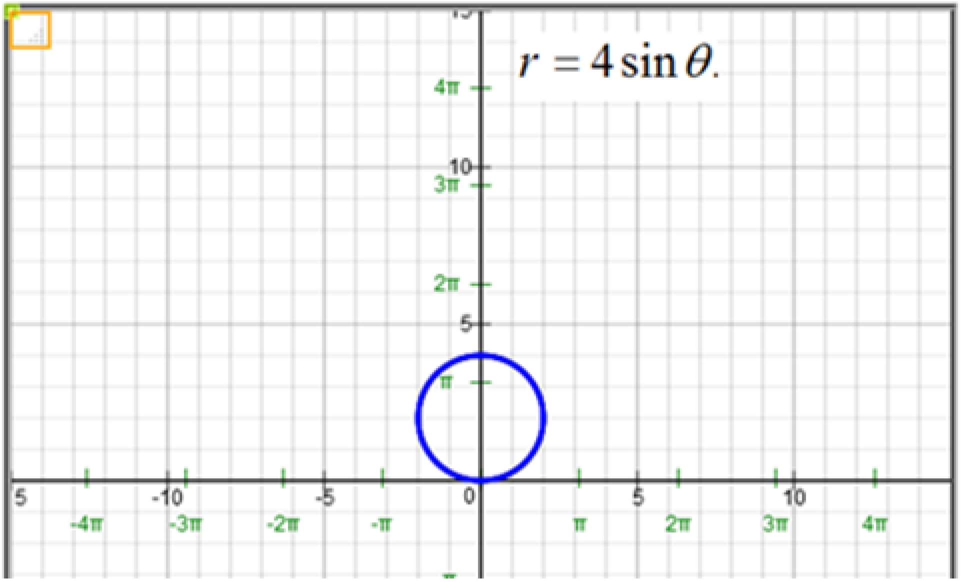
$$\text{Area of Polar Region} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{2}r_i^2\Delta\theta_i = \int_{\alpha}^{\beta}\frac{1}{2}r^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta$$

Therefore, area of the polar region for $r = 3\sin(\theta) + 5$ with $\pi/6 \le \theta \le \pi/2$ is:

$$\text{Area of Polar Region} = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\pi/6}^{\pi/2}[3\sin(\theta) + 5]^2\,d\theta = 29.410823254041$$

Example 1:

Find the area of the interior of $r = 4\sin\theta$.



First graph $r = 4\sin\theta$ with $0 \le \theta \le \pi$.

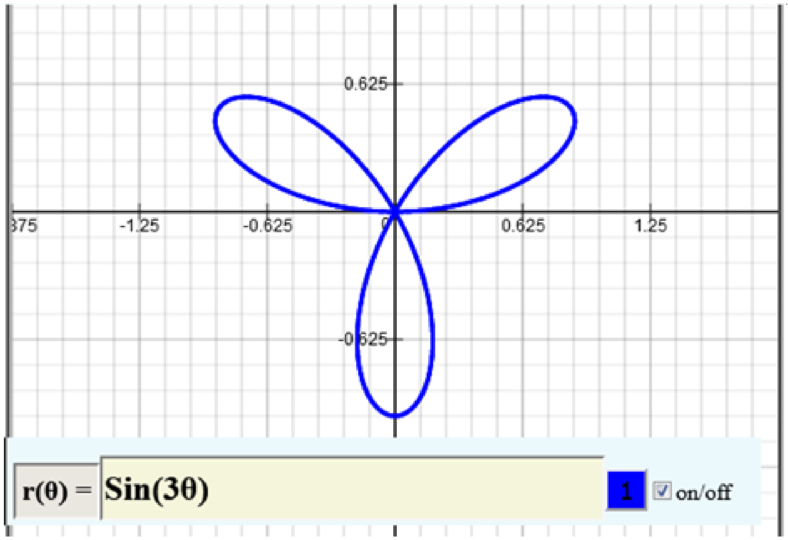
$$\text{Hence, Area} = \frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2\,d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2\,d\theta$$

$$= \frac{1}{2}\int_{0}^{\pi}[4\sin\theta]^2\,d\theta = 12.56637122$$

To evaluate definite integral, use numerical integration method like Trapezoid's Rule.

Example 2:

Find the area of one petal of $r = \sin 3\theta$.



From graph of $r = \sin 3\theta$ with $0 \le \theta \le 2\pi$ we see that there are three petals.

Note that each petal starts and ends at the pole (origin); and at the pole, $r = 0$.

To find where one petal starts and ends, set $r = 0$.

Hence $r = \sin 3\theta$

$$0 = \sin 3\theta$$

To solve this equation, we note that $\sin(0) = 0$; $\sin(\pi) = 0$; $\sin(2\pi) = 0$; and so on.

Hence, $3\theta = 0, \pi, 2\pi, 3\pi, \ldots$

$$\Rightarrow \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \ldots$$

Now graph $r = \sin 3\theta$ with $0 \leq \theta \leq \pi / 3$.

We can see from graph that we have one petal.

Hence, one petal starts at $\theta = 0$ and ends at $\theta = \dfrac{\pi}{3}$'

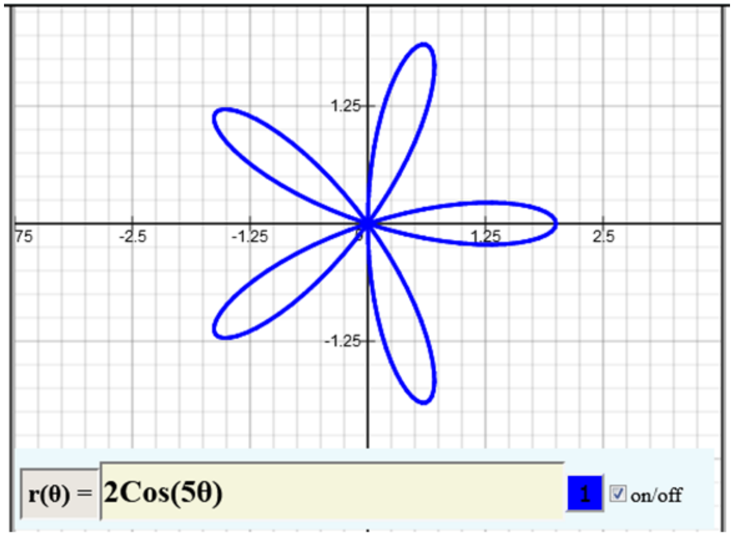
Area of one petal = $\dfrac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \dfrac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$

Area of one petal = $\dfrac{1}{2}\int_{0}^{\pi/3} [\sin 3\theta]^2 d\theta = 0.2617994$

To evaluate definite integral, use numerical integration method like Trapezoid's Rule.

Example 3:

Find the area of one petal of $r = 2\cos 5\theta$.



From graph of $r = 2\cos 5\theta$ with $0 \le \theta \le 2\pi$, we can see that there are five petals.

Note that each petal starts and ends at the pole (origin); and at the pole, $r = 0$.

To find where one petal starts and ends, set $r = 0$.

set $r = 2\cos 5\theta$

$$\Rightarrow \cos 5\theta = 0$$

To solve this equation, note that $\cos(\pi/2) = 0;\ \cos(3\pi/2) = 0;\ \cos(5\pi/2) = 0;$ and so on.

Hence, $5\theta = \pi/2,\ 3\pi/2,\ 5\pi/2,\ 7\pi/2,\ 9\pi/2...$

$$\Rightarrow \theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}, ...$$

Now graph $r = 2\cos 5\theta$ with $\dfrac{\pi}{10} \le \theta \le \dfrac{3\pi}{10}$.

From graph we can see that we have one petal.

Therefore,

Area of one petal = $\dfrac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta = \dfrac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$:

Area of one petal = $\dfrac{1}{2}\int_{\pi/10}^{3\pi/10} [2\cos 5\theta]^2 d\theta = 0.628318$

First graph $r = 2 + 5\cos\theta$ with $0 \le \theta \le 2\pi$.

Note that both inner and outer loops start and end at the pole (origin); and at the pole, $r = 0$. To find where the loops start and end, set $r = 0$.

Hence, $0 = 2+5\cos\theta$

Solving the equation $0 = 2 + 5\cos\theta$ using calculator:

$0 = 2 + 5\cos\theta$

$\cos\theta = -2/5$

$\theta = \cos^{-1}\left(-2/5\right) = 1.9823131728623846$ rad

and $\theta = \pi - \cos^{-1}\left(-2/5\right) = 2\pi - 1.9823131728623846 = 4.300872134317202$ rad

Also,

$\theta = 1.9823131728623846 + 2\pi = 8.265498480041972$ rad,

$\theta = 4.300872134317202 + 2\pi = 10.584057441496789$ rad

Now graph $r = 4 + 6\sin\theta$ with $1.9823131728623846 \le \theta \le 4.300872134317202$.

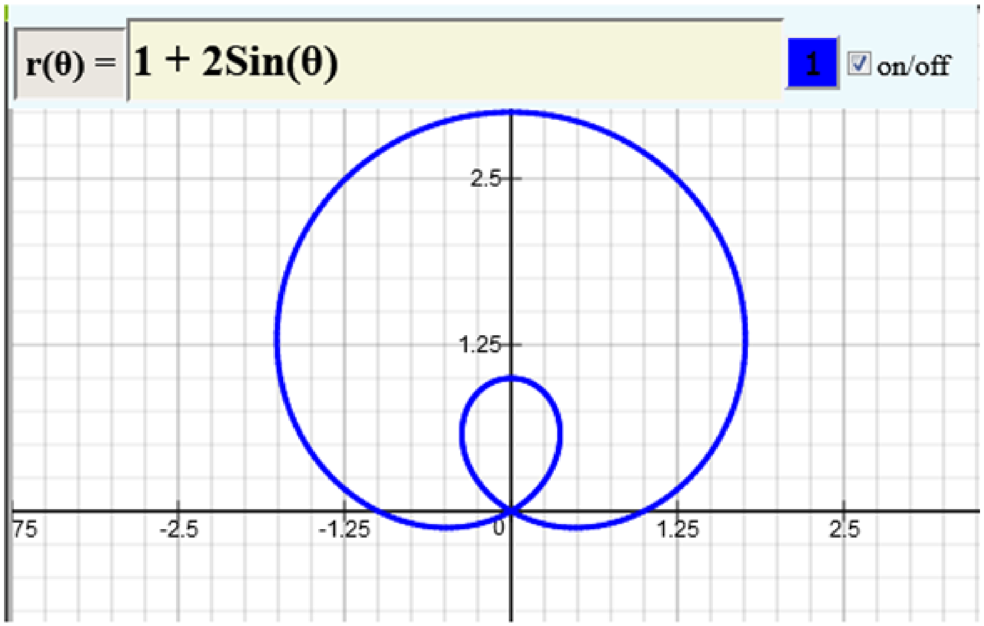
We have the inner loop.

Now graph $r = 4 + 6\sin\theta$ with $4.300872134317202 \le \theta \le 8.265498480041972$.

We have the outer loop.

Example 4:

Find the area of the inner loop of $r = 1 + 2\sin\theta$



$r = 1 + 2\sin\theta$ with $0 \le \theta \le 2\pi$.

Note that the inner loop starts and ends at the pole (origin); and at the pole, $r = 0$.

To find where the inner loop starts and ends, set $r = 0$.

Hence $r = 1 + 2\sin\theta$

$\Leftrightarrow$ $1 + 2\sin\theta = 0$

$\Leftrightarrow$ $\sin\theta = -1/2$

To solve this equation, note that from Trigonometric Table of Values $\sin(7\pi/6) = -1/2$ and $\sin(11\pi/6) = -1/2$.

$\Rightarrow \theta = 7\pi/6,\ 11\pi/6,\ 7\pi/6 + 2\pi = 19\pi/6,\ 11\pi/6 + 2\pi = 23\pi/6,\ \ldots$

Now graph $r = 1 + 2\sin\theta$ with $7\pi/6 \le \theta \le 11\pi/6$.

We can see from graph that we have the inner loop.

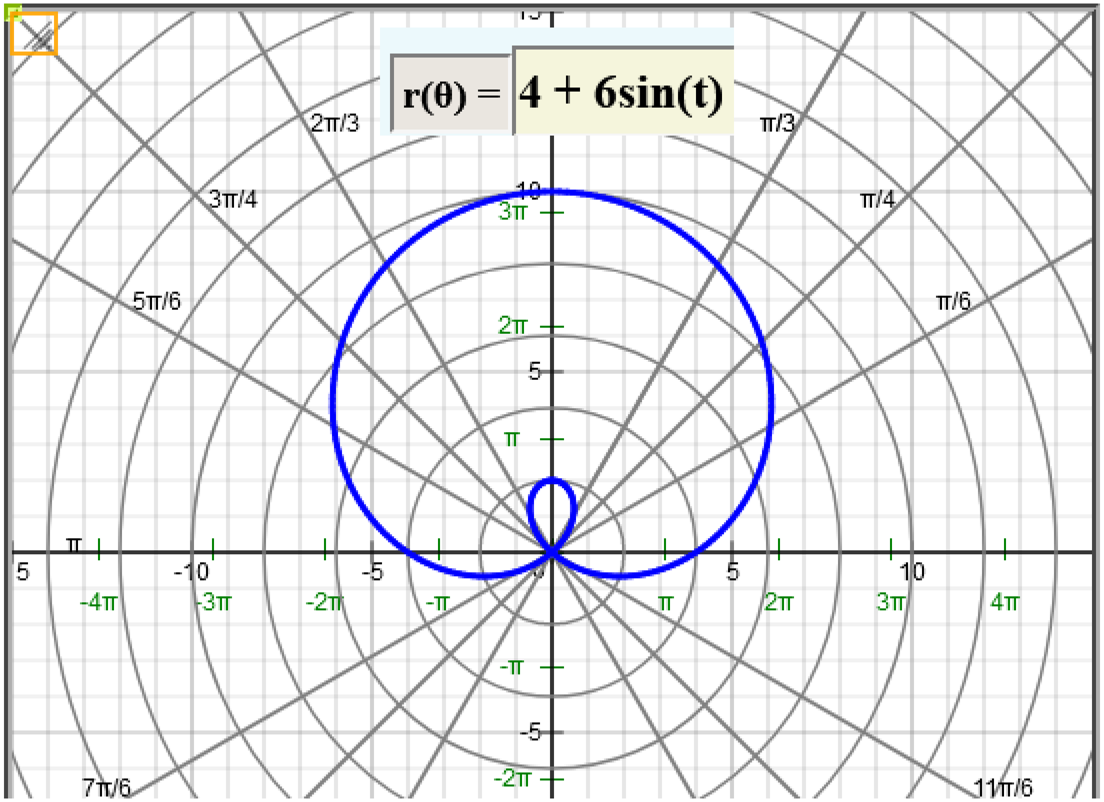
Therefore, the inner loop starts at $\theta = \dfrac{7\pi}{6}$ and ends at $\theta = \dfrac{11\pi}{6}$.

Area of the inner loop= $\dfrac{1}{2}\displaystyle\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \dfrac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$:

Area of inner loop = $\dfrac{1}{2}\displaystyle\int_{7\pi/6}^{11\pi/6}[1 + 2\sin\theta]^2 d\theta = 0.54351646$

Example 5:

Find the area of the inner loop of $r = 4 + 6\sin\theta$.



Note that the inner loop starts and ends at the pole (origin); and at the pole, $r = 0$. To find where the inner loop starts and ends, set $r = 0$.

Hence, $0 = 4 + 6\sin\theta$

First graph $r = 4 + 6\sin\theta$ with $0 \le \theta \le 2\pi$.

Note that the inner loop starts and ends at the pole (origin); and at the pole, $r = 0$. To find where the inner loop starts and ends, set $r = 0$.

Hence, $0 = 4 + 6\sin\theta$

Solving the equation $0 = 4 + 6\sin\theta$ using calculator:

$0 = 4 + 6\sin\theta$

$\sin\theta = -4/6$

$\theta = \sin^{-1}(-4/6) =$ -0.7297276562269663 rad

and $\theta = \pi - \sin^{-1}(-4/6) = \pi - -0.7297276562269663 = 3.8713203098167597$ rad

Also,

$\theta = -0.7297276562269663 + 2\pi = 5.55345765095262$ rad,

$\theta = 3.8713203098167597 + 2\pi = 10.154505616996346$ rad

Now graph $r = 4 + 6\sin\theta$ with $3.8713203098167597 \le \theta \le 5.55345765095262$.

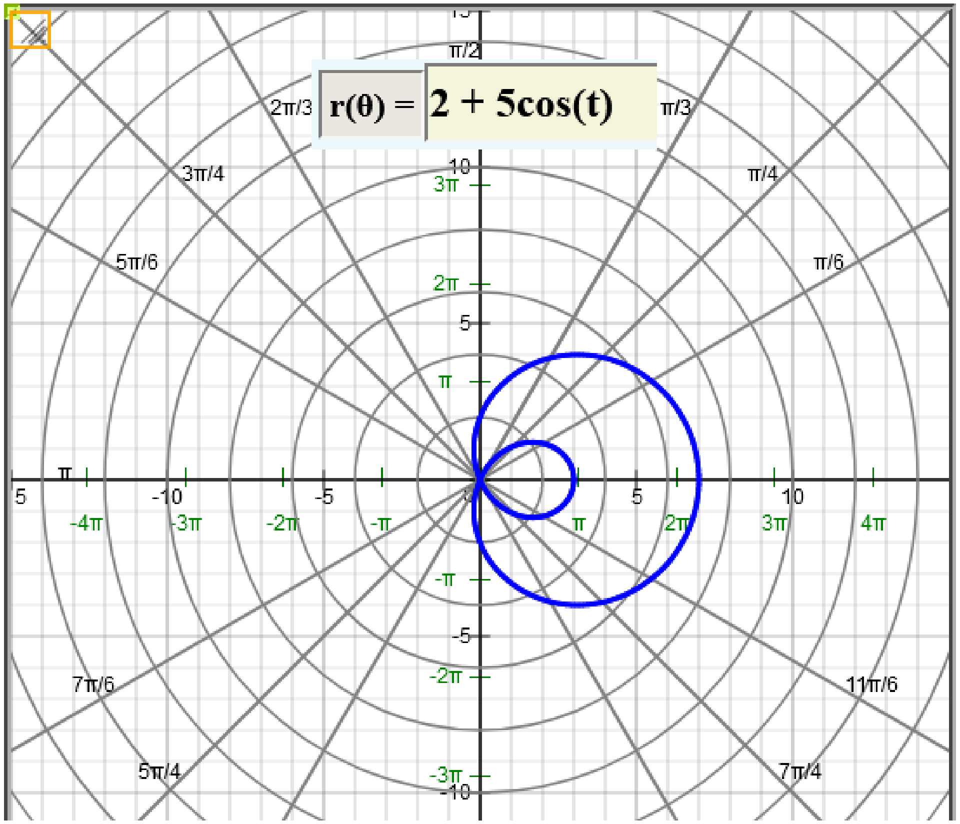
We can see from graph, the inner loop starts at $\theta = 3.8713203098167597$ and ends at $\theta = 5.55345765095262$.

Area of the inner loop= $\frac{1}{2}\int_{\alpha}^{\beta}[f(\theta)]^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} r^2 d\theta$:

Area of inner loop = $\int_{3.8713203098167597}^{5.55345765095262} \frac{1}{2}[4 + 6\sin\theta]^2 d\theta = 1.763518902619$

Example 6:

Find the area between the inner loop and outer loop of $r = 2 + 5\cos\theta$.



$r = 2 + 5\cos\theta$ with $0 \le \theta \le 2\pi$.

Note that both inner and outer loops start and end at the pole (origin); and at the pole, $r = 0$. To find where the loops start and end, set $r = 0$.

Hence, $0 = 2+5\cos\theta$

Solving the equation $0 = 2 + 5\cos\theta$ using calculator:

$0 = 2 + 5\cos\theta$

$\cos\theta = -2/5$

$\theta = \cos^{-1}\left(-2/5\right) = 1.9823131728623846$ rad

and $\theta = \pi - \cos^{-1}\left(-2/5\right) = 2\pi - 1.9823131728623846 = 4.300872134317202$ rad

Also,

$\theta = 1.9823131728623846 + 2\pi = 8.265498480041972$ rad,

$\theta = 4.300872134317202 + 2\pi = 10.584057441496789$ rad

Now graph $r = 4 + 6\sin\theta$ with $1.9823131728623846 \le \theta \le 4.300872134317202$.

We have the inner loop.

Now graph $r = 4 + 6\sin\theta$ with $4.300872134317202 \le \theta \le 8.265498480041972$.

We have the outer loop.

Area of inner loop = $\displaystyle\int_{1.9823131728623846}^{4.300872134317202} \frac{1}{2}[2 + 5\cos\theta]^2\, d\theta = 5.380384259649$

Area of outer loop = $\displaystyle\int_{4.300872134317202}^{8.265498480041972} \frac{1}{2}[2 + 5\cos\theta]^2\, d\theta = 46.455892628869$

Area between inner and outer loops = $46.455892628869 - 5.380384259649 = 41.07550836922$

Arc Length of Polar Region

$$\text{Arc Length} = \int_{\alpha}^{\beta} \sqrt{\left[f(\theta)\right]^2 + \left[f'(\theta)\right]^2}\, d\theta \;=\; \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example 7:

Find the length of the arc for $r = 4\sin\theta \qquad 0 \le \theta \le \pi$

a) Graph $r = 2\cos 5\theta \qquad 0 \le \theta \le \pi$ (see below).

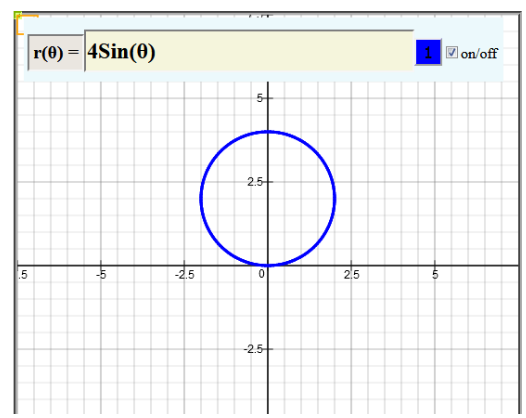
b) Find Arc Length $= \displaystyle\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$:

$$\frac{dr}{d\theta} = 4\cos\theta$$

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{\pi} \sqrt{\left[4\sin\theta\right]^2 + \left[4\cos\theta\right]^2}\, d\theta$$

$$= \int_{0}^{\pi} \sqrt{16\left[\sin\theta\right]^2 + 16\left[\cos\theta\right]^2}\, d\theta \;= \int_{0}^{\pi} \sqrt{16\left(\left[\sin\theta\right]^2 + \left[\cos\theta\right]^2\right)}\, d\theta$$

$$= \int_{0}^{\pi} \sqrt{16(1)}\, d\theta = \int_{0}^{\pi} 4\, d\theta = 4\theta\Big|_0^{\pi} = 4\pi$$



Example 8:

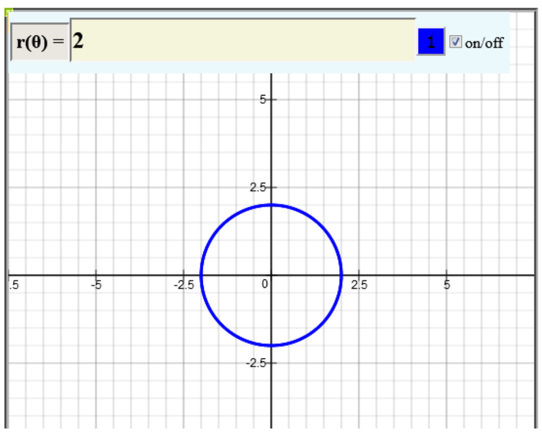
Find the length of the arc for $r = 2 \qquad 0 \le \theta \le 2\pi$

a) Graph $r = 2 \qquad 0 \le \theta \le 2\pi$ (see below).

b) Find Arc Length = $\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$:

$$\frac{dr}{d\theta} = 0$$

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2^2 + 0^2}\, d\theta = \int_0^{2\pi} 2\, d\theta = 2\theta\Big|_0^{2\pi} = 4\pi$$



Example 9:

Find the length of the arc for $r = 2 + 3\sin\theta \qquad 0 \le \theta \le \pi/4$

a) Graph $r = 2 + 3\sin\theta \qquad 0 \le \theta \le \pi/4 \qquad$ (see below).

b) Find Arc Length = $\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$:

$$\frac{dr}{d\theta} = 3\cos\theta$$

$$\int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{0}^{\pi/4} \sqrt{(2 + 3\sin\theta)^2 + (3\cos\theta)^2}\, d\theta = 3.274805725$$

To evaluate definite integral, we can use numerical integration method like Trapezoid's Rule.