Cross Product of Two Space Vectors in Space

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2)\mathbf{i} - (u_1 v_3 - u_3 v_1)\mathbf{j} + (u_1 v_2 - u_2 v_1)\mathbf{k}$$

and

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2}$$

Example 1:

Let $\mathbf{u} = \langle 1,\ 2,\ 3 \rangle$ and $\mathbf{v} = \langle 5,\ 7,\ 9 \rangle$

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(u_2v_3 - u_3v_2\right)\mathbf{i} - \left(u_1v_3 - u_3v_1\right)\mathbf{j} + \left(u_1v_2 - u_2v_1\right)\mathbf{k}$$

$$= \left((2)(9) - (3)(7)\right)\mathbf{i} - \left((1)(9) - (3)(5)\right)\mathbf{j} + \left((1)(7) - (2)(5)\right)\mathbf{k}$$

$$= -3\mathbf{i} - \left(-6\right)\mathbf{j} + \left(-3\right)\mathbf{k} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} \quad = \langle \text{-}3,\ 6,\ \text{-}3 \rangle$$

$$\mathbf{v}\times\mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \left((3)(7) - (2)(9)\right)\mathbf{i} - \left((3)(5) - (1)(9)\right)\mathbf{j} + \left((2)(5) - (1)(7)\right)\mathbf{k}$$

$$= 3\mathbf{i} - \left(6\right)\mathbf{j} + \left(3\right)\mathbf{k} = 3\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} \quad = \langle 3,\ \text{-}6,\ 3 \rangle$$

Hence, $\mathbf{u}\times\mathbf{v} = -\left(\mathbf{u}\times\mathbf{v}\right)$

Example 2:

Let $\mathbf{u} = \mathbf{i} = \langle 1,\ 0,\ 0 \rangle$ and $\mathbf{v} = \mathbf{k} = \langle 0,\ 0,\ 1 \rangle$

$$\mathbf{u} \times \mathbf{v} = \left(u_2 v_3 - u_3 v_2\right)\mathbf{i} - \left(u_1 v_3 - u_3 v_1\right)\mathbf{j} + \left(u_1 v_2 - u_2 v_1\right)\mathbf{k}$$

$$= \left((0)(1) - (0)(0)\right)\mathbf{i} - \left((1)(1) - (0)(0)\right)\mathbf{j} + \left((1)(0) - (0)(0)\right)\mathbf{k}$$

$$= 0\mathbf{i} - \left(1\right)\mathbf{j} + 0\mathbf{k}$$

$$= -\mathbf{j}$$

Example 3: Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Claim: $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$; $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{v}$

Let $\mathbf{u} = \langle 1,\ 2,\ 3 \rangle$ and $\mathbf{v} = \langle 5,\ 7,\ 9 \rangle$

$$\mathbf{u} \times \mathbf{v} = \left(u_2 v_3 - u_3 v_2\right)\mathbf{i} - \left(u_1 v_3 - u_3 v_1\right)\mathbf{j} + \left(u_1 v_2 - u_2 v_1\right)\mathbf{k}$$

$$= \left((2)(9) - (3)(7)\right)\mathbf{i} - \left((1)(9) - (3)(5)\right)\mathbf{j} + \left((1)(7) - (2)(5)\right)\mathbf{k}$$

$$= -3\mathbf{i} - \left(-6\right)\mathbf{j} + \left(-3\right)\mathbf{k} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} = \langle -3,\ 6,\ -3 \rangle$$

Note:

$$\left(\mathbf{u} \times \mathbf{v}\right) \cdot \mathbf{u} = \langle -3,\ 6,\ -3 \rangle \cdot \langle 1,\ 2,\ 3 \rangle = -3 + 12 + -9 = 0$$

$$\left(\mathbf{u} \times \mathbf{v}\right) \cdot \mathbf{v} = \langle -3,\ 6,\ -3 \rangle \cdot \langle 5,\ 7,\ 9 \rangle = -15 + 42 + -27 = 0$$

Therefore:

$\left(\mathbf{u} \times \mathbf{v}\right)$ and $\mathbf{u}$ are orthogonal.

$\left(\mathbf{u} \times \mathbf{v}\right)$ and $\mathbf{v}$ are orthogonal.

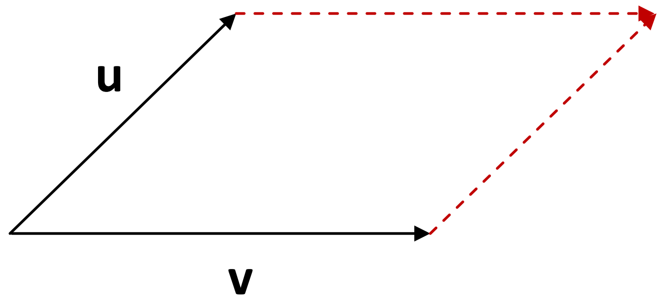
Find a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

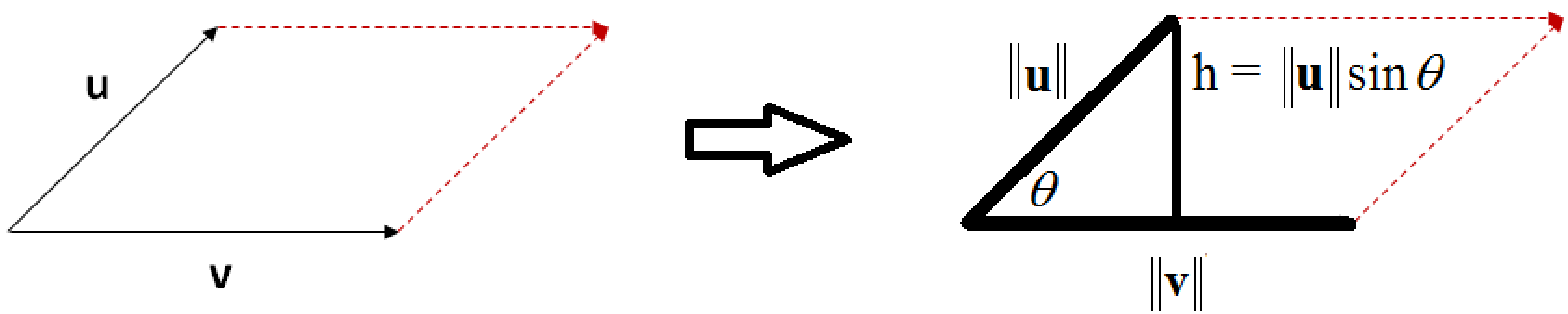
A unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ $= \dfrac{\mathbf{u}\times\mathbf{v}}{\|\mathbf{u}\times\mathbf{v}\|} = \dfrac{1}{\|\mathbf{u}\times\mathbf{v}\|}(\mathbf{u}\times\mathbf{v})$

$$= \frac{1}{\|\langle -3,\ 6,\ -3\rangle\|}\langle -3,\ 6,\ -3\rangle = \frac{1}{\sqrt{54}}\langle -3,\ 6,\ -3\rangle = \left\langle \frac{-3}{\sqrt{54}},\ \frac{6}{\sqrt{54}},\ \frac{-3}{\sqrt{54}} \right\rangle$$

Note: $\|\mathbf{u}\times\mathbf{v}\| = \|\langle -3,\ 6,\ -3\rangle\| = \sqrt{(-3)^2+(6)^2+(-3)^2} = \sqrt{54}$

Area of parallelogram:





Note: $\sin\theta = \dfrac{h}{\|\mathbf{u}\|}$; $\mathrm{h} = \|\mathbf{u}\|\sin\theta$

Area of parallelogram = (base)(height) = $\left(\|\mathbf{v}\|\right)\left(\|\mathbf{u}\|\sin\theta\right)$

Claim: Area of parallelogram = $\|\mathbf{u}\times\mathbf{v}\|$

Recall: $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}$

$\sin^2\theta + \cos^2\theta = 1$; $\sin^2\theta = 1-\cos^2\theta$; $\sin\theta = \sqrt{1-\cos^2\theta}$

$\|\mathbf{u}\| = \sqrt{u_1^2+u_2^2+u_3^2}$; $\|\mathbf{u}\|^2 = \left(\sqrt{u_1^2+u_2^2+u_3^2}\right)^2 = u_1^2+u_2^2+u_3^2$

Hence, $\|\mathbf{v}\|\|\mathbf{u}\|\sin\theta = \|\mathbf{v}\|\|\mathbf{u}\|\sqrt{1-\cos^2\theta} = \|\mathbf{v}\|\|\mathbf{u}\|\sqrt{1-\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)^2} = \sqrt{\|\mathbf{v}\|^2\|\mathbf{u}\|^2}\sqrt{1-\left(\dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)^2}$

$$= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2\left[1-\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}\right)^2\right]} = \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2\|\mathbf{v}\|^2\cdot\frac{(\mathbf{u}\cdot\mathbf{v})^2}{\left(\|\mathbf{u}\|\|\mathbf{v}\|\right)^2}}$$

$$= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\cdot\mathbf{v})^2} = \sqrt{\left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-(\mathbf{u}\cdot\mathbf{v})^2}$$

$$= \sqrt{\left(u_1^2+u_2^2+u_3^2\right)\left(v_1^2+v_2^2+v_3^2\right)-\left(u_1v_1+u_2v_2+u_3v_3\right)^2}$$

$$= \sqrt{\left(u_2v_3-u_3v_2\right)^2+\left(u_1v_3-u_3v_1\right)^2+\left(u_1v_2-u_2v_1\right)^2} = \|\mathbf{u}\times\mathbf{v}\|$$

Area = (base)(height) = $\left(\|\mathbf{u}\|\right)\left(\|\mathbf{v}\|\sin\theta\right) = \|\mathbf{u}\times\mathbf{v}\|$

Example 4: Let $\mathbf{u} = \langle 1,\ 2,\ 3 \rangle$ and $\mathbf{v} = \langle 5,\ 7,\ 9 \rangle$

$$\mathbf{u} \times \mathbf{v} = \left(u_2 v_3 - u_3 v_2\right)\mathbf{i} - \left(u_1 v_3 - u_3 v_1\right)\mathbf{j} + \left(u_1 v_2 - u_2 v_1\right)\mathbf{k}$$

$$= \left((2)(9) - (3)(7)\right)\mathbf{i} - \left((1)(9) - (3)(5)\right)\mathbf{j} + \left((1)(7) - (2)(5)\right)\mathbf{k}$$

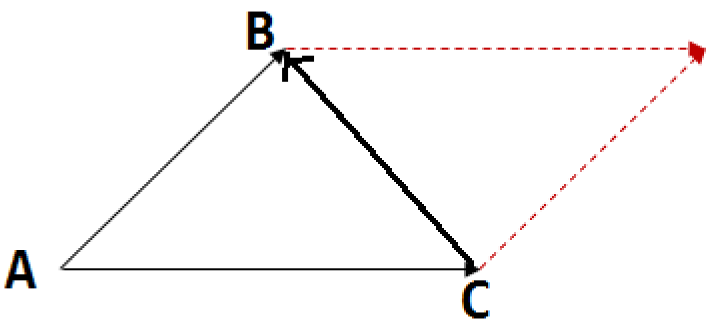
$$= -3\mathbf{i} - \left(-6\right)\mathbf{j} + \left(-3\right)\mathbf{k}$$

$$= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

$$\left\|\mathbf{u} \times \mathbf{v}\right\| = \left\|\langle -3,\ 6,\ -3 \rangle\right\| = \sqrt{\left(-3\right)^2 + \left(6\right)^2 + \left(-3\right)^2} = \sqrt{54}$$

Area of parallelogram = $\left\|\mathbf{u} \times \mathbf{v}\right\| = \sqrt{54}$

Find area of triangle



$$\text{Area of Triangle} = \frac{1}{2}\left\| \overrightarrow{AC} \times \overrightarrow{AB} \right\|$$

Example 5: Find area of triangle with the following vertices: A(1,2,3), B(4, 3, 3), C(4,7,8).

Let $\overrightarrow{AC} = \langle 4-1, 7-2, 8-3 \rangle = \langle 3,5,5 \rangle$

Let $\overrightarrow{AB} = \langle 4-1, 3-2, 3-3 \rangle = \langle 3,1,0 \rangle$

$\overrightarrow{AC} \times \overrightarrow{AB} = \langle 3,5,5 \rangle \times \langle 3,1,0 \rangle =$

$= ((5)(0) - (5)(1))\mathbf{i} - ((3)(0) - (5)(3))\mathbf{j} + ((3)(1) - (5)(3))\mathbf{k}$

$= (-5)\mathbf{i} - (-15)\mathbf{j} + (-12)\mathbf{k}$

$= -5\mathbf{i} + 15\mathbf{j} - 12\mathbf{k} = \langle -5, 15, -12 \rangle$

$\left\| \overrightarrow{AC} \times \overrightarrow{AB} \right\| = \sqrt{(-5)^2 + (15)^2 + (-12)^2} = \sqrt{394}$

Area of Triangle = $\frac{1}{2} \left\| \overrightarrow{AC} \times \overrightarrow{AB} \right\| = \frac{1}{2} \cdot \sqrt{394} = 9.92$