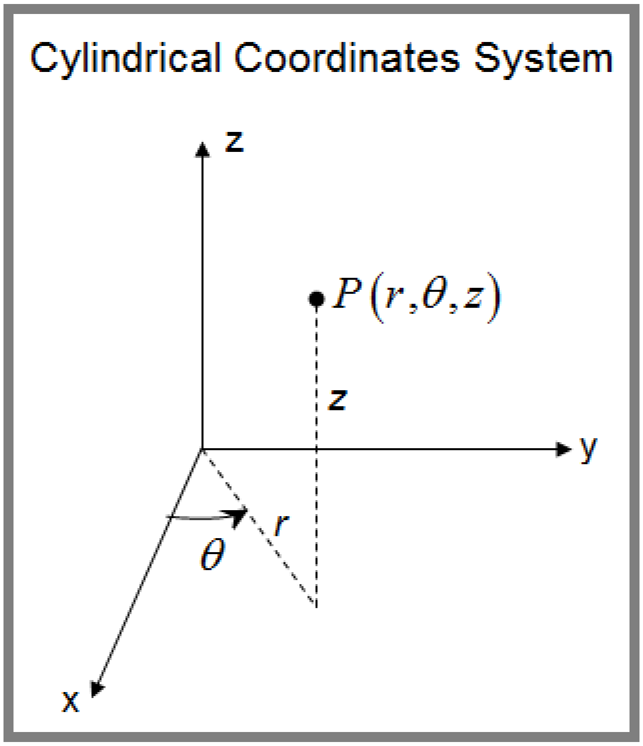
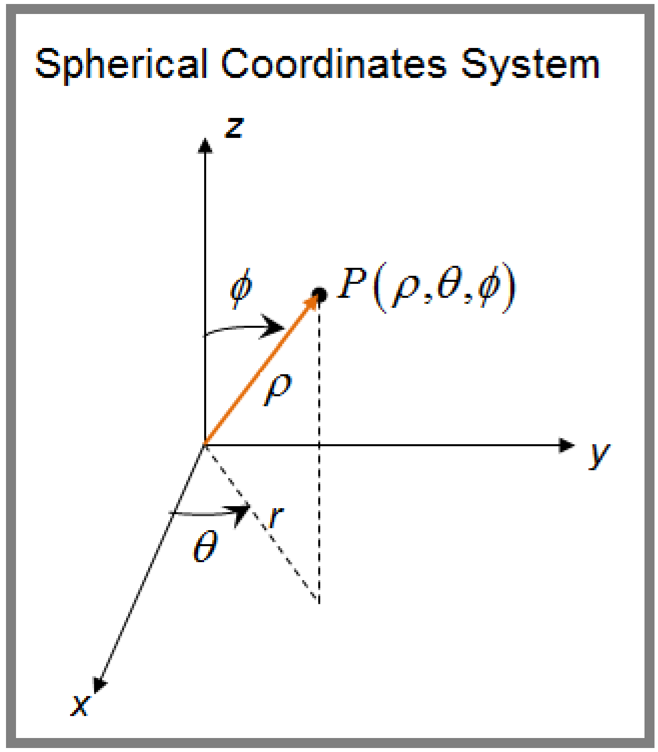
# Cylindrical and Spherical Systems



Cylindrical System: $(r, \theta, z)$ *vs* Rectangular System: $(x, y, z)$

$x = r\cos\theta$; $\quad r = r\sin\theta$; $\quad z = z$; $\quad x^2 + y^2 = r^2$; $\quad \tan\theta = \dfrac{y}{x}$

Cylindrical System: $(\rho, \theta, \phi)$ *vs* Rectangular System: $(x, y, z)$

$x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta, \quad z = \rho\cos\phi$

$\rho^2 = x^2 + y^2 + z^2, \quad \tan\theta = \dfrac{y}{x}, \quad \phi = \cos^{-1}\left(\dfrac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$

Cylindrical $(r, \theta, z)$ *vs* Spherical Coordinates $(\rho, \theta, \phi)$:

$r^2 = \rho^2\sin^2\phi, \quad z = \rho\cos\phi \quad \rho = \sqrt{r^2 + z^2}, \quad \phi = \cos^{-1}\left(\dfrac{z}{\sqrt{r^2 + z^2}}\right)$

(Note: Both systems use $\theta$.)

Example 1: Convert $(x, y, z) = (1, \sqrt{3}, 2)$ to Cylindrical.

Find $r$:

$$x^2 + y^2 = r^2 \quad \Leftrightarrow \quad (1)^2 + \left(\sqrt{3}\right)^2 = r^2 \quad \Leftrightarrow \quad r^2 = 5 \quad \Leftrightarrow \quad r = \pm\sqrt{5}$$

Find $\theta$:

$$\tan\theta = \frac{y}{x} \Leftrightarrow \tan\theta = \frac{\sqrt{3}}{1} \Leftrightarrow \theta = \tan^{-1}\left(\sqrt{3}\right) = \frac{\pi}{3}$$

Hence, $(r, \theta, z) = \left(\sqrt{5}, \dfrac{\pi}{3}, 2\right) = \left(-\sqrt{5}, \dfrac{4\pi}{3}, 2\right)$

Example 2: Convert $(x, y, z) = (2\sqrt{2}, -2\sqrt{2}, 4)$ to Cylindrical.

Find $r$:

$x^2 + y^2 = r^2 \quad \Leftrightarrow \quad x^2 + y^2 = r^2 \quad \Leftrightarrow \quad (2\sqrt{2})^2 + \left(-2\sqrt{2}\right)^2 = r^2 \quad \Leftrightarrow \quad r = \pm\sqrt{16} = \pm 4$

Find $\theta$:

$$\tan\theta = \frac{y}{x} \Leftrightarrow \tan\theta = \frac{-2\sqrt{2}}{2\sqrt{2}} = -1 \Leftrightarrow \theta = \tan^{-1}(-1) = \frac{3\pi}{4}$$

Hence, $(r, \theta, z) = \left(4, \frac{3\pi}{4}, 4\right)$

Example 3: Convert $(r, \theta, z) = (2, -\pi, -4)$ to Rectangular.

Find $x, y, z$:

$x = r\cos\theta = 2\cos(-\pi) = 2(-1) = -2$

$x = r\sin\theta = 2\sin(-\pi) = 2(0) = 0$

$z = -4$

Hence, $(x, y, z) = (\text{-}2, 0, -4)$

Example 4: Convert $(r, \theta, z) = (6, -\pi/4, 2)$ to Rectangular.

Find $x, y, z$:

$x = r\cos\theta = 6\cos(-\pi/4) = 6(\sqrt{2}/2) = 3\sqrt{2}$

$x = r\sin\theta = 6\sin(-\pi/4) = 6(-\sqrt{2}/2) = -3\sqrt{2}$

$z = 2$

Hence, $(x, y, z) = \left(3\sqrt{2}, -3\sqrt{2}, 2\right)$

Example 5: Convert the equation $x = 9$ in Rectangular System to an equation in Cylindrical System .

$x = 9$ Equation in Rectangular System

$r\cos\theta = 9$

$r = \dfrac{9}{\cos\theta} = 9\sec\theta$ Equation in Cylindrical System

Example 6: Convert the equation $z = x^2 + y^2 - 11$ in Rectangular System to an equation in Cylindrical System .

$z = x^2 + y^2 - 11$ Equation in Rectangular System

Note: $r^2 = x^2 + y^2$

$z = r^2 - 11$ Equation in Cylindrical System

Example 7: Convert the equation $x^2 + y^2 = 8x$ in Rectangular System to an equation in Cylindrical System .

$x^2 + y^2 = 8x$ Equation in Rectangular System

Note: $r^2 = x^2 + y^2$ and $x = r\cos\theta$

$r^2 = 8r\cos\theta$

$r^2 - 8r\cos\theta = 0$ Equation in Cylindrical System

Example 8: Convert the equation $x^2 + y^2 + z^2 - 3z = 0$ in Rectangular System to an equation in Cylindrical System .

$x^2 + y^2 + z^2 - 3z = 0$ Equation in Rectangular System

Note: $r^2 = x^2 + y^2$

$r^2 + z^2 - 3z = 0$ Equation in Cylindrical System

Example 9: Convert the equation $z = 2$ in Cylindrical System to an equation in Rectangular System .

$z = 2$ Equation in Cylindrical System

$z = 2$ Equation in Rectangular System

Example 10: Convert the equation $r = \frac{1}{2}z$ in Cylindrical System to an equation in Rectangular System .

$r = \frac{1}{2}z$ Equation in Cylindrical System

$$r^2 = \left(\frac{1}{2}z\right)^2 = \frac{1}{4}z^2$$

$x^2 + y^2 = \frac{1}{4}z^2$ Equation in Rectangular System

Example 11: Convert the equation $z = r^2\cos^2\theta$ in Cylindrical System to an equation in Rectangular System .

$z = r^2\cos^2\theta$ Equation in Cylindrical System

$z = r^2\cos^2\theta = (r\cos\theta)^2$

$z = x^2$ Equation in Rectangular System

Example 12: Convert the equation $r = 2\cos\theta$ in Cylindrical System to an equation in Rectangular System .

$r = 2\cos\theta$ Equation in Cylindrical System

$r \cdot r = (2\cos\theta)\cdot r$

$r^2 = 2r\cos\theta$

$x^2 + y^2 = 2x$ Equation in Rectangular System

Relationship Between Spherical System and Rectangular System

Cylindrical System: $(\rho,\theta,\phi)$

Rectangular System: $(x, y, z)$

$x = \rho\sin\phi\cos\theta;$ $y = \rho\sin\phi\sin\theta;$ $z = \rho\cos\phi;$

$\rho^2 = x^2 + y^2 + z^2;$ $\tan\theta = \dfrac{y}{x};$ $\phi = \dfrac{z}{\sqrt{x^2 + y^2 + z^2}}$

Example 13: Convert $\left(x,y,z\right)=(-4,0,0)$ to Spherical.

$$\rho^2=x^2+y^2+z^2=\left(-4\right)^2+\left(0\right)^2+\left(0\right)^2=16 \quad\Leftrightarrow\quad \rho=4$$

$$\tan\theta=\frac{y}{x}=\frac{0}{-4}=0 \quad\Leftrightarrow\quad \theta=\tan^{-1}\left(0\right)=0$$

$$\phi=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\cos^{-1}\left(\frac{0}{\sqrt{\left(-4\right)^2+\left(0\right)^2+\left(0\right)^2}}\right)=\cos^{-1}\left(0\right)=\frac{\pi}{2}$$

Cylindrical System: $\left(\rho,\theta,\phi\right)=\left(4,0,\frac{\pi}{2}\right)$

Example 14: Convert $\left(x,y,z\right)=(2,2,4\sqrt{2})$ to Spherical.

$$\rho^2=x^2+y^2+z^2=\left(2\right)^2+\left(2\right)^2+\left(4\sqrt{2}\right)^2=40 \quad\Leftrightarrow\quad \rho=\sqrt{40}$$

$$\tan\theta=\frac{y}{x}=\frac{2}{2}=1 \quad\Leftrightarrow\quad \theta=\tan^{-1}\left(1\right)=\frac{\pi}{4}$$

$$\phi=\cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)=\cos^{-1}\left(\frac{4\sqrt{2}}{\sqrt{\left(2\right)^2+\left(2\right)^2+\left(4\sqrt{2}\right)^2}}\right)=\cos^{-1}\left(\frac{4\sqrt{2}}{\sqrt{40}}\right)$$

Cylindrical System: $\left(\rho,\theta,\phi\right)=\left(\sqrt{40},\frac{\pi}{4},\cos^{-1}\left(\frac{4\sqrt{2}}{\sqrt{40}}\right)\right)$

Example 15: Convert $\left(\rho,\theta,\phi\right)=(12,3\pi/4,\pi/9)$ to Rectangular.

$$x=\rho\sin\phi\cos\theta=12\sin\left(\pi/9\right)\cos\left(3\pi/4\right);$$

$$y=\rho\sin\phi\sin\theta=12\sin\left(\pi/9\right)\sin\left(3\pi/4\right);$$

$$z=\rho\cos\phi=12\cos\left(\pi/9\right);$$

$$\left(x,y,z\right)=\left(12\sin\left(\pi/9\right)\cos\left(3\pi/4\right),12\sin\left(\pi/9\right)\sin\left(3\pi/4\right),12\cos\left(\pi/9\right)\right)$$

Example 16: Convert the equation $z = 6$ in Rectangular System to an equation in Spherical System .

$z = 6$ Equation in Rectangular System

$z = \rho \cos\phi$

$6 = \rho \cos\phi$

$\rho = \dfrac{6}{\cos\phi}$ Equation in Spherical System

Example 17: Convert the equation $x^2 + y^2 - 3z^2 = 0$ in Rectangular System to an equation in Spherical System .

$x^2 + y^2 - 3z^2 = 0$ Equation in Rectangular System

$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 - 3(\rho \cos\phi)^2 = 0$

$(\rho \sin\phi)^2 \left((\cos\theta)^2 + (\sin\theta)^2\right) - 3(\rho \cos\phi)^2 = 0$

$Note: (\cos\theta)^2 + (\sin\theta)^2 = 1$

$(\rho \sin\phi)^2 (1) - 3(\rho \cos\phi)^2 = 0$

$\rho^2 \left[(\sin\phi)^2 - 3(\cos\phi)^2\right] = 0$ Equation in Spherical System

Example 18: Convert the equation $\theta = \frac{3\pi}{4}$ in Spherical System to

an equation in Rectangular System .

$\theta = \frac{3\pi}{4}$ Equation in Spherical System

$$\tan\theta = \frac{y}{x}$$

$$\tan\frac{3\pi}{4} = \frac{y}{x}$$

$$-1 = \frac{y}{x}$$

$y = -x$ Equation in Rectangular System

Example 19: Convert the equation $\phi = \frac{\pi}{2}$ in Spherical System to

an equation in Rectangular System .

$\phi = \frac{\pi}{2}$ Equation in Spherical System

$$\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \Leftrightarrow \frac{\pi}{2} = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\Leftrightarrow \cos\frac{\pi}{2} = \frac{z}{\sqrt{x^2 + y^2 + z^2}} \Leftrightarrow 0 = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\Leftrightarrow z = 0$$

$z = 0$ Equation in Rectangular System

Example 20: Convert the equation $\rho = 2\sec\phi$ in Spherical System to an equation in Rectangular System .

$\rho = 2\sec\phi$ Equation in Spherical System

$$\rho^2 = (2\sec\phi)^2 = 4(\sec\phi)^2$$

$$x^2 + y^2 + z^2 = \frac{4}{(\cos\phi)^2}$$

Note: $\phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \Rightarrow \cos\phi = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

$$x^2 + y^2 + z^2 = \frac{4}{\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)^2}$$

$$x^2 + y^2 + z^2 = \frac{4}{\left(\frac{z^2}{x^2 + y^2 + z^2}\right)} = 4 \cdot \frac{x^2 + y^2 + z^2}{z^2}$$

$1 = \frac{4}{z^2}$ Divide left side and right side by $x^2 + y^2 + z^2$

$$z^2 = 4$$

$$z = 2$$

$z = 2$ Equation in Rectangular System

Relationship Between Spherical System and Cylindrical System

Cylindrical System: $(\rho, \theta, \phi)$ and Spherical System: $(r, \theta, z)$

$r^2 = \rho^2 \sin^2 \phi; \qquad z = \rho \cos \phi$

$\rho = \sqrt{r^2 + z^2}; \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$

Example 21: Convert $(r, \theta, z) = (3, -\pi/4, 0)$ to Spherical.

Cylindrical System: $(\rho, \theta, \phi)$ and Spherical System: $(r, \theta, z)$

$\rho = \sqrt{r^2 + z^2} = \sqrt{3^2 + 0^2} = 3$

$\theta = -\pi/4$

$\phi = \cos^{-1}\left(\frac{z}{\sqrt{r^2 + z^2}}\right) = \cos^{-1}\left(\frac{0}{3}\right) = \cos^{-1}(0) = \pi/2$

Spherical System: $(r, \theta, z) = (3, -\pi/4, \pi/2)$

Example 22: Convert $(\rho, \theta, \phi) = (4, \pi/18, \pi/2)$ to Cylindrical.

Cylindrical System: $(r, \theta, z)$ and Spherical System: $(\rho, \theta, \phi)$

$r^2 = \rho^2 \sin^2 \phi = (4)^2 (\sin \pi/2)^2 = 16 \cdot 1 = 16 \quad \Leftrightarrow \quad r = 4$

$\theta = \pi/18$

$z = \rho \cos \phi = 4 \cos \pi/2 = 4(0) = 0$

Cylindrical System: $(r, \theta, z) = (4, \pi/18, 0)$

Example 23: Convert the equation $4(x^2 + y^2) = z^2$ in the Rectangular System to an equation in the Cylindrical System and and equation in theSpherical System.

To Cylindrical:

Note: $x^2 + y^2 = r^2$

$4r^2 = z^2$

$2r = z$ equation in the Cylindrical System

To Spherical:

$4(x^2 + y^2) = z^2$

$4\left[(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2\right] = (\rho\cos\phi)^2$

$4\left[(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2\right] - (\rho\cos\phi)^2 = 0$

$\rho^2\left[4(\sin\phi\cos\theta)^2 + 4(\sin\phi\sin\theta)^2 - (\cos\phi)^2\right] = 0$

$\rho^2\left[4(\sin\phi)^2\left[(\cos\theta)^2 + (\sin\theta)^2\right] - (\cos\phi)^2\right] = 0$

$\rho^2\left[4(\sin\phi)^2[1] - (\cos\phi)^2\right] = 0$ Note: $(\cos\theta)^2 + (\sin\theta)^2 = 1$

$\rho^2\left[4(\sin\phi)^2 - (\cos\phi)^2\right] = 0 = 0$

Example 24: Convert the equation $x^2 + y^2 = 36$ in the Rectangular System to an equation in the Cylindrical System and and equation in theSpherical System.

To Cylindrical:

Note: $x^2 + y^2 = r^2$

$x^2 + y^2 = 36$

$r^2 = 36$

$r = 6$ equation in the Cylindrical System

To Spherical:

$x^2 + y^2 = 36$

$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = 36$

$\rho^2 (\sin\phi)^2 \left[(\cos\theta)^2 + (\sin\theta)^2\right] = 36$

$\rho^2 (\sin\phi)^2 [1] = 36$ Note: $(\cos\theta)^2 + (\sin\theta)^2 = 1$

$\rho^2 (\sin\phi)^2 = 36$

$\rho(\sin\phi) = 6$