

Cross Product of Two Space Vectors in Space

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}\end{aligned}$$

and

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2}$$

Let $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 5, 7, 9 \rangle$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$= ((2)(9) - (3)(7))\mathbf{i} - ((1)(9) - (3)(5))\mathbf{j} + ((1)(7) - (2)(5))\mathbf{k}$$

$$= -3\mathbf{i} - (-6)\mathbf{j} + (-3)\mathbf{k}$$

$$= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}$$

Let $\mathbf{u} = \mathbf{i} = \langle 1, 0, 0 \rangle$ and $\mathbf{v} = \mathbf{k} = \langle 0, 0, 1 \rangle$

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

$$= ((0)(1) - (0)(0))\mathbf{i} - ((1)(1) - (0)(0))\mathbf{j} + ((1)(0) - (0)(0))\mathbf{k}$$

$$= 0\mathbf{i} - (1)\mathbf{j} + 0\mathbf{k}$$

$$= -\mathbf{j}$$

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Claim: $\mathbf{u} \times \mathbf{v}$ is orthogonal to \mathbf{u} ; $\mathbf{u} \times \mathbf{v}$ is orthogonal to \mathbf{v}

Let $\mathbf{u} = \langle 1, 2, 3 \rangle$ and $\mathbf{v} = \langle 5, 7, 9 \rangle$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\ &= ((2)(9) - (3)(7))\mathbf{i} - ((1)(9) - (3)(5))\mathbf{j} + ((1)(7) - (2)(5))\mathbf{k} \\ &= -3\mathbf{i} - (-6)\mathbf{j} + (-3)\mathbf{k} = -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k} = \langle -3, 6, -3 \rangle\end{aligned}$$

Note:

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{u} = \langle -3, 6, -3 \rangle \cdot \langle 1, 2, 3 \rangle = -3 + 12 + -9 = 0$$

$$(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{v} = \langle -3, 6, -3 \rangle \cdot \langle 5, 7, 9 \rangle = -15 + 42 + -27 = 0$$

Therefore:

$(\mathbf{u} \times \mathbf{v})$ and \mathbf{u} are orthogonal.

$(\mathbf{u} \times \mathbf{v})$ and \mathbf{v} are orthogonal.

Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

A unit vector that is orthogonal to both \mathbf{u} and \mathbf{v}
is a unit vector that is parallel to $\mathbf{u} \times \mathbf{v}$.

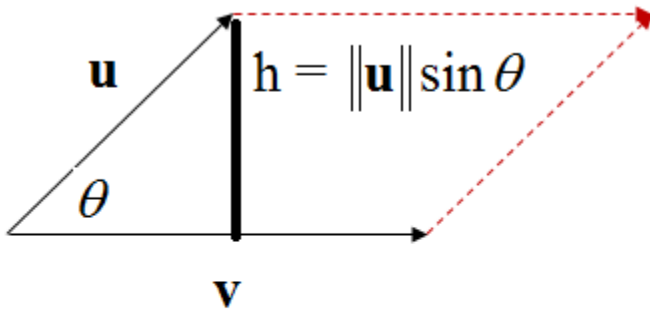
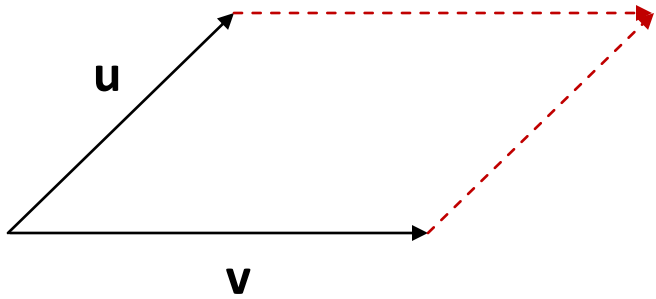
Hence,

unit vector that is orthogonal to both \mathbf{u} and \mathbf{v}

$$\begin{aligned} &= \frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = \frac{1}{\|\mathbf{u} \times \mathbf{v}\|} (\mathbf{u} \times \mathbf{v}) \\ &= \frac{1}{\|\langle -3, 6, -3 \rangle\|} \langle -3, 6, -3 \rangle = \frac{1}{\sqrt{54}} \langle -3, 6, -3 \rangle = \left\langle \frac{-3}{\sqrt{54}}, \frac{6}{\sqrt{54}}, \frac{-3}{\sqrt{54}} \right\rangle \end{aligned}$$

Note: $\|\mathbf{u} \times \mathbf{v}\| = \|\langle -3, 6, -3 \rangle\| = \sqrt{(-3)^2 + (6)^2 + (-3)^2} = \sqrt{54}$

Area of parallelogram:



Note: $\sin \theta = \frac{h}{\|\mathbf{u}\|}$; $h = \|\mathbf{u}\| \sin \theta$

$$\text{Area} = (\text{base})(\text{height}) = (\|\mathbf{v}\|)(\|\mathbf{u}\| \sin \theta)$$

Claim: Area of parallelogram = $\|\mathbf{u} \times \mathbf{v}\|$

$$\text{Recall: } \cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

$$\sin^2 \theta + \cos^2 \theta = 1; \quad \sin^2 \theta = 1 - \cos^2 \theta; \quad \sin \theta = \sqrt{1 - \cos^2 \theta}$$

$$\|\mathbf{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}; \quad \|\mathbf{u}\|^2 = \left(\sqrt{u_1^2 + u_2^2 + u_3^2}\right)^2 = u_1^2 + u_2^2 + u_3^2$$

$$\text{Hence, } \|\mathbf{v}\| \|\mathbf{u}\| \sin \theta = \|\mathbf{v}\| \|\mathbf{u}\| \sqrt{1 - \cos^2 \theta}$$

$$= \|\mathbf{v}\| \|\mathbf{u}\| \sqrt{1 - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)^2} = \sqrt{\|\mathbf{v}\|^2 \|\mathbf{u}\|^2} \sqrt{1 - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)^2}$$

$$= \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \left[1 - \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}\right)^2\right]} = \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \cdot \frac{(\mathbf{u} \cdot \mathbf{v})^2}{(\|\mathbf{u}\| \|\mathbf{v}\|)^2}}$$

$$= \sqrt{\|\mathbf{u}\|^2 \|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} = \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (\mathbf{u} \cdot \mathbf{v})^2}$$

$$= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2}$$

$$= \sqrt{(u_2 v_3 - u_3 v_2)^2 + (u_1 v_3 - u_3 v_1)^2 + (u_1 v_2 - u_2 v_1)^2} = \|\mathbf{u} \times \mathbf{v}\|$$

$$\text{Area} = (\text{base})(\text{height}) = (\|\mathbf{u}\|)(\|\mathbf{v}\|\sin\theta) = \|\mathbf{u} \times \mathbf{v}\|$$

Example :

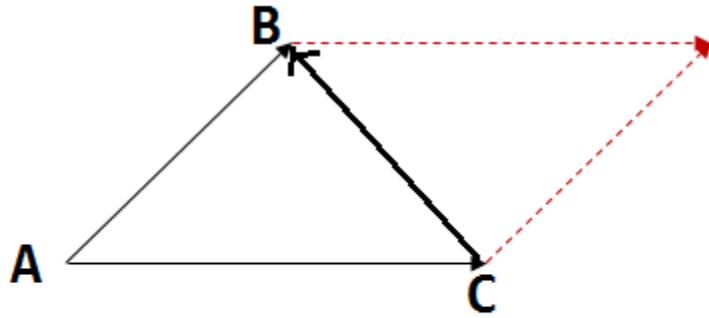
$$\text{Let } \mathbf{u} = \langle 1, 2, 3 \rangle \text{ and } \mathbf{v} = \langle 5, 7, 9 \rangle$$

$$\begin{aligned}\mathbf{u} \times \mathbf{v} &= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k} \\ &= ((2)(9) - (3)(7))\mathbf{i} - ((1)(9) - (3)(5))\mathbf{j} + ((1)(7) - (2)(5))\mathbf{k} \\ &= -3\mathbf{i} - (-6)\mathbf{j} + (-3)\mathbf{k} \\ &= -3\mathbf{i} + 6\mathbf{j} - 3\mathbf{k}\end{aligned}$$

$$\|\mathbf{u} \times \mathbf{v}\| = \|\langle -3, 6, -3 \rangle\| = \sqrt{(-3)^2 + (6)^2 + (-3)^2} = \sqrt{54}$$

$$\text{Area of parallelogram} = \|\mathbf{u} \times \mathbf{v}\| = \sqrt{54}$$

Find area of triangle



$$\text{Area of Triangle} = \frac{1}{2} \|\vec{AC} \times \vec{AB}\|$$

Find area of triangle with the following vertices:

$$A(1,2,3), \quad B(4, 3, 3), \quad C(4,7,8)$$

$$A(1,2,3), \quad B(4, 3, 3), \quad C(4,7,8)$$

$$\overrightarrow{AC} = \langle 4-1, 7-2, 8-3 \rangle = \langle 3, 5, 5 \rangle$$

$$\overrightarrow{AB} = \langle 4-1, 3-2, 3-3 \rangle = \langle 3, 1, 0 \rangle$$

$$\overrightarrow{AC} \times \overrightarrow{AB} = \langle 3, 5, 5 \rangle$$

$$\times \langle 3, 1, 0 \rangle =$$

$$= ((5)(0) - (5)(1))\mathbf{i} - ((3)(0) - (5)(3))\mathbf{j} + ((3)(1) - (5)(3))\mathbf{k}$$

$$= (-5)\mathbf{i} - (-15)\mathbf{j} + (-12)\mathbf{k}$$

$$= -5\mathbf{i} + 15\mathbf{j} - 12\mathbf{k} = \langle -5, 15, -12 \rangle$$

$$\|\overrightarrow{AC} \times \overrightarrow{AB}\| = \sqrt{(-5)^2 + (15)^2 + (-12)^2} = \sqrt{394}$$

$$\text{Area of Triangle} = \frac{1}{2} \|\overrightarrow{AC} \times \overrightarrow{AB}\| = \frac{1}{2} \cdot \sqrt{394}$$