Dot Product of Two Vectors

Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$

Then: $\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 v_2$

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Then: $\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 v_2 + u_3 v_3$

Note: Dot product of two vectors is a scalar quantity.

Let $\mathbf{u} = \langle 2, 3 \rangle$ and $\mathbf{v} = \langle 4, 5 \rangle$

Then: $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 4 + 3 \cdot 5 = 23$

Let $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 4, 5, 6 \rangle$

Then: $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$

Note: Dot product of two vectors is a scalar quantity.

Claim: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Proof:

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Then: $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$

$$= u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3$$

$$= v_1 \cdot u_1 + v_2 \cdot u_2 + v_3 \cdot u_3$$

$$= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle$$

$$= \mathbf{v} \cdot \mathbf{u}$$

Claim: $\mathbf{v}\cdot\mathbf{v}=\|\mathbf{v}\|^2$

Proof:

Let $\mathbf{v}=\langle v_1, v_2, v_3\rangle$

Note: $\|\mathbf{v}\|=\sqrt{(v_1)^2+(v_2)^2+(v_3)^2}$

$$\|\mathbf{v}\|^2=\left(\sqrt{(v_1)^2+(v_2)^2+(v_3)^2}\right)^2=(v_1)^2+(v_2)^2+(v_3)^2$$

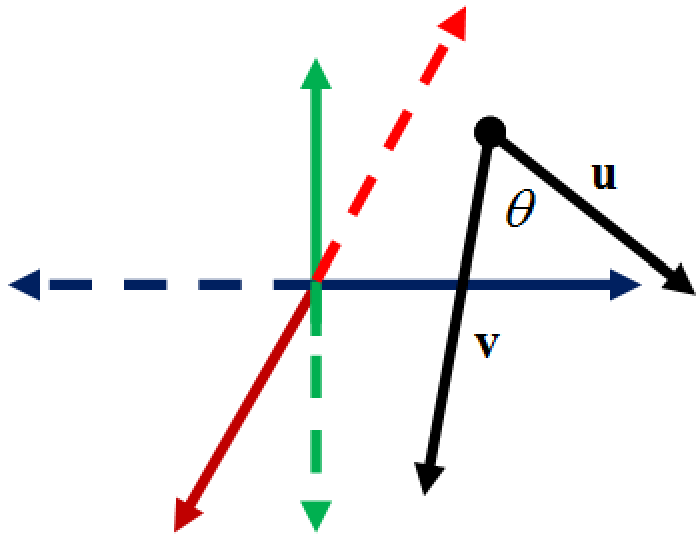
Then: $\mathbf{v}\cdot\mathbf{v}=\langle v_1, v_2, v_3\rangle\cdot\langle v_1, v_2, v_3\rangle$

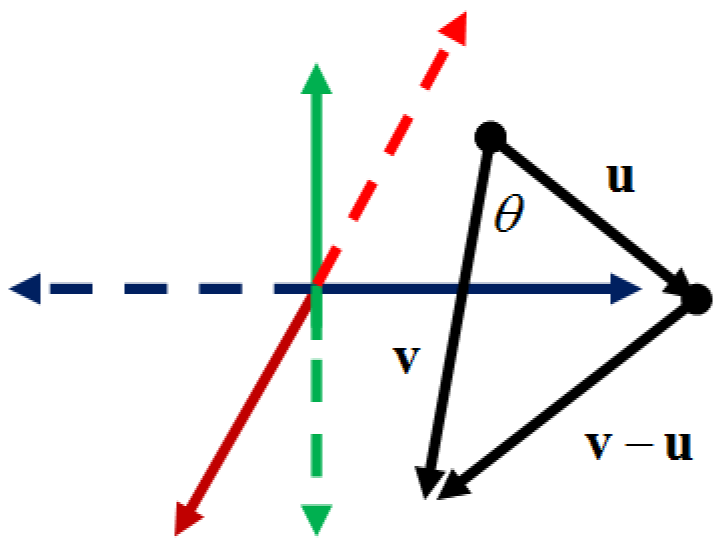
$$= v_1\cdot v_1+v_2\cdot v_2+v_3\cdot v_3$$

$$= (v_1)^2+(v_2)^2+(v_3)^2$$

$$= \|\mathbf{v}\|^2$$

Angle Between Two Vectors





Claim: Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$, where $0 \le \theta \le \pi$.

Then $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$ or $\cos\theta = \dfrac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$

Proof:

From Law of the cosines: $\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$

$$
\begin{aligned}
\|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\
&= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v}) - (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{u}) \\
&= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\
&= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} \\
&= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2
\end{aligned}
$$

Hence,

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

$$\|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

$$-2\mathbf{u} \cdot \mathbf{v} = -2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos\theta$$

Let $\mathbf{v} = \langle 1,2,3 \rangle$ and $\mathbf{u} = \langle 4,6,9 \rangle$

Then $\mathbf{u} \cdot \mathbf{v} = (1)(4) + (2)(6) + (3)(9) = 43$

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\mathbf{v}\| = \sqrt{4^2 + 6^2 + 9^2} = \sqrt{133}$$

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{43}{\sqrt{14}\sqrt{133}} = 0.9965030155175412$$

$$\theta = \cos^{-1}\left(0.9965030155175412\right)$$

$$= 0.083654342807899 \text{ radian} = 4.79304078°$$

Definition of Orthogonal: $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular) if $\mathbf{u} \cdot \mathbf{v} = 0$

Let $\mathbf{u} = \langle 1,2,3 \rangle$ and $\mathbf{v} = \langle 4,7,9 \rangle$

$\mathbf{u} \cdot \mathbf{v} = 4 + 14 + 27 = 45$

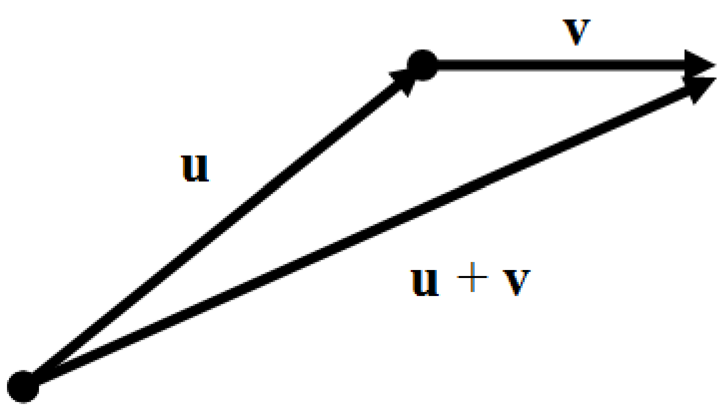
Hence, $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal.

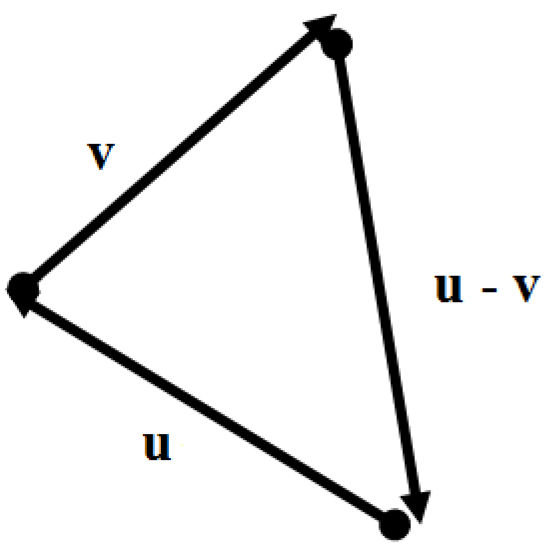
Let $\mathbf{u} = \langle 1,0,0 \rangle$ and $\mathbf{v} = \langle 0,1,0 \rangle$

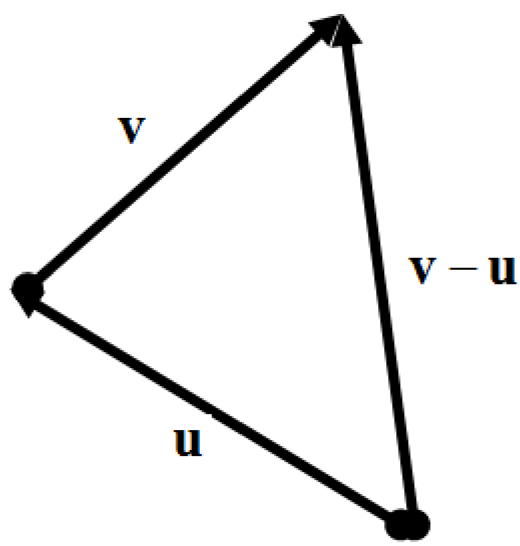
$\mathbf{u} \cdot \mathbf{v} = 0 + 0 + 0 = 0$

Hence, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

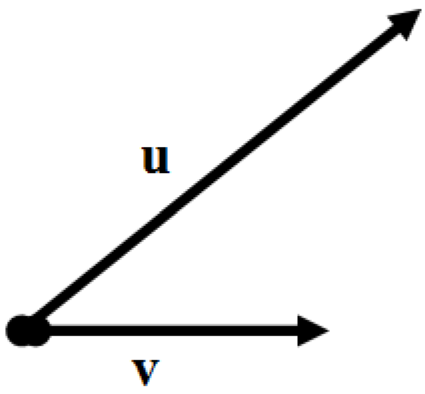
Vector Addition and Subtraction

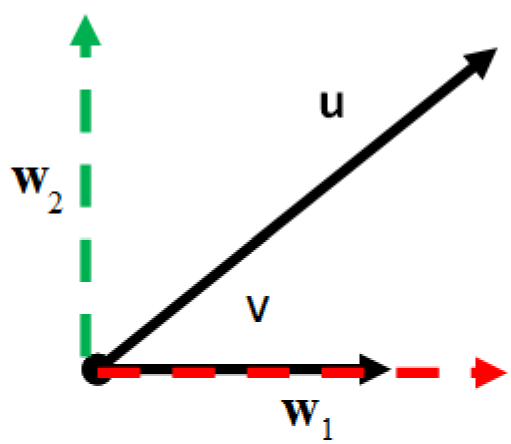


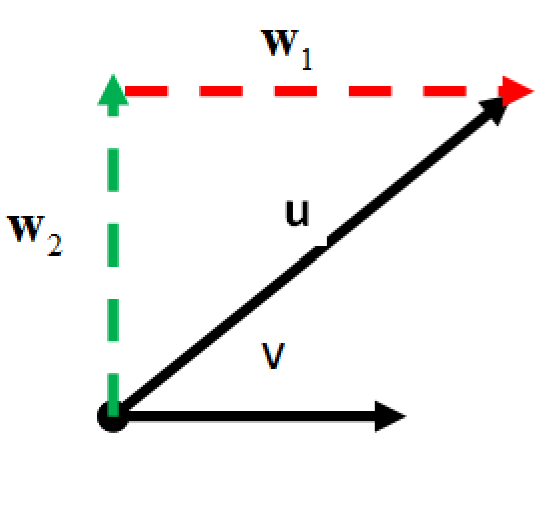




Projection and Vector Components







Note: $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$

$\mathbf{w}_1 =$ projection of $\mathbf{u}$ onto v = $\text{proj}_{\mathbf{v}}\mathbf{u}$.

$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 =$ vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$

Claim: $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$

Proof:

Note that $\mathbf{w}_1$ and $\mathbf{v}$ parallel. Hence, $\mathbf{w}_1 = k\mathbf{v}$.

Note that $\mathbf{w}_2$ and $\mathbf{v}$ orthogonal. Hence, $\mathbf{w}_2 \cdot \mathbf{v} = 0$

$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = k\mathbf{v} + \mathbf{w}_2$

$\mathbf{u}\cdot\mathbf{v} = (k\mathbf{v} + \mathbf{w}_2)\cdot\mathbf{v}$ Taking dot product of both sides with $\mathbf{v}$

$\mathbf{u}\cdot\mathbf{v} = k\mathbf{v}\cdot\mathbf{v} + \mathbf{w}_2\cdot\mathbf{v}$

$\mathbf{u}\cdot\mathbf{v} = k\mathbf{v}\cdot\mathbf{v} + 0$

$\mathbf{u}\cdot\mathbf{v} = k\mathbf{v}\cdot\mathbf{v}$

$k = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}$ Note: $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$

$$\mathbf{w}_1 = k\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}$$

Example: Let $\mathbf{u} = \langle 3,4 \rangle$ and $\mathbf{v} = \langle 1,7 \rangle$

$\mathbf{w}_1 =$ projection of $\mathbf{u}$ onto v = $\text{proj}_{\mathbf{v}}\mathbf{u} =$

$$= \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{(3+28)}{\sqrt{1^2+7^2}}\right)\mathbf{v} = \left(\frac{31}{\sqrt{50}}\right)\langle 1,7\rangle$$

$$= \left\langle \frac{31}{\sqrt{50}}, \frac{217}{\sqrt{50}} \right\rangle$$

$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 =$ vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$

$$= \langle 3,4\rangle - \left\langle \frac{31}{\sqrt{50}}, \frac{217}{\sqrt{50}} \right\rangle = \left\langle 3 - \frac{31}{\sqrt{50}}, 4 - \frac{217}{\sqrt{50}} \right\rangle$$