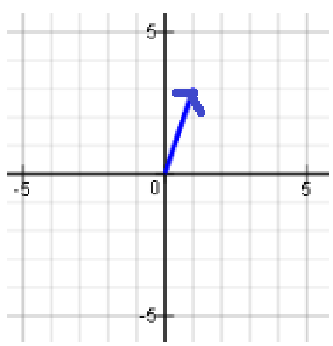
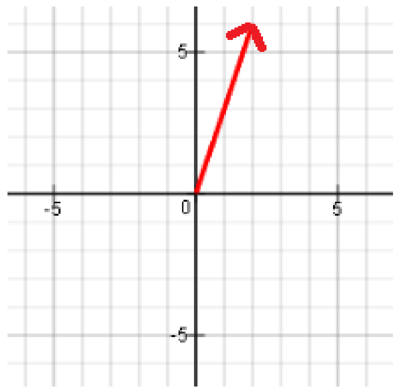
Lines and Planes in Space

Parallel Vectors:

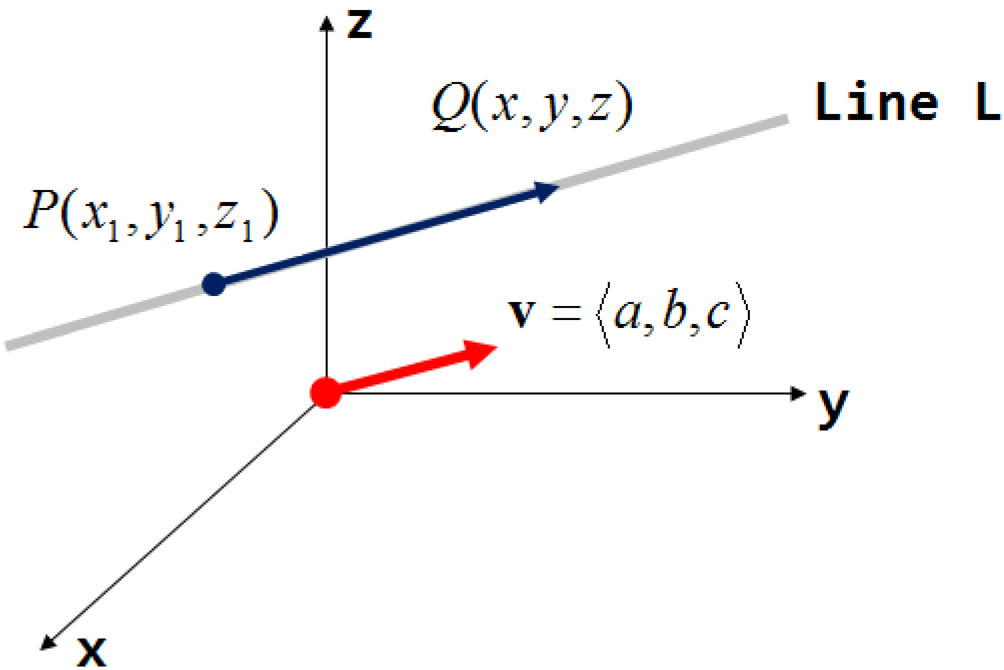
Let $\mathbf{u} = \langle 1,3 \rangle$ and $\mathbf{v} = 2\mathbf{u} = \langle 2,6 \rangle$

If vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel, then $\mathbf{u} = t\mathbf{v}$; where t is a real number.





Lines in space



Note: Vectors $\overrightarrow{PQ}$ and $\mathbf{v}$ are parallel. Hence, $\overrightarrow{PQ} = t\mathbf{v}$; where t is a real number.

$$\langle x - x_1, y - y_1, z - z_1 \rangle = t\langle a, b, c \rangle$$

$$\langle x - x_1, y - y_1, z - z_1 \rangle = \langle ta, tb, tc \rangle$$

Line L is represented by the parametric equations:

$$x - x_1 = ta$$

$$y - y_1 = tb$$

$$z - z_1 = tc$$

Example 1: Find parametric equations of the line L

that passes through (1, 3, 4) and is parallel to $\mathbf{v} = \langle 4, 5, 6 \rangle$.

Line L is represented by the parametric equations:

$x - x_1 = ta, \quad y - y_1 = tb, \quad z - z_1 = tc$

$x - 1 = t(4), \quad y - 3 = t(5), \quad z - 4 = t(6)$

$x - 1 = 4t, \quad y - 3 = 5t, \quad z - 4 = 6t$ parametric equations for line L

$x = 4t + 1, \quad y = 5t + 3, \quad z = 6t + 4$

3D Calculator Input: curve($4t + 1$, $5t + 3$, $6t + 4$, t, -10, 10)

$x - 1 = 4t, \quad y - 3 = 5t, \quad z - 4 = 6t$

$$t = \frac{x-1}{4}; \quad t = \frac{y-3}{5}; \quad t = \frac{z-4}{6}$$

$\dfrac{x-1}{4} = \dfrac{y-3}{5} = \dfrac{z-4}{6}$ symmetric equations for line L

3D Calculator Input: $\dfrac{x-1}{4} = \dfrac{y-3}{5} = \dfrac{z-4}{6}$

List two points on line L.

$x-1=4t, \quad y-3=5t, \quad z-4=6t$

For $t=2$: $x-1=4(2), \quad y-3=5(2), \quad z-4=6(2)$

$\Rightarrow x=9,\ y=13, \quad z=16$

Point on line L: $(9,13,\ 16)$

For $t=4$: $x-1=4(4), \quad y-3=5(4), \quad z-4=6(4)$

$\Rightarrow x=15,\ y=17, \quad z=20$

Point on line L: $(15,17,\ 20)$

Example 2:

Find parametric equations of the line L that passes through $P(1, 3, 4)$ and $Q(4,7,8)$.

Let $\mathbf{v}$ be the vector with initial point at the origin and parallel to the vector $\overrightarrow{PQ}$.

$$\mathbf{v} = \langle 4-1, 7-3, 8-4 \rangle = \langle 3,4,4 \rangle$$

Line L is represented by the parametric equations:

$$x - x_1 = ta, \quad y - y_1 = tb, \quad z - z_1 = tc$$

$$\langle a,b,c \rangle = \langle 3,4,4 \rangle; \quad \langle x_1, y_1, z_1 \rangle = \langle 1,3,,4 \rangle$$

$$x - 1 = t(3), \quad y - 3 = t(4), \quad z - 4 = t(4)$$

$$x - 1 = 4t, \quad y - 3 = 5t, \quad z - 4 = 6t$$

Symmetric Equations:

$$x - 1 = 4t, \quad y - 3 = 5t, \quad z - 4 = 6t$$

$$t = \frac{x-1}{4}, \quad t = \frac{y-3}{5}, \quad t = \frac{z-4}{6}$$

$$\frac{x-1}{4} = \frac{y-3}{5} = \frac{z-4}{6}$$

Example 3:

Find the coordinates of a point P on the line and a vector **v** parallel to the line.

$$x = 4t; \quad y = 5 - t; \quad z = 4 + 7t$$

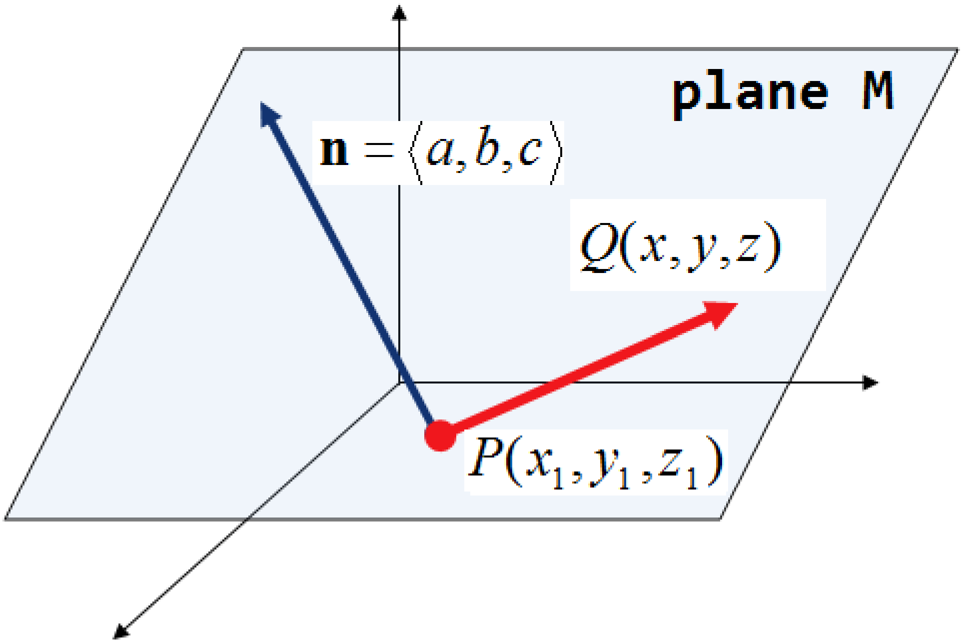
$$x - 0 = 4t; \quad y - 5 = -t; \quad z - 4 = 7t$$

$$x - x_1 = ta, \qquad y - y_1 = tb, \qquad z - z_1 = tc$$

Vector $\mathbf{v} =$ direction vector $= \langle a, b, c \rangle = \langle 4, -1, 7 \rangle;$

Point $P(x_1, y_1, z_1) = (0, 5, 4)$

Planes in space



Let $P(x_1, y_1, z_1)$ and $Q(x, y, z)$ be on plane M.

Let $\mathbf{n} = \langle a, b, c \rangle$ be a vector that is orthogonal to plane M.

Note: Vectors $\overrightarrow{\mathrm{PQ}}$ and $\mathbf{n}$ are orthogonal (perpendicular).

Hence, $\overrightarrow{\mathrm{PQ}} \cdot \mathbf{n} = 0$

$$\langle x - x_1, y - y_1, z - z_1 \rangle \cdot \langle a, b, c \rangle = 0$$

$$\Rightarrow a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Equation of plane M : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Example 4:

Find equation of the plane containing $P(x_1, y_1, z_1) = (2,3,4)$

and perpendicular to the vector $\mathbf{n} = \langle a,b,c \rangle = \langle 6,9,1 \rangle$

Equation of plane : $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Equation of plane : $6(x - 2) + 9(y - 3) + 1(z - 4) = 0$

$$\Rightarrow \quad 6x - 12 + 9y - 27 + z - 4 = 0$$

$$\Rightarrow \quad 6x + 9y + z - 43 = 0$$

3D Calculator Input: $6x + 9y + z - 43 = 0$

Example 5:

Find equation of the plane containing $P(x_1, y_1, z_1) = (2,3,4)$

and perpendicular to the vector $\mathbf{n} = \mathbf{k} = \langle a,b,c \rangle = \langle 0,0,1 \rangle$

Equation of plane: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Equation of plane: $0(x-2) + 0(y-3) + 1(z-4) = 0$

$\Rightarrow \quad 1(z-4) = 0$

$\Rightarrow \quad z = 4$

Example 6: Find equation of the plane containing A(5,9,0), B(2,0,3), and C(4,5,9).

Let $\overrightarrow{\mathrm{AB}} = \langle 2\text{-}5, 0\text{-}9, 3\text{-}0 \rangle = \langle \text{-}3, \text{-}9, 3 \rangle$

Let $\overrightarrow{\mathrm{AC}} = \langle 4\text{-}5, 5\text{-}9, 9\text{-}0 \rangle = \langle \text{-}1, \text{-}4, 9 \rangle$

Note: Vectors $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$ are on the plane.

A vector that is orthogonal (perpendicular) to $\overrightarrow{\mathrm{AB}}$ and $\overrightarrow{\mathrm{AC}}$ is the cross product $\overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}}$.

$$\textit{Hence},\ \overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \left[(-9)(9) - (3)(-4)\right]\mathbf{i} - \left[(-3)(9) - (3)(-1)\right]\mathbf{j} + \left[(-3)(-4) - (-9)(-1)\right]\mathbf{k}$$

$$= \left[-69\right]\mathbf{i} - \left[-24\right]\mathbf{j} + \left[3\right]\mathbf{k} = -69\mathbf{i} + 24\mathbf{j} + 3\mathbf{k} = \langle -69, 24, 3 \rangle$$

Let $\mathbf{n} = \overrightarrow{\mathrm{AB}} \times \overrightarrow{\mathrm{AC}} = \langle a, b, c \rangle = \langle -69, 24, 3 \rangle$

Let point P(x_1,y_1,z_1)= A(5,9,0)

Equation of plane: $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

Equation of plane: $\text{-}69(x-5) + 24(y-9) + 3(z-0) = 0$

3D Calculator Input: $\text{-}69(x-5) + 24(y-9) + 3(z-0) = 0$

Distance Between Plane and Point.

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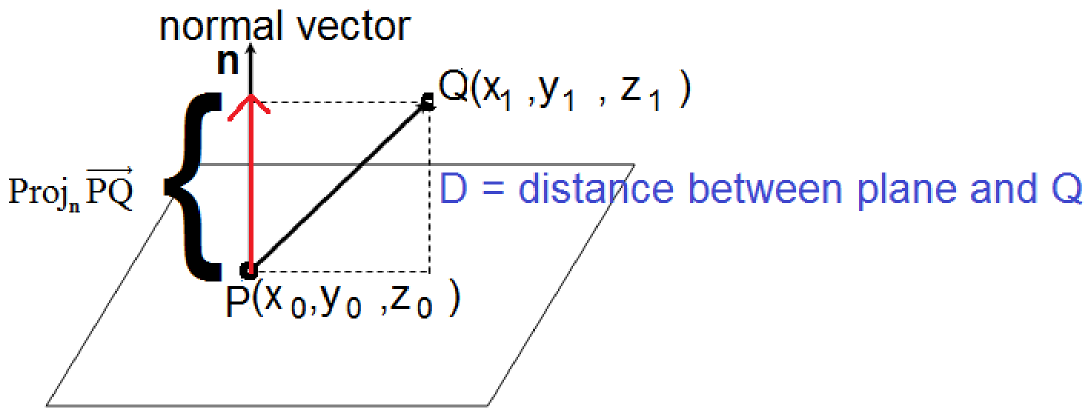


$\mathrm{Proj}_{\mathbf{n}}\overrightarrow{PQ} =$ projection of $\overrightarrow{PQ}$ along $\mathbf{n}$

P is a point on the plane.
We want to distance between the plane and point Q.

$\mathrm{Proj}_{\mathbf{n}}\overrightarrow{PQ} =$ projection of $\overrightarrow{PQ}$ along $\mathbf{n}$

$$\text{Recall: } \mathrm{Proj}_{\mathbf{n}}\overrightarrow{PQ} = \left(\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\cdot\mathbf{n}$$

$$D = \left\|\mathrm{Proj}_{\mathbf{n}}\overrightarrow{PQ}\right\| = \left\|\left(\frac{\overrightarrow{PQ}\cdot\mathbf{n}}{\|\mathbf{n}\|^2}\right)\cdot\mathbf{n}\right\| = \frac{\left\|\overrightarrow{PQ}\cdot\mathbf{n}\right\|}{\|\mathbf{n}\|^2}\|\mathbf{n}\| = \frac{\left|\overrightarrow{PQ}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$$

Example 7:

Find the distance between the point Q(0,0,0) and the plane $2x+3y+z-12=0$

Distance between point and plane = $\dfrac{\left|\overrightarrow{\mathrm{PQ}}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}$

Equation of Plane: $ax+by+cz+d=0 \quad \Rightarrow \quad$ Normal vector $\mathbf{n}=\langle a,b,c\rangle=\langle 2,3,1\rangle$

A point the plane $2x+3y+z-12=0$ is any ordered triple (x,y,z) that satifies this equation.

We can let $x=0, y=0, z=12$. Hence, $(0,0,12)$ satisfies the equation $2x+3y+z-12=0$

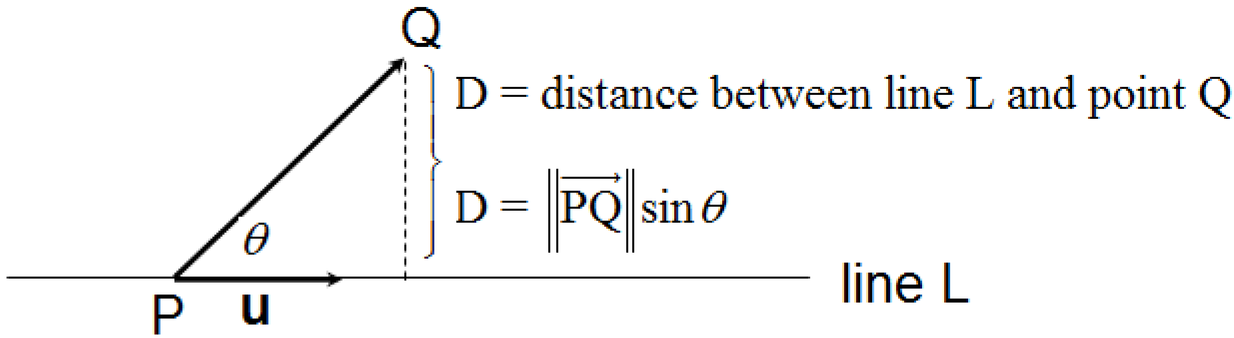
and a point $\mathrm{P}(x_1,y_1,z_1)=(0,0,12)$ is on plane.

$\overrightarrow{\mathrm{PQ}}=\langle 0\text{-}0, 0\text{-}0, 0-12\rangle=\langle 0,0,-12\rangle$

$\overrightarrow{\mathrm{PQ}}\cdot\mathbf{n}=\langle 0,0,-12\rangle\cdot\langle 2,3,1\rangle=0+0+-12=-12$

$\|\mathbf{n}\|=\sqrt{2^2+3^2+1^2}=\sqrt{14}$

Distance between point and plane = $\dfrac{\left|\overrightarrow{\mathrm{PQ}}\cdot\mathbf{n}\right|}{\|\mathbf{n}\|}=\dfrac{|-12|}{\sqrt{14}}=\dfrac{12}{\sqrt{14}}$



Note: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{D}{\left\| \overrightarrow{PQ} \right\|} \quad \Rightarrow \quad D = \left\| \overrightarrow{PQ} \right\| \sin\theta$

D = distance between line L and point Q

$D = \left\| \overrightarrow{PQ} \right\| \sin\theta$

Note: $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{D}{\left\| \overrightarrow{PQ} \right\|} \quad \Rightarrow \quad D = \left\| \overrightarrow{PQ} \right\| \sin\theta$

$$\left\| \overrightarrow{PQ} \times \mathbf{u} \right\| = \left\| \mathbf{u} \times \overrightarrow{PQ} \right\| = \left\| \overrightarrow{PQ} \right\| \left\| \mathbf{u} \right\| \sin\theta \quad \Rightarrow \quad \left\| \overrightarrow{PQ} \right\| \sin\theta = \frac{\left\| \overrightarrow{PQ} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|}$$

So $D = \left\| \overrightarrow{PQ} \right\| \sin\theta = \dfrac{\left\| \overrightarrow{PQ} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|}$

Example 8:

Find the distance between the point Q(1,-2,4) and the line $x = 2t; y = t - 3; z = 2t + 2$

Distance between point and plane = $\dfrac{\left\| \overrightarrow{\mathrm{PQ}} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|}$

Equation of line: $x - x_1 = at; y - y_1 = bt; z - z_1 = ct;$

$Note: \; x = 2t; y = t - 3; z = 2t + 2 \Leftrightarrow x - 0 = 2t; y + 3 = t; z - 2 = 2t$

Direction vector for the line: $\mathbf{u} = \langle 2,1,2 \rangle$.

To find a point on the line, let $t = 0: \quad x = 2t; y = t - 3; z = 2t + 2 \quad \Rightarrow \quad x = 0; \quad y = -3; \quad z = 2$

Hence, a point on the line is P(0,-3,2).

Hence, $\overrightarrow{\mathrm{PQ}} = \langle 1\text{-}0,\ \text{-}2 - \text{-}3, 4 - 2 \rangle = \langle 1,1,2 \rangle$

$\overrightarrow{\mathrm{PQ}} \times \mathbf{u} = \langle 1,1,2 \rangle \times \langle 2,1,2 \rangle = 0\mathbf{i} + 2\mathbf{j} + -1\mathbf{k} = \langle 0,2,-1 \rangle$

$\left\| \overrightarrow{\mathrm{PQ}} \times \mathbf{u} \right\| = \left\| \langle 0,2,-1 \rangle \right\| = \sqrt{(0)^2 + (2)^2 + (-1)^2} = \sqrt{5}$

$\left\| \mathbf{u} \right\| = \left\| \langle 2,1,2 \rangle \right\| = \sqrt{(2)^2 + (1)^2 + (2)^2} = \sqrt{9} = 3$

Distance between point and plane = $\dfrac{\left\| \overrightarrow{\mathrm{PQ}} \times \mathbf{u} \right\|}{\left\| \mathbf{u} \right\|} = \dfrac{\sqrt{5}}{3}$

Example 9:

Plane 1: $3x + 2y - 4z = 12$

Plane 2: $x + 2y + 12z = 1$

Are Plane 1 and Plane 2 parallel, perpendicular, or neither?

Solution:

Let $\mathbf{n_1}$ = normal vector for Plane 1 = $\langle 3,2,-4 \rangle$

Let $\mathbf{n}_2$ = normal vector for Plane 2 = $\langle 1,2,12 \rangle$

Determine if $\mathbf{n_1}$ and $\mathbf{n_2}$ are parallel:

Since $\frac{3}{1} = \frac{2}{2} = \frac{-4}{12}$ is false, $\mathbf{n_1}$ and $\mathbf{n_2}$ are not parallel $\Rightarrow$ Plane 1 and Plane 2 are not parallel.

Determine if $\mathbf{n_1}$ and $\mathbf{n_2}$ are perpendicular:

$\mathbf{n_1} \cdot \mathbf{n_2} = \langle 3,2,-4 \rangle \cdot \langle 1,2,12 \rangle = 3 + 4 + -48 = -41$;

since $\mathbf{n_1} \cdot \mathbf{n_2}$ is not equal to 0, Plane 1 and Plane 2 are not perpendicular.