

# Dot Product of Two Vectors

Let  $\mathbf{u} = \langle u_1, u_2 \rangle$  and  $\mathbf{v} = \langle v_1, v_2 \rangle$

Then:  $\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 v_2$

Let  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Then:  $\mathbf{u} \cdot \mathbf{v} = u_1 \cdot v_1 + u_2 v_2 + u_3 v_3$

*Note* : Dot product of two vectors is a scalar quantity.

Let  $\mathbf{u} = \langle 2, 3 \rangle$  and  $\mathbf{v} = \langle 4, 5 \rangle$

Then:  $\mathbf{u} \cdot \mathbf{v} = 2 \cdot 4 + 3 \cdot 5 = 23$

Let  $\mathbf{u} = \langle 1, 2, 3 \rangle$  and  $\mathbf{v} = \langle 4, 5, 6 \rangle$

Then:  $\mathbf{u} \cdot \mathbf{v} = 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 = 32$

*Note* : Dot product of two vectors is a scalar quantity.

Claim:  $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

Proof:

Let  $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$  and  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Then:  $\mathbf{u} \cdot \mathbf{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle$

$$= u_1 \cdot v_1 + u_2 \cdot v_2 + u_3 \cdot v_3$$

$$= v_1 \cdot u_1 + v_2 \cdot u_2 + v_3 \cdot u_3$$

$$= \langle v_1, v_2, v_3 \rangle \cdot \langle u_1, u_2, u_3 \rangle$$

$$= \mathbf{v} \cdot \mathbf{u}$$

Claim:  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

Proof:

Let  $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Note:  $\|\mathbf{v}\| = \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2}$

$$\|\mathbf{v}\|^2 = \left( \sqrt{(v_1)^2 + (v_2)^2 + (v_3)^2} \right)^2 = (v_1)^2 + (v_2)^2 + (v_3)^2$$

Then:  $\mathbf{v} \cdot \mathbf{v} = \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = v_1 \cdot v_1 + v_2 \cdot v_2 + v_3 \cdot v_3$   
 $= (v_1)^2 + (v_2)^2 + (v_3)^2 = \|\mathbf{v}\|^2$

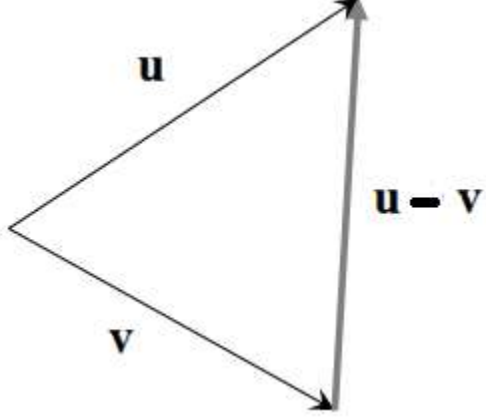
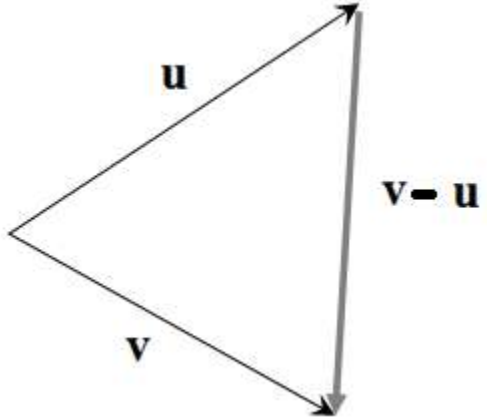
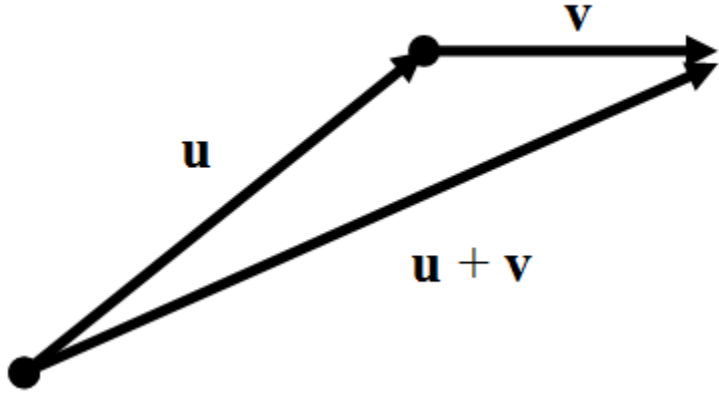
Hence,

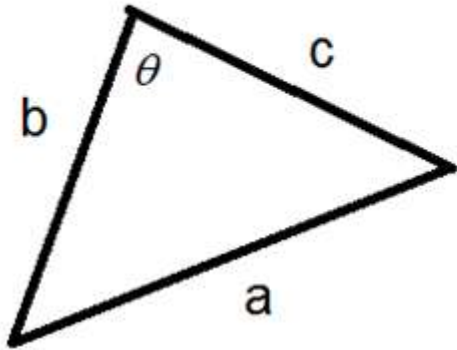
$$\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$$

$$(\mathbf{u} - \mathbf{v})(\mathbf{u} - \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|^2$$

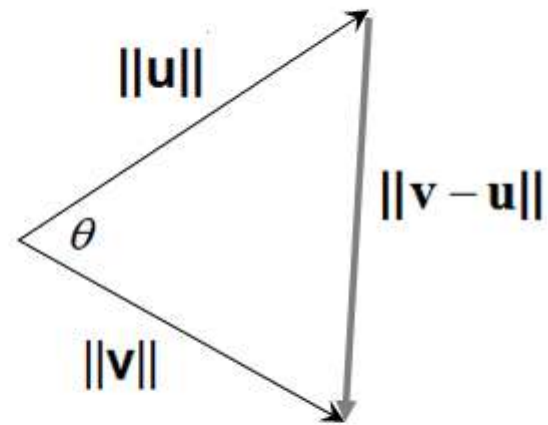
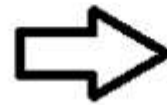
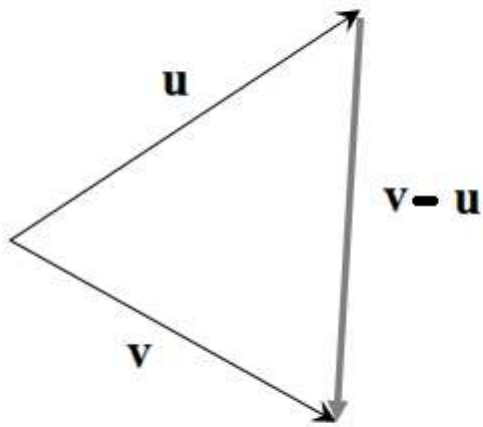
$$(\mathbf{v} - \mathbf{u})(\mathbf{v} - \mathbf{u}) = \|\mathbf{v} - \mathbf{u}\|^2$$

# Vector Addition and Subtraction





Law of the cosines:  $a^2 = b^2 + c^2 - 2bc \cos \theta$



Law of the cosines:  $\|v - u\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos \theta$

## Angle Between Two Vectors

Claim: Let  $\theta$  be the angle between  $\mathbf{u}$  and  $\mathbf{v}$ , where  $0 \leq \theta \leq \pi$ .

$$\text{Then } \mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta \quad \text{or} \quad \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|}$$

Proof:

$$\text{From Law of the cosines: } \|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$$

$$\begin{aligned} \|\mathbf{v} - \mathbf{u}\|^2 &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v} - \mathbf{u}) \\ &= (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{v}) - (\mathbf{v} - \mathbf{u}) \cdot (\mathbf{u}) \\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{u} \\ &= \mathbf{v} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{u} \\ &= \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 \end{aligned}$$

Hence,

$$\|\mathbf{v} - \mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$$

$$\|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v} + \|\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$$

$$-2\mathbf{u} \cdot \mathbf{v} = -2\|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$$

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\| \cdot \|\mathbf{v}\| \cos \theta$$



Let  $\mathbf{v} = \langle 1, 2, 3 \rangle$  and  $\mathbf{u} = \langle 4, 6, 9 \rangle$

Then  $\mathbf{u} \cdot \mathbf{v} = (1)(4) + (2)(6) + (3)(9) = 43$

$$\|\mathbf{u}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

$$\|\mathbf{v}\| = \sqrt{4^2 + 6^2 + 9^2} = \sqrt{133}$$

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \cdot \|\mathbf{v}\|} = \frac{43}{\sqrt{14}\sqrt{133}} = 0.9965030155175412$$

$$\theta = \cos^{-1}(0.9965030155175412)$$

$$= 0.083654342807899 \text{ radian} = 4.79304078^\circ$$

Definition of Orthogonal:  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal (perpendicular) if  $\mathbf{u} \cdot \mathbf{v} = 0$

Let  $\mathbf{u} = \langle 1, 2, 3 \rangle$  and  $\mathbf{v} = \langle 4, 7, 9 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = 4 + 14 + 27 = 45$$

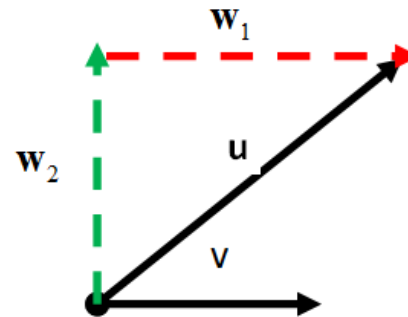
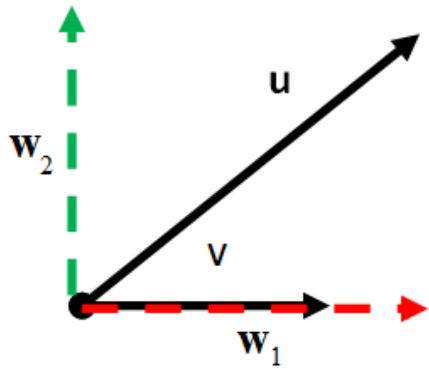
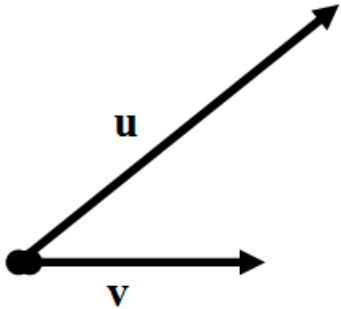
Hence,  $\mathbf{u}$  and  $\mathbf{v}$  are not orthogonal.

Let  $\mathbf{u} = \langle 1, 0, 0 \rangle$  and  $\mathbf{v} = \langle 0, 1, 0 \rangle$

$$\mathbf{u} \cdot \mathbf{v} = 0 + 0 + 0 = 0$$

Hence,  $\mathbf{u}$  and  $\mathbf{v}$  are orthogonal.

## Projection and Vector Components



Note:  $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$

$\mathbf{w}_1 = \text{projection of } \mathbf{u} \text{ onto } \mathbf{v} = \text{proj}_{\mathbf{v}} \mathbf{u}$ .

$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \text{vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{v}$

$$\text{Claim: } \mathbf{w}_1 = \text{proj}_{\mathbf{v}} \mathbf{u} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

Proof:

Note that  $\mathbf{w}_1$  and  $\mathbf{v}$  parallel. Hence,  $\mathbf{w}_1 = k\mathbf{v}$ .

Note that  $\mathbf{w}_2$  and  $\mathbf{v}$  orthogonal. Hence,  $\mathbf{w}_2 \cdot \mathbf{v} = 0$

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2 = k\mathbf{v} + \mathbf{w}_2$$

$$\mathbf{u} \cdot \mathbf{v} = (k\mathbf{v} + \mathbf{w}_2) \cdot \mathbf{v} \quad \text{Taking dot product of both sides with } \mathbf{v}$$

$$\mathbf{u} \cdot \mathbf{v} = k\mathbf{v} \cdot \mathbf{v} + \mathbf{w}_2 \cdot \mathbf{v}$$

$$\mathbf{u} \cdot \mathbf{v} = k\mathbf{v} \cdot \mathbf{v} + 0$$

$$\mathbf{u} \cdot \mathbf{v} = k\mathbf{v} \cdot \mathbf{v}$$

$$k = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}$$

$$\text{Note: } \mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$$

$$\mathbf{w}_1 = k\mathbf{v} = \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v}$$

Example: Let  $\mathbf{u} = \langle 3, 4 \rangle$  and  $\mathbf{v} = \langle 1, 7 \rangle$

$\mathbf{w}_1 = \text{projection of } \mathbf{u} \text{ onto } \mathbf{v} = \text{proj}_{\mathbf{v}} \mathbf{u} =$

$$= \left( \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2} \right) \mathbf{v} = \left( \frac{3 + 28}{(\sqrt{1^2 + 7^2})^2} \right) \mathbf{v} = \left( \frac{31}{50} \right) \langle 1, 7 \rangle$$

$$= \left\langle \frac{31}{50}, \frac{217}{50} \right\rangle = \langle 0.62, 4.34 \rangle$$

$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = \text{vector component of } \mathbf{u} \text{ orthogonal to } \mathbf{v}$

$$= \langle 3, 4 \rangle - \left\langle \frac{31}{50}, \frac{217}{50} \right\rangle = \left\langle 3 - \frac{31}{50}, 4 - \frac{217}{50} \right\rangle$$

$$= \langle 2.38, -0.34 \rangle$$

