Vectors in the Plane

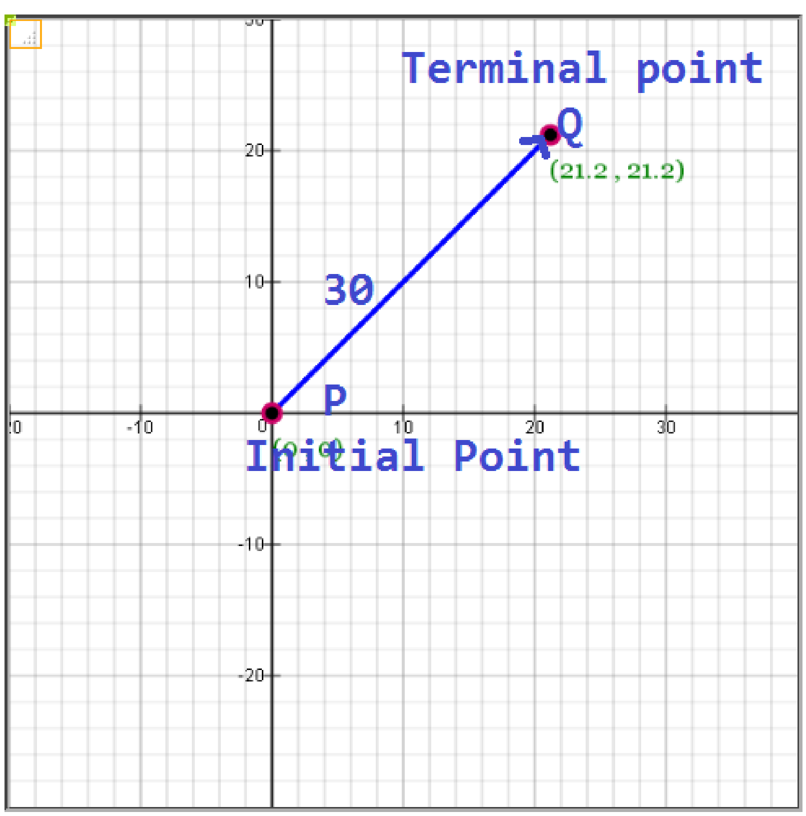
Scalar Quantities: area, volume, height, temperature, time,...

Vector Quantity is a quantity of something which possesses both magnitude and direction.

Examples Vector Quantities: force, velocity, acceleration, ...

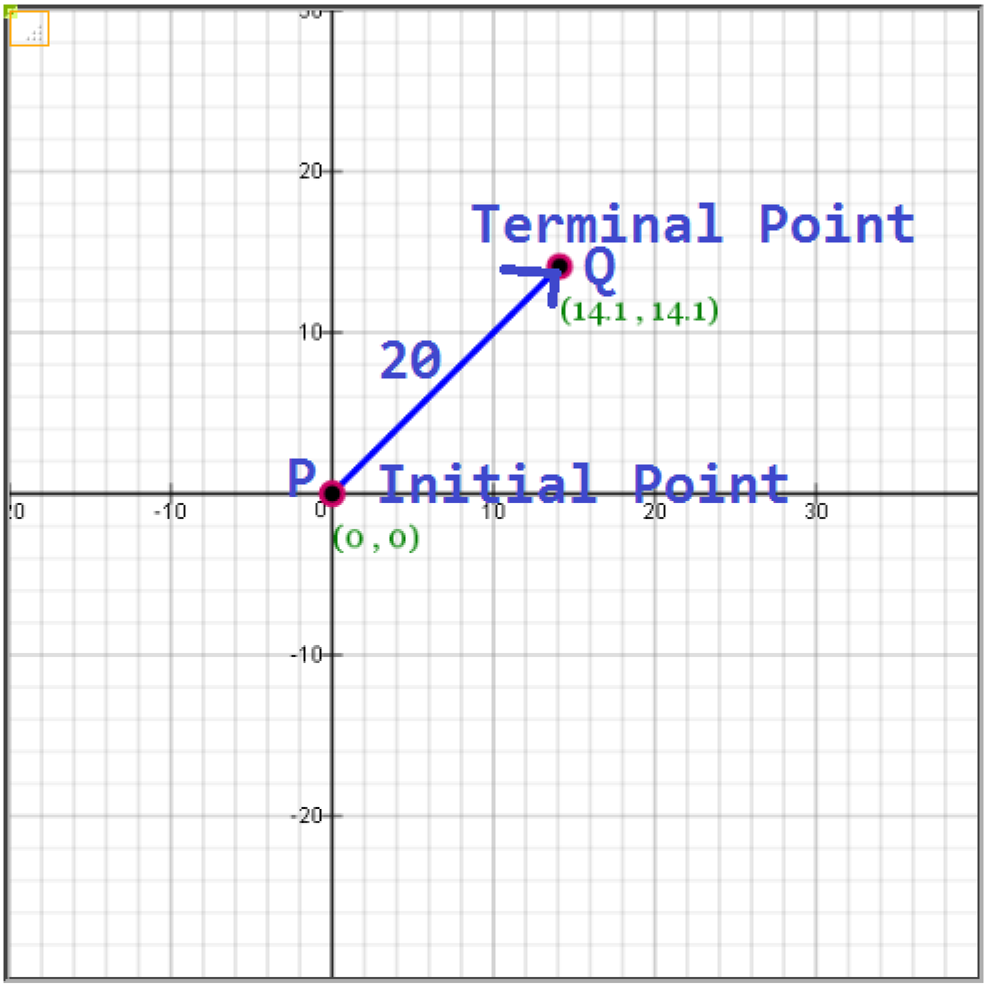
Example 1: Representing Vector with directed line segment.

A car's velocity of 30 miles per hour, going northeast.



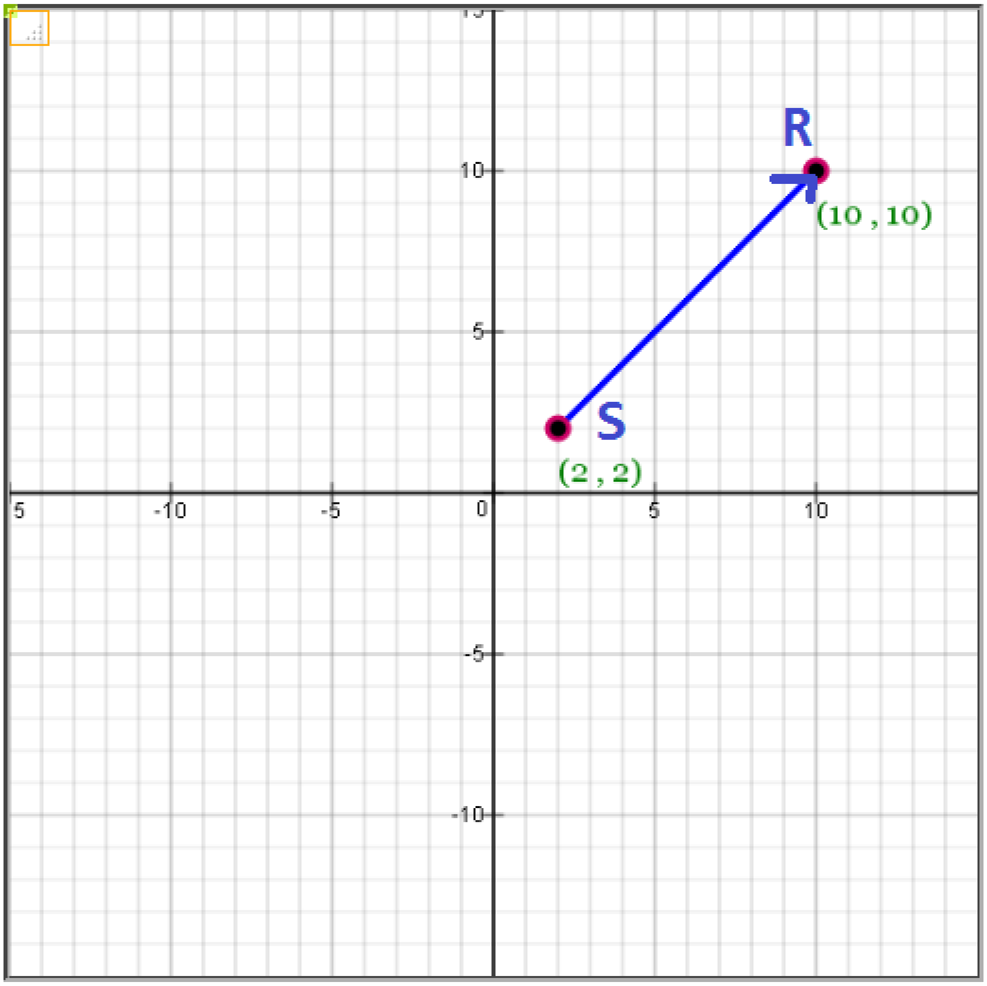
$$\text{Length or Magnitude of Vector} = \left\| \overrightarrow{PQ} \right\| = 30$$

Example 2: A car's acceleration of 20 miles per hour, going northeast.



$$\text{Length or Magnitude of Vector} = \left\| \overrightarrow{PQ} \right\| = 20$$

Example 3: Find the length of vector $\overrightarrow{SR}$.

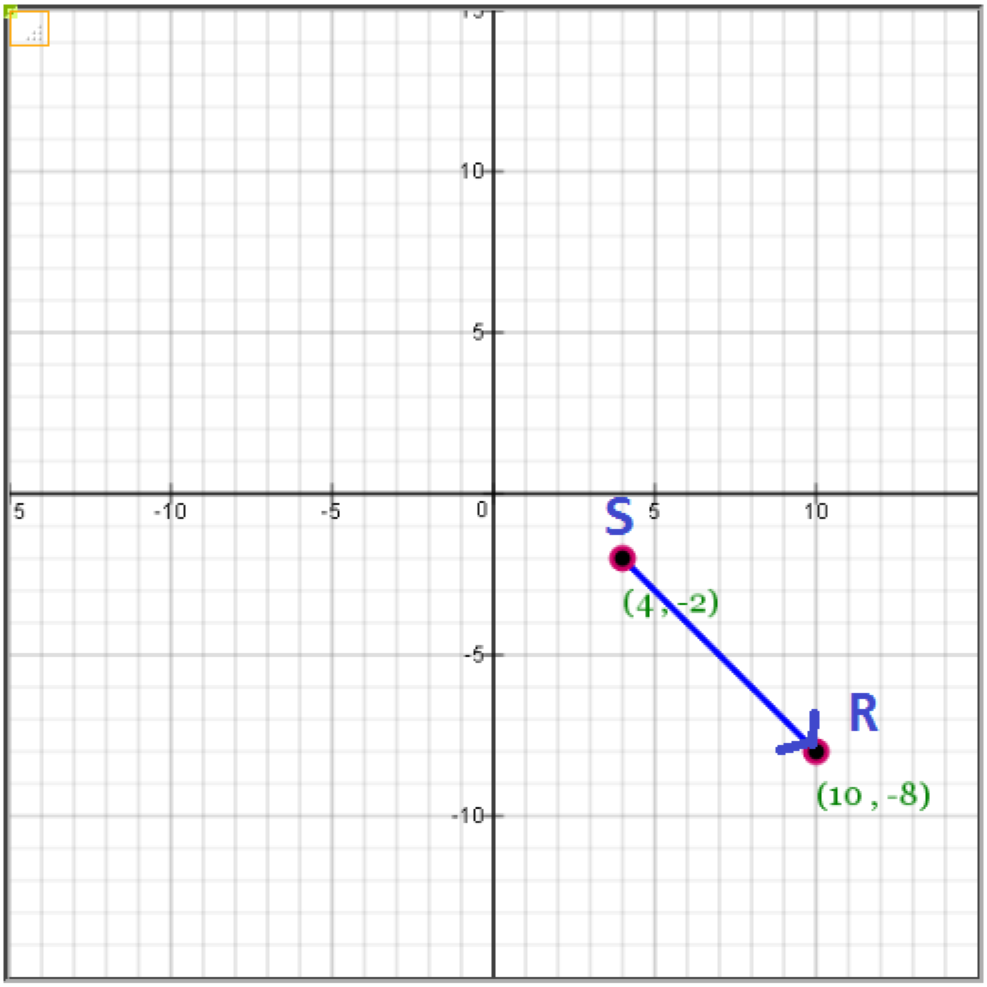


Length or Magnitude of Vector = $\left\| \overrightarrow{SR} \right\|$

$=$ distance between R and S

$$= \sqrt{(10-2)^2 + (10-2)^2} = 11.3137$$

Example 4: Find the length of vector $\overrightarrow{SR}$.

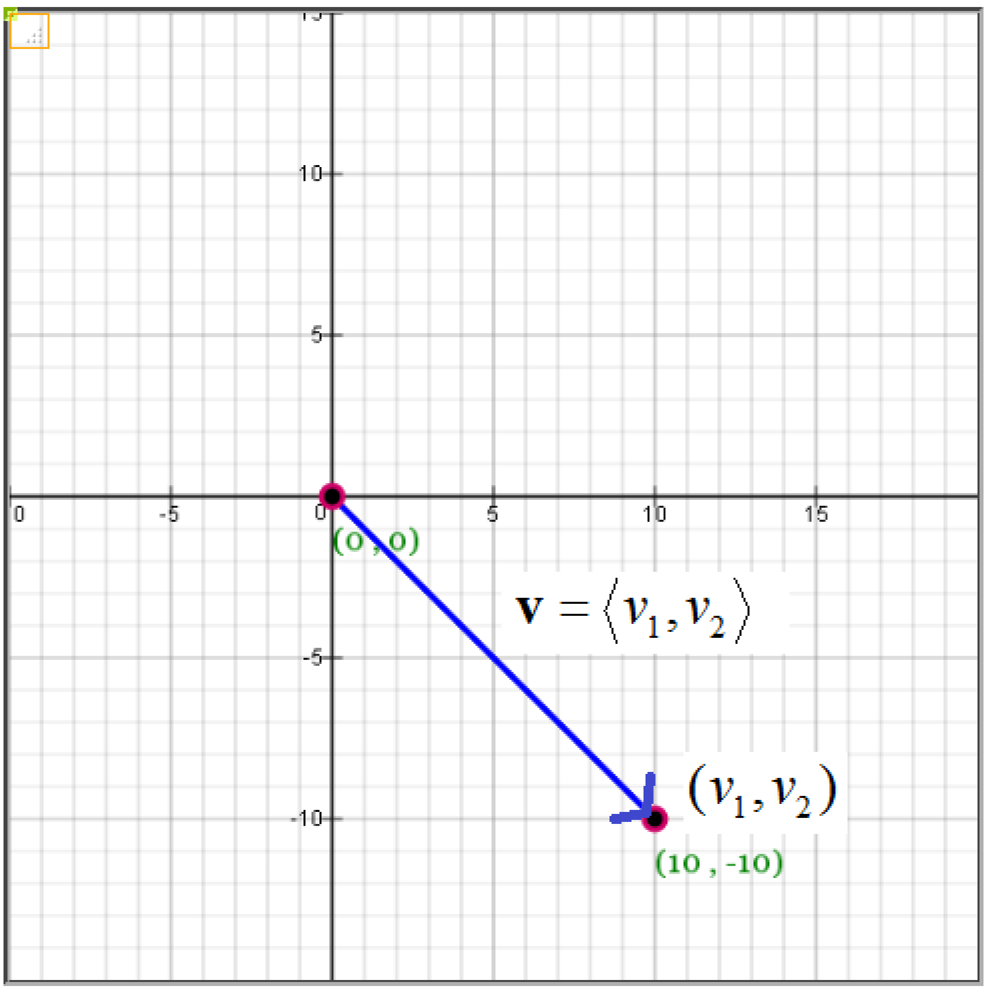


Length or Magnitude of Vector = $\left\|\overrightarrow{SR}\right\|$

= distance between R and S

$= \sqrt{(10-4)^2 + (-8 - -2)^2} = 8.4852813$

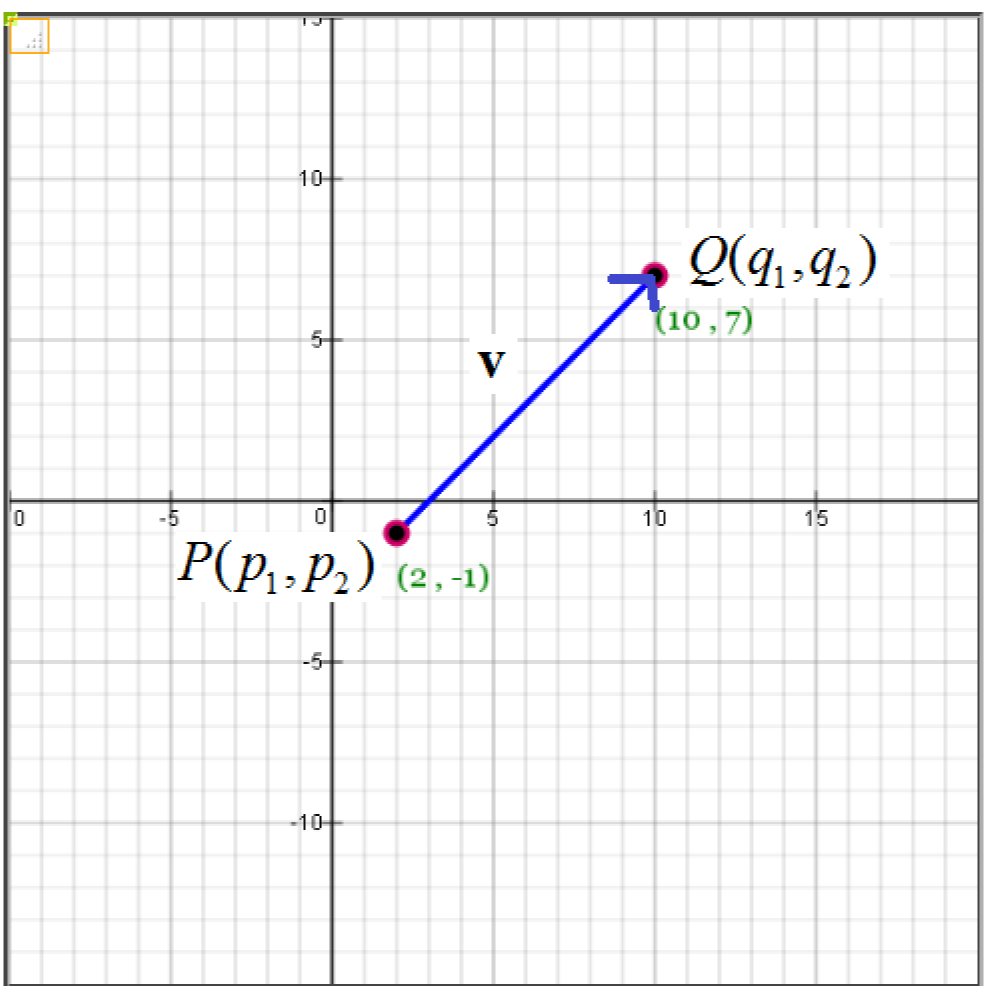
Vector in Standard Position: Vector with initial point at origin.



Components of vector $\mathbf{v}$ are v_1 and v_2.

Magnitude of $\mathbf{v} = \|\mathbf{v}\| = \|\langle v_1, v_2 \rangle\| = \sqrt{(10)^2 + (10)^2} = \sqrt{200}$

Vector in Non-Standard Position:



Components of vector **v** are v_1 and v_2 :

$$\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle 10 - 2, 7 - -1 \rangle = \langle 8,8 \rangle$$

Magnitude of **v** :

$$\text{Magnitude of } \mathbf{v} = \|\mathbf{v}\| = \|\langle v_1, v_2 \rangle\| = \sqrt{(8)^2 + (8)^2} = \sqrt{128}$$

Vector Operations:

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$

a) $\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$

b) $\mathbf{-u} = \langle -u_1, -u_2 \rangle$; $-\mathbf{v} = \langle -v_1, -v_2 \rangle$

c) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1, u_2 \rangle + \langle -v_1, -v_2 \rangle = \langle u_1 + -v_1, u_2 + -v_2 \rangle$

d) Scalar Multiple:

$$b\mathbf{u} = b\langle u_1, u_2 \rangle = \langle bu_1, bu_2 \rangle$$

$$c\mathbf{v} = c\langle v_1, v_2 \rangle = \langle cv_1, cv_2 \rangle$$

Examples:

Let $\mathbf{u} = \langle 3, 7 \rangle$ and $\mathbf{v} = \langle 1, 8 \rangle$

a) $\mathbf{u} + \mathbf{v} = \langle 3, 7 \rangle + \langle 1, 8 \rangle = \langle 4, 15 \rangle$

b) $\mathbf{-u} = \langle -3, -7 \rangle$; $-\mathbf{v} = \langle -1, -8 \rangle$

c) $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle 3, 7 \rangle + \langle -1, -8 \rangle = \langle 2, -1 \rangle$

d) Scalar Multiple:

$$9\mathbf{u} = 9\langle 3, 7 \rangle = \langle 27, 63 \rangle$$

$$\text{-}3\mathbf{v} = -3\langle 1, 8 \rangle = \langle -3, -24 \rangle$$

$e)\ 3\mathbf{u}+5\mathbf{v}=3\langle 3,7\rangle+5\langle 1,8\rangle=\langle 9,21\rangle+\langle 5,40\rangle=\langle 14,61\rangle$

$f)\ 8\mathbf{u}-6\mathbf{v}=8\mathbf{u}+\left(-6\mathbf{v}\right)=8\langle 3,7\rangle+(-6)\langle 1,8\rangle$

$$=\langle 24,56\rangle+\langle -6,-48\rangle=\langle 18,8\rangle$$

Unit Vector = vector with length of 1.

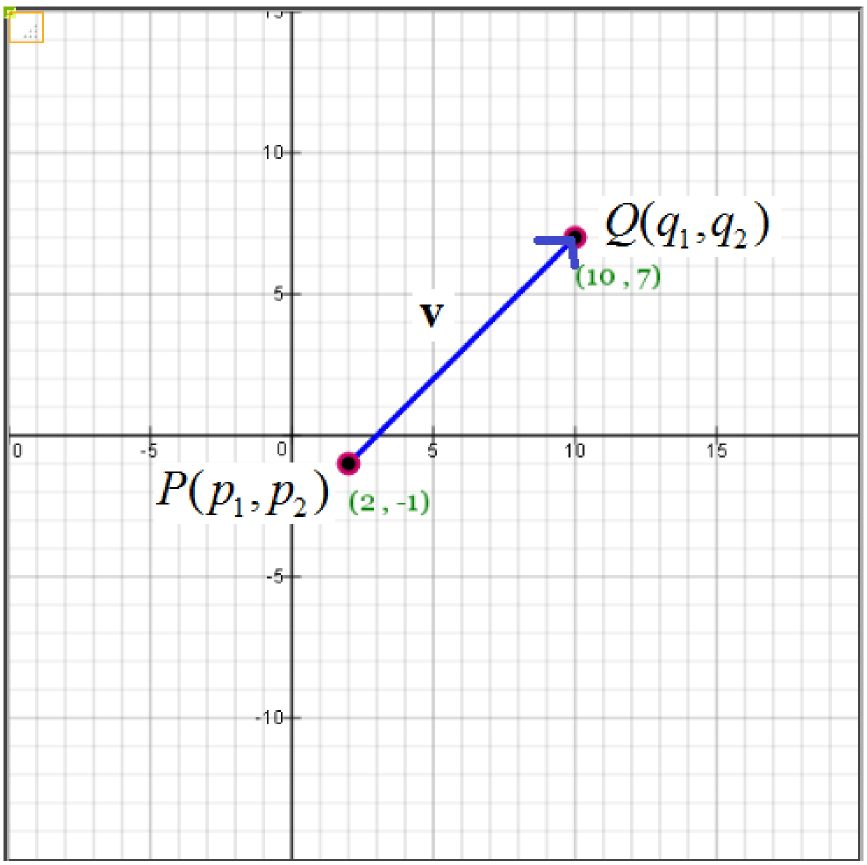
Let $\mathbf{u}$ be a unit vector in the direction of $\mathbf{v}$:

Then $\mathbf{u}=\dfrac{\mathbf{v}}{\|\mathbf{v}\|}=\dfrac{1}{\|\mathbf{v}\|}\mathbf{v}$ Note: $\mathbf{v}$ is a vector; $\|\mathbf{v}\|$ is a scalar.

Show that $\|\mathbf{u}\|=1$.

$$\|\mathbf{u}\|=\left\|\frac{1}{\|\mathbf{v}\|}\mathbf{v}\right\|=\frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\|=\frac{\|\mathbf{v}\|}{\|\mathbf{v}\|}=1$$

Finding unit vector in the same direction as vector **v**



Components of vector $\mathbf{v}$ are v_1 and v_2 :

$$\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle = \langle 10 - 2, 7 - -1 \rangle = \langle 8, 8 \rangle$$

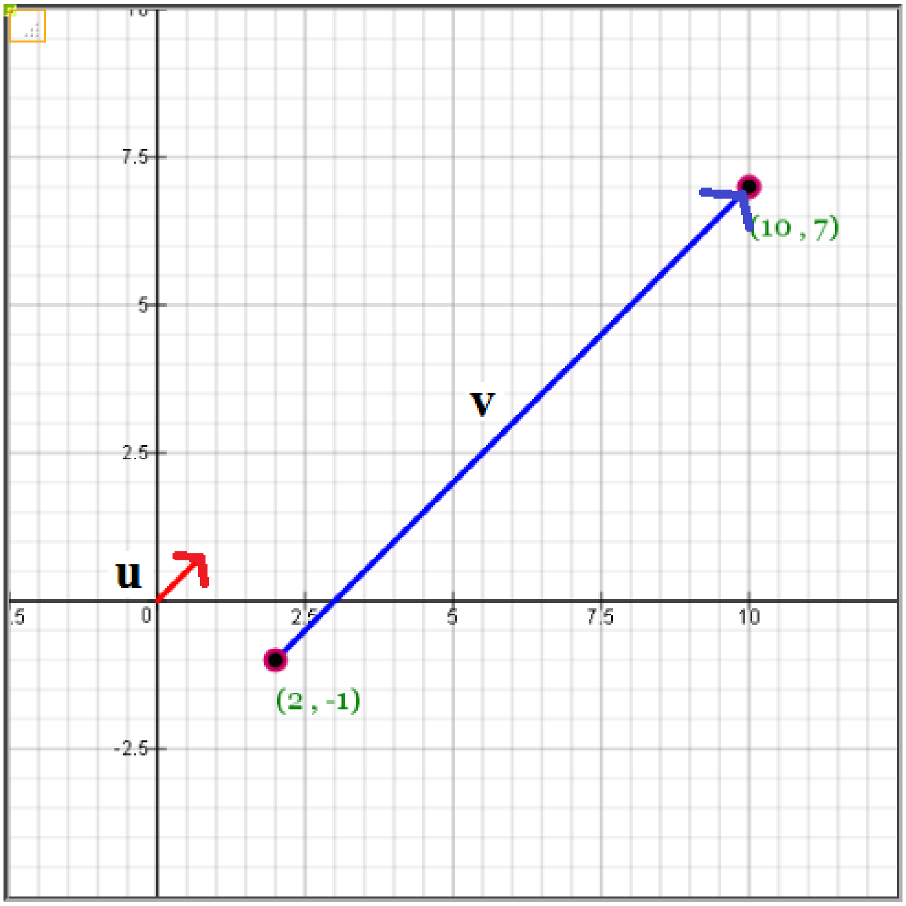
Magnitude of $\mathbf{v}$:

$$\text{Magnitude of } \mathbf{v} = \|\mathbf{v}\| = \|\langle v_1, v_2 \rangle\| = \sqrt{(8)^2 + (8)^2} = \sqrt{128}$$

$$\text{Let } \mathbf{u} = \frac{\mathbf{1}}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{\sqrt{128}}\langle 8, 8 \rangle = \left\langle \frac{8}{\sqrt{128}}, \frac{8}{\sqrt{128}} \right\rangle$$

$$\|\mathbf{u}\| = \left\| \left\langle \frac{8}{\sqrt{128}}, \frac{8}{\sqrt{128}} \right\rangle \right\| = \sqrt{\left(\frac{8}{\sqrt{128}}\right)^2 + \left(\frac{8}{\sqrt{128}}\right)^2} = 1$$

Hence, $\mathbf{u}$ is a unit vector in the same direction as $\mathbf{v}$.



u has length of 1 and **u** is the same direction as **v**.

Standard Unit Vectors:

$\mathbf{i} = \langle 1, 0 \rangle$ and $\mathbf{j} = \langle 0, 1 \rangle$

Let $\mathbf{v} = \langle v_1, v_2 \rangle$

Show that $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$:

$$v_1\mathbf{i} + v_2\mathbf{j} = v_1\langle 1, 0 \rangle + v_2\langle 0, 1 \rangle = \langle v_1, 0 \rangle + \langle v_2, 0 \rangle = \langle v_1, v_2 \rangle = \mathbf{v}$$

Example: Let $\mathbf{v} = \langle 2, 7 \rangle$

Then $\mathbf{v}$ can be expressed with unit vectors as follows;

$\mathbf{v} = 2\mathbf{i} + 7\mathbf{j}$

Example: Let $\mathbf{u} = \langle -2, 5 \rangle$

Then $\mathbf{u}$ can be expressed with unit vectors as follows;

$\mathbf{u} = -2\mathbf{i} + 5\mathbf{j}$

Example: Let $\mathbf{u} = 7\mathbf{i} + 9\mathbf{j}$

Find magnitude of **u**.

Note: $\mathbf{u} = 7\mathbf{i} + 9\mathbf{j} = \langle 7, 9 \rangle$

$$\|\mathbf{u}\| = \sqrt{(7)^2 + (9)^2} = \sqrt{112}$$

Example: Let $\mathbf{u} = \langle 7, 9 \rangle$ and $\mathbf{v} = \langle 1, 2 \rangle$

$$\|\mathbf{u}\| = \sqrt{(7)^2 + (9)^2} = \sqrt{130}$$

$$\|\mathbf{v}\| = \sqrt{(1)^2 + (2)^2} = \sqrt{5}$$

$$\mathbf{u} + \mathbf{v} = \langle 8, 11 \rangle$$

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{(8)^2 + (121)^2} = \sqrt{185}$$

$\mathit{Note}: \|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$ (or $\sqrt{185} \le \sqrt{130} + \sqrt{5}$)

$$\left\|\frac{\mathbf{u}}{\|\mathbf{u}\|}\right\| = \left\|\frac{1}{\|\mathbf{u}\|} \cdot \mathbf{u}\right\| = \frac{1}{\|\mathbf{u}\|}\|\mathbf{u}\| = \frac{1}{\sqrt{130}}\sqrt{130} = 1$$

$$\left\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\right\| = \left\|\frac{1}{\|\mathbf{v}\|} \cdot \mathbf{v}\right\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = \frac{1}{\sqrt{5}}\sqrt{5} = 1$$

$$\left\|\frac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|}\right\| = \left\|\frac{1}{\|\mathbf{u} + \mathbf{v}\|} \cdot (\mathbf{u} + \mathbf{v})\right\| = \frac{1}{\|\mathbf{u} + \mathbf{v}\|}\|\mathbf{u} + \mathbf{v}\| = \frac{1}{\sqrt{185}}\sqrt{185} = 1$$

Find vector **v** with given magnitude that has same direction as **u**:

Let $\|\mathbf{v}\| = 3$ and $\mathbf{u} = \langle 4, 7 \rangle$

Note: $\|\mathbf{u}\| = \sqrt{4^2 + 7^2} = \sqrt{65}$

Let **w** be the unit vector in the same direction as **u**

$$\mathbf{w} = \frac{1}{\|\mathbf{u}\|}\mathbf{u} = \frac{1}{\sqrt{65}}\langle 4, 7 \rangle = \left\langle \frac{4}{\sqrt{65}}, \frac{7}{\sqrt{65}} \right\rangle$$

Hence, **w** and **u** are in the same direction.

Then, $3\mathbf{w}$ would be a vector of magnitude 3 and has has same direction as **w** and **u**.

$$3\mathbf{w} = 3\left\langle \frac{4}{\sqrt{65}}, \frac{7}{\sqrt{65}} \right\rangle = \left\langle \frac{12}{\sqrt{65}}, \frac{21}{\sqrt{65}} \right\rangle$$

$$\|3\mathbf{w}\| = \left\| \left\langle \frac{12}{\sqrt{65}}, \frac{21}{\sqrt{65}} \right\rangle \right\| = \sqrt{\left(\frac{12}{\sqrt{65}}\right)^2 + \left(\frac{21}{\sqrt{65}}\right)^2} = 3$$

Example of Linear Combination

Let $\mathbf{u} = \langle 1,\ 2 \rangle$, $\mathbf{v} = \langle 7, 8 \rangle$, and $\mathbf{w} = \langle -3, 4 \rangle$.

Find scalars a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$.

Let $\mathbf{u} = \langle 1,\ 2 \rangle$, $\mathbf{v} = \langle 7, 8 \rangle$, and $\mathbf{w} = \langle -3, 4 \rangle$.

Note: $\mathbf{u} = \langle 1,\ 2 \rangle = \mathbf{i} + 2\mathbf{j}$ and $\mathbf{w} = \langle -3, 4 \rangle = -3\mathbf{i} + 4\mathbf{j}$.

$\mathbf{v} = a\mathbf{u} + b\mathbf{w} = a(\mathbf{i} + 2\mathbf{j}) + b(-3\mathbf{i} + 4\mathbf{j})$

$= a\mathbf{i} + 2\mathrm{a} - 3b\mathbf{i} + 4b\mathbf{j}$

$= a\mathbf{i} - 3b\mathbf{i} + 2a\mathbf{j} + 4b\mathbf{j}$

$= (a - 3b)\mathbf{i} + (2a + 4b)\mathbf{j}$

$\langle 7, 8 \rangle = \langle a - 3b, 2a + 4b \rangle$

$a - 3b = 7$ (1)

$2a + 4b = 8$ (2)

Solving system of two equations:

(1) $a = 3b + 7$

(2) $2(3b + 7) + 4b = 8$

10b + 14 = 8

b = -6/10= -3/5

$a = 3(-3/5) + 7 = 26/5$

Summary:

$\mathbf{v} = a\mathbf{u} + b\mathbf{w}$

$$\langle 7, 8 \rangle = \frac{-3}{5}\langle 1,\ 2 \rangle + \frac{26}{5}\langle -3, 4 \rangle$$