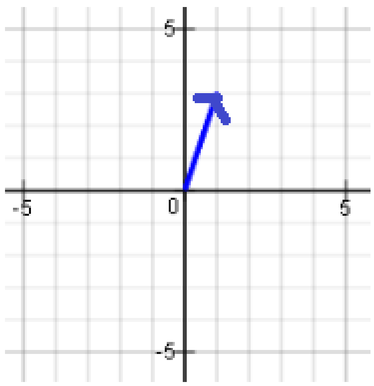
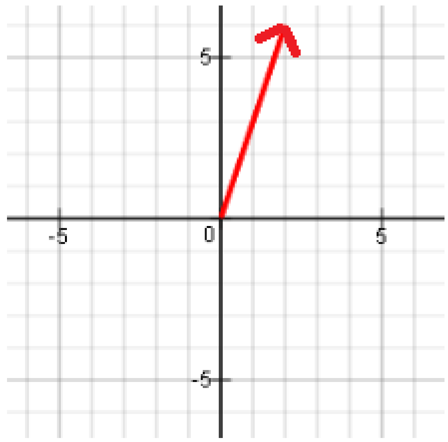
Lines and Planes in Space

Parallel Vectors:

Let $\mathbf{u} = \langle 1,3 \rangle$ and $\mathbf{v} = 2\mathbf{u} = \langle 2,6 \rangle$

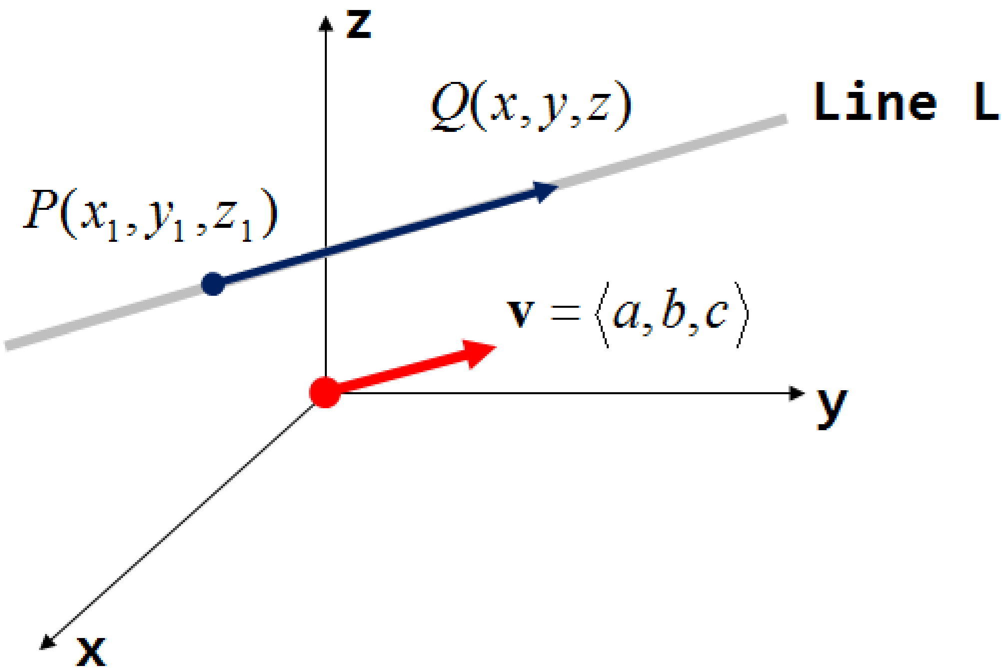




If vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{u} = t\mathbf{v}$;

where t is a real number.

Lines in space



Note: Vectors $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel.

Hence, $\overrightarrow{PQ} = t\mathbf{v}$; where t is a real number.

$$\langle x - x_1, y - y_1, z - z_1 \rangle = t \langle a, b, c \rangle$$

$$\langle x - x_1, y - y_1, z - z_1 \rangle = \langle ta, tb, tc \rangle$$

Line L is represented by the parametric equations:

$$x - x_1 = ta, \quad y - y_1 = tb, \quad z - z_1 = tc$$

Example 1: Find parametric equations of the line L

that passes through (1, 3, 4) and is parallel to $\mathbf{v} = \langle 4, 5, 6 \rangle$.

Line L is represented by the parametric equations:

$x - x_1 = ta, \quad y - y_1 = tb, \quad z - z_1 = tc$

$x - 1 = t(4), \quad y - 3 = t(5), \quad z - 4 = t(6)$

$x - 1 = 4t, \quad y - 3 = 5t, \quad z - 4 = 6t$

Find symmetric equations of the line L that passes

through (1, 3, 4) and is parallel to $\mathbf{v} = \langle 4, 5, 6 \rangle$.

$x - 1 = 4t, \quad y - 3 = 5t, \quad z - 4 = 6t$

$$t = \frac{x-1}{4}; \quad t = \frac{y-3}{5}; \quad t = \frac{z-4}{6}$$

$\Rightarrow \dfrac{x-1}{4} = \dfrac{y-3}{5} = \dfrac{z-4}{6}$ symmetric equations

List two points on line L.

$x-1=4t, \quad y-3=5t, \quad z-4=6t$

For $t=2$: $x-1=4(2), \quad y-3=5(2), \quad z-4=6(2)$

$\Rightarrow x=9,\ y=13, \quad z=16$

Point on line L: $(9,13,\ 16)$

For $t=4$: $x-1=4(4), \quad y-3=5(4), \quad z-4=6(4)$

$\Rightarrow x=15,\ y=17, \quad z=20$

Point on line L: $(15,17,\ 20)$

Find parametric equations of the line L that passes

through $P(1, 3, 4)$ and $Q(4,7,8)$

Let **v** be the vector with initial point at the origin

and parallel to the vector $\overrightarrow{PQ}$.

$$\mathbf{v} = \langle 4-1, 7-3, 8-4 \rangle = \langle 3, 4, 4 \rangle$$

Line L is represented by the parametric equations:

$$x - x_1 = ta, \quad y - y_1 = tb, \quad z - z_1 = tc$$

$$\langle a, b, c \rangle = \langle 3, 4, 4 \rangle; \quad \langle x_1, y_1, z_1 \rangle = \langle 1, 3, , 4 \rangle$$

$$x - 1 = t(3), \quad y - 3 = t(4), \quad z - 4 = t(4)$$

$$x - 1 = 4t, \quad y - 3 = 5t, \quad z - 4 = 6t$$

Example 2:

Find symmetric equations of the line L that passes through $P(1, 3, 4)$ and $Q(4,7,8)$

Parametric Equations:

$$x-1=4t, \quad y-3=5t, \quad z-4=6t$$

Symmetric Equations:

$$x-1=4t, \quad y-3=5t, \quad z-4=6t$$

$$t=\frac{x-1}{4}, \quad t=\frac{y-3}{5}, \quad t=\frac{z-4}{6}$$

Example 3:

Find the coordinates of a point P on the line and a vector v parallel to the line.

$$x = 4t; \quad y = 5 - t; \quad z = 4 + 7t$$

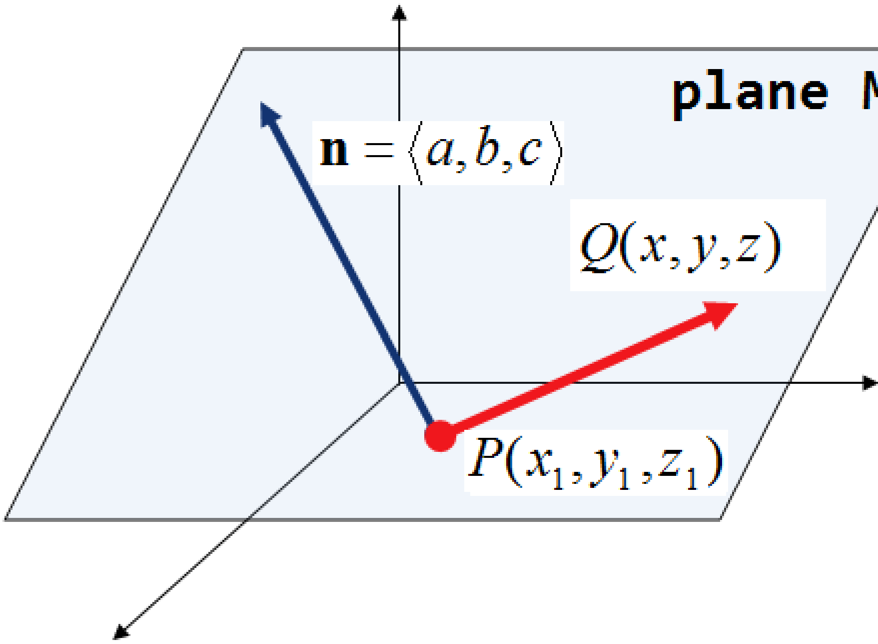
$$x - 0 = 4t; \quad y - 5 = -t; \quad z - 4 = 7t$$

$$x - x_1 = ta, \qquad y - y_1 = tb, \qquad z - z_1 = tc$$

$$\text{Vector } \mathbf{v} = \langle a, b, c \rangle = \langle 4, -1, 7 \rangle;$$

$$\text{Point } P(x_1, y_1, z_1) = (0, 5, 4)$$

Planes in space



Let $P(x_1, y_1, z_1)$ and $Q(x, y, z)$ be on plane M.

Let $\mathbf{n} = \langle a, b, c \rangle$ be a vector that is orthogonal to plane M.

Note: Vectors $\overrightarrow{\mathrm{PQ}}$ and $\mathbf{n}$ are orthogonal (perpendicular).

Hence, $\overrightarrow{\mathrm{PQ}} \cdot \mathbf{n} = 0$

$\langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle a, b, c \rangle = 0$

$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Equation of plane M : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Example 4:

Find equation of the plane containing $P(x_1, y_1, z_1) = (2,3,4)$

and perpendicular to the vector $\mathbf{n} = \langle a, b, c \rangle = \langle 6, 9, 1 \rangle$

$$\text{Equation of plane: } \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$\text{Equation of plane: } \quad 6(x - 2) + 9(y - 3) + 1(z - 4) = 0$$

$$\Rightarrow \qquad 6x - 12 + 9y - 27 + z - 4 = 0$$

$$\Rightarrow \qquad 6x + 9y + z - 43 = 0$$

Example 5:

Find equation of the plane containing $P(x_1, y_1, z_1) = (2,3,4)$

and perpendicular to the vector $\mathbf{n} = \mathbf{k} = \langle a, b, c \rangle = \langle 0, 0, 1 \rangle$

$$\text{Equation of plane: } \quad a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$\text{Equation of plane: } \quad 0(x - 2) + 0(y - 3) + 1(z - 4) = 0$$

$$\Rightarrow \qquad 1(z - 4) = 0$$

$$\Rightarrow \qquad z = 4$$

Example 6:

Find equation of the plane containing A(5,9,0),

B(2,0,3), and C(4,5,9).

$\overrightarrow{AB} = \langle 2\text{-}5, 0\text{-}9, 3\text{-}0 \rangle = \langle -3, -9, 3 \rangle$

$\overrightarrow{AC} = \langle 4\text{-}5, 5\text{-}9, 9\text{-}0 \rangle = \langle -1, -4, 9 \rangle$

Note: Vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are on the plane.

A vector that is orthogonal (perpendicular) to $\overrightarrow{AB}$ and $\overrightarrow{AC}$

is the cross product $\overrightarrow{AB} \times \overrightarrow{AC}$.

$Hence,\ \overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$

$$= \left[(-9)(9) - (3)(-4)\right]\mathbf{i} - \left[(-3)(9) - (3)(-1)\right]\mathbf{j}$$

$$+ \left[(-3)(-4) - (-9)(-1)\right]\mathbf{k}$$

$$= \left[-69\right]\mathbf{i} - \left[-24\right]\mathbf{j} + \left[3\right]\mathbf{k} = -69\mathbf{i} + 24\mathbf{j} + 3\mathbf{k}$$

$$= \left\langle -69, 24, 3 \right\rangle$$

Let $\mathbf{n} = \overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \left\langle a, b, c \right\rangle = \left\langle -69, 24, 3 \right\rangle$

Let point P(x_1,y_1,z_1)= A(5,9,0)

Equation of plane: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Equation of plane: $-69(x-5) + 24(y-9) + 3(z-0) = 0$

Example 7:

Find the distance between the point Q(0,0,0) and

the plane $2x+3y+z-12=0$

Distance between point and plane = $\dfrac{\left|\overrightarrow{\mathrm{PQ}}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$

Equation of Plane: $ax+by+cz+d=0$

Normal vector $\mathbf{n} = \langle a,b,c\rangle = \langle 2,3,1\rangle$

A point the plane $2x+3y+z-12=0$ is any

ordered triple (x,y,z) that satifies this equation.

We can let $x=0, y=0, z=12.$

Hence, $(0,0,12)$ satisfies the equation $2x+3y+z-12=0$

and a point $\mathrm{P}(x_1,y_1,z_1) = (0,0,12)$ is on plane.

$\overrightarrow{\mathrm{PQ}} = \langle 0\text{-}0, 0\text{-}0, 12-0\rangle = \langle 0,0,12\rangle$

$\overrightarrow{\mathrm{PQ}}\cdot\mathbf{n} = \langle 0,0,12\rangle\cdot\langle 2,3,1\rangle = 0+0+12 = 12$

$\|\mathbf{n}\| = \sqrt{2^2+3^2+1^2} = \sqrt{14}$

Distance between point and plane = $\dfrac{\left|\overrightarrow{\mathrm{PQ}}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|} = \dfrac{|12|}{\sqrt{14}} = \dfrac{12}{\sqrt{14}}$

Example 8:

Find the distance between the point Q(1,-2,4) and

the line $x = 2t; y = t - 3; z = 2t + 2$

Distance between point and plane = $\dfrac{\left\| \overrightarrow{\mathrm{PQ}} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|}$

Equation of line: $x - x_1 = at; y - y_1 = bt; z - z_1 = ct;$

$Note:\ x = 2t; y = t - 3; z = 2t + 2 \Leftrightarrow x - 0 = 2t; y + 3 = t; z - 2 = 2t$

Direction vector for the line: $\mathbf{u} = \langle 2,1,2 \rangle$

To find a point on the line, let $t = 0$:

$x = 2t; y = t - 3; z = 2t + 2$

$x = 0; \quad y = -3; \quad z = 2$

Point P on the line: (0,-3,2)

Hence, $\overrightarrow{\mathrm{PQ}} = \langle \text{1-0, -2 - -3}, 4 - 2 \rangle = \langle 1,1,2 \rangle$

$\overrightarrow{\mathrm{PQ}} \times \mathbf{u} = \langle 1,1,2 \rangle \times \langle 2,1,2 \rangle = 0\mathbf{i} + 2\mathbf{j} + -1\mathbf{k} = \langle 0,2,-1 \rangle$

$\left\| \overrightarrow{\mathrm{PQ}} \times \mathbf{u} \right\| = \left\| \langle 0,2,-1 \rangle \right\| = \sqrt{(0)^2 + (2)^2 + (-1)^2} = \sqrt{5}$

$\left\| \mathbf{u} \right\| = \left\| \langle 2,1,2 \rangle \right\| = \sqrt{(2)^2 + (1)^2 + (2)^2} = \sqrt{9} = 3$

Distance between point and plane = $\dfrac{\left\| \overrightarrow{\mathrm{PQ}} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|} = \dfrac{\sqrt{5}}{3}$