

Derivative of Vector-Valued Functions

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

Example 1: Let $\mathbf{r}(t) = 3t\mathbf{i} + t^2\mathbf{j}$

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\left[3(t + \Delta t)\mathbf{i} + (t + \Delta t)^2\mathbf{j}\right] - \left[3t\mathbf{i} + t^2\mathbf{j}\right]}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\left[3t\mathbf{i} + 3\Delta t\mathbf{i} + t^2\mathbf{j} + 2t\Delta t\mathbf{j} + (\Delta t)^2\mathbf{j}\right] - \left[3t\mathbf{i} + t^2\mathbf{j}\right]}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \frac{\left[3\Delta t\mathbf{i} + 2t\Delta t\mathbf{j} + (\Delta t)^2\mathbf{j}\right]}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left[3\mathbf{i} + 2t\mathbf{j} + (\Delta t)\mathbf{j}\right] \\ &= 3\mathbf{i} + 2t\mathbf{j}\end{aligned}$$

Example 2: Let $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}$. Find $\mathbf{r}'(t)$ and $\mathbf{r}'(2)$.

$$\mathbf{r}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$\mathbf{r}'(t) = D_t(t)\mathbf{i} + D_t(t^2)\mathbf{j} + D_t\left(\frac{3}{2}\right)\mathbf{k}$$

$$\mathbf{r}'(t) = 1\mathbf{i} + 2t\mathbf{j} + 0\mathbf{k} = \langle 1, 2t, 0 \rangle$$

$$\mathbf{r}'(2) = \langle 1, 2t, 0 \rangle = \langle 1, 2(2), 0 \rangle = \langle 1, 4, 0 \rangle$$

Example 3: Let $\mathbf{r}(t) = 4\sqrt{t}\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}$. Find $\mathbf{r}'(t)$ and $\mathbf{r}'(2)$.

$$\mathbf{r}'(t) = D_t(4\sqrt{t})\mathbf{i} + D_t(t^2)\mathbf{j} + D_t(\ln t)\mathbf{k}$$

$$\mathbf{r}'(t) = 2t^{-1/2}\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k} = \left\langle 2t^{-1/2}, 2t, \frac{1}{t} \right\rangle$$

$$\mathbf{r}'(2) = \left\langle 2t^{-1/2}, 2t, \frac{1}{t} \right\rangle = \left\langle 2(2)^{-1/2}, 4, \frac{1}{2} \right\rangle$$

Example 4: Let $\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j}$. Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.

$$\mathbf{r}'(t) = D_t(t^2 + 1)\mathbf{i} + D_t(t^2 - 1)\mathbf{j}$$

$$\mathbf{r}'(t) = 2t\mathbf{i} + 2t\mathbf{j} = \langle 2t, 2t \rangle$$

$$\mathbf{r}''(t) = 2\mathbf{i} + 2\mathbf{j} = \langle 2, 2 \rangle$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 2t \cdot 2 + 2t \cdot 2 = 8t$$

Example 5: Let $\mathbf{r}(t) = (3\cos t)\mathbf{i} + (3\sin t)\mathbf{j}$ be a circular path of a particle.

$$\text{Velocity of particle at time } t = \mathbf{r}'(t) = \langle -3\sin t, 3\cos t \rangle$$

$$\text{Speed of particle at time } t = \|\mathbf{r}'(t)\| = \|\langle -3\sin t, 3\cos t \rangle\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2} = \sqrt{9} = 3$$

$$\text{Acceleration of particle at time } t = \mathbf{r}''(t) = \langle -3\cos t, 3\sin t \rangle$$

Example 6: Let $\mathbf{r}(t) = t^3\mathbf{i} + (2t^2 + 3)\mathbf{j} + (3t - 5)\mathbf{k}$.

Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

$$\mathbf{r}'(t) = D_t(t^3)\mathbf{i} + D_t(2t^2 + 3)\mathbf{j} + D_t(3t - 5)\mathbf{k}$$

$$\mathbf{r}'(t) = 3t^2\mathbf{i} + 4t\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{r}''(t) = 6t\mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = 3t^2 \cdot 6t + 4t \cdot 4 + 3 \cdot 0 = 18t^3 + 16t$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = [(4t)(0) - (3)(4)]\mathbf{i} - [(3t^2)(0) - (3)(6t)]\mathbf{j} + [(3t^2)(4) - (4t)(6t)]\mathbf{k}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = [-12]\mathbf{i} - [-18t]\mathbf{j} + [-12t^2]\mathbf{k}$$

Example 7: Let $\mathbf{r}(t) = e^{-t}\mathbf{i} + t^2\mathbf{j} + \tan t\mathbf{k}$.

Find $\mathbf{r}'(t)$, $\mathbf{r}''(t)$, $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$, $\mathbf{r}'(t) \times \mathbf{r}''(t)$.

$$\mathbf{r}'(t) = D_t(e^{-t})\mathbf{i} + D_t(t^2)\mathbf{j} + D_t(\tan t)\mathbf{k}$$

$$\mathbf{r}'(t) = -e^{-t}\mathbf{i} + 2t\mathbf{j} + \sec^2 t\mathbf{k}$$

$$\mathbf{r}''(t) = e^{-t}\mathbf{i} + 2\mathbf{j} + (2\sec^2 t \tan t)\mathbf{k}$$

$$\mathbf{r}'(t) \cdot \mathbf{r}''(t) = (-e^{-t})(e^{-t}) + (2t)(2) + (\sec^2 t)(2\sec^2 t \tan t) = -e^{-2t} + 4t + 2\sec^3 t \tan t$$

$$\begin{aligned}\mathbf{r}'(t) \times \mathbf{r}''(t) &= \left[(2t)(2\sec^2 t \tan t) - (\sec^2 t)(2) \right] \mathbf{i} \\ &\quad - \left[(-e^{-t})(2\sec^2 t \tan t) - (\sec^2 t)(e^{-t}) \right] \mathbf{j} \\ &\quad + \left[(-e^{-t})(2) - (2t)(e^{-t}) \right] \mathbf{k}\end{aligned}$$

Example 8: Find $\int \left[(t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + (\cos t)\mathbf{k} \right] dt$

$$\begin{aligned}\int \left[(t^2 + 1)\mathbf{i} + (t^2 - 1)\mathbf{j} + (\cos t)\mathbf{k} \right] dt &= \left[\int (t^2 + 1) dt \right] \mathbf{i} + \left[\int (t^2 - 1) dt \right] \mathbf{j} + \left[\int (\cos t) dt \right] \mathbf{k} \\ &= \left[\frac{t^3}{3} + t + C_1 \right] \mathbf{i} + \left[\frac{t^3}{3} + t + C_2 \right] \mathbf{j} + [\sin t + C_3] \mathbf{k}\end{aligned}$$

Example 9: Find $\int \left[\ln t \mathbf{i} + \frac{1}{t} \mathbf{j} + \mathbf{k} \right] dt$

$$\begin{aligned} \int \left[\ln t \mathbf{i} + \frac{1}{t} \mathbf{j} + \mathbf{k} \right] dt &= \left[\int (\ln t) dt \right] \mathbf{i} + \left[\int \left(\frac{1}{t} \right) dt \right] \mathbf{j} + \left[\int 1 dt \right] \mathbf{k} \\ &= \left[\frac{1}{t} + C_1 \right] \mathbf{i} + \left[\frac{-1}{t^2} + C_2 \right] \mathbf{j} + [t + C_3] \mathbf{k} \end{aligned}$$

Example 10: Find $\int \left[e^t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k} \right] dt$

$$\begin{aligned} \int \left[e^t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k} \right] dt &= \left[\int (e^t) dt \right] \mathbf{i} + \left[\int (\sin t) dt \right] \mathbf{j} + \left[\int (\cos t) dt \right] \mathbf{k} \\ &= [e^t + C_1] \mathbf{i} + [-\cos t + C_2] \mathbf{j} + [\sin t + C_3] \mathbf{k} \end{aligned}$$

Example 11: Find $\int_{-1}^1 \left[t \mathbf{i} + t^3 \mathbf{j} + \sqrt[3]{t} \mathbf{k} \right] dt$

$$\begin{aligned} \int_{-1}^1 \left[t \mathbf{i} + t^3 \mathbf{j} + \sqrt[3]{t} \mathbf{k} \right] dt &= \left[\int (t) dt \right] \mathbf{i} + \left[\int (t^3) dt \right] \mathbf{j} + \left[\int (t^{1/3}) dt \right] \mathbf{k} \\ &= \left(\left[\frac{t^2}{2} \right] \mathbf{i} + \left[\frac{t^4}{4} \right] \mathbf{j} + \left[\frac{t^{4/3}}{4/3} \right] \mathbf{k} \right) \Big|_{-1}^1 \\ &= \left(\left[\frac{1}{2} \right] \mathbf{i} + \left[\frac{1}{4} \right] \mathbf{j} + \left[\frac{3}{4} \right] \mathbf{k} \right) - \left(\left[\frac{1}{2} \right] \mathbf{i} + \left[\frac{1}{4} \right] \mathbf{j} + \left[\frac{3}{4} \right] \mathbf{k} \right) = \mathbf{0} = \langle 0, 0, 0 \rangle \end{aligned}$$

Example 12: Let $\mathbf{r}'(t) = 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}$; and $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$. Find $\mathbf{r}(t)$.

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t) dt = \left[\int 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k} \right] dt = \left[\int 0\mathbf{i} + 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k} \right] dt \\ &= \left[\int 0 dt \right] \mathbf{i} + \left[\int 3t^2 dt \right] \mathbf{j} + \left[\int 6\sqrt{t} dt \right] \mathbf{k} \\ &= [0 + C_1] \mathbf{i} + [t^3 + C_2] \mathbf{j} + [4t^{3/2} + C_3] \mathbf{k}\end{aligned}$$

$$\mathbf{r}(t) = [0 + C_1] \mathbf{i} + [t^3 + C_2] \mathbf{j} + [4t^{3/2} + C_3] \mathbf{k}$$

$$\mathbf{r}(0) = [C_1] \mathbf{i} + [C_2] \mathbf{j} + [C_3] \mathbf{k}$$

$$\mathbf{i} + 2\mathbf{j} = [C_1] \mathbf{i} + [C_2] \mathbf{j} + [C_3] \mathbf{k}$$

$$C_1 = 1; \quad C_2 = 2; \quad C_3 = 0$$

$$\mathbf{r}(t) = [0 + C_1] \mathbf{i} + [t^3 + C_2] \mathbf{j} + [4t^{3/2} + C_3] \mathbf{k}$$

$$\mathbf{r}(t) = [0 + 1] \mathbf{i} + [t^3 + 2] \mathbf{j} + [4t^{3/2} + 0] \mathbf{k}$$

$$\mathbf{r}(t) = \mathbf{i} + [t^3 + 2] \mathbf{j} + [4t^{3/2}] \mathbf{k}$$

Example 13: Let $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}$; and $\mathbf{r}(0) = 4\mathbf{j}$; and $\mathbf{r}'(0) = 3\mathbf{k}$. Find $\mathbf{r}(t)$.

$$\begin{aligned}\mathbf{r}'(t) &= \int \mathbf{r}''(t) dt = \left[\int -4\cos t\mathbf{j} - 3\sin t\mathbf{k} \right] dt = \left[\int 0\mathbf{i} - 4\cos t\mathbf{j} - 3\sin t\mathbf{k} \right] dt \\ &= (0 + C_1)\mathbf{i} + (-4\sin t + C_2)\mathbf{j} + (3\cos t + C_3)\mathbf{k} \quad \text{Note: } \int \cos t dt = \sin t; \int \sin t dt = -\cos t;\end{aligned}$$

$$\mathbf{r}'(t) = (0 + C_1)\mathbf{i} + (-4\sin t + C_2)\mathbf{j} + (3\cos t + C_3)\mathbf{k}$$

$$\mathbf{r}'(0) = (0 + C_1)\mathbf{i} + (0 + C_2)\mathbf{j} + (3 + C_3)\mathbf{k}$$

$$3\mathbf{k} = (0 + C_1)\mathbf{i} + (0 + C_2)\mathbf{j} + (3 + C_3)\mathbf{k}$$

$$C_1 = 0; \quad C_2 = 0; \quad 3 + C_3 = 3 \Leftrightarrow C_3 = 0$$

$$\mathbf{r}'(t) = (0)\mathbf{i} + (-4\sin t)\mathbf{j} + (3\cos t)\mathbf{k}$$

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t) dt = \left[\int (-4\sin t)\mathbf{j} + (3\cos t)\mathbf{k} \right] dt = \left[\int 0\mathbf{i} + (-4\sin t)\mathbf{j} + (3\cos t)\mathbf{k} \right] dt \\ &= [0 + C_4]\mathbf{i} + [4\cos t + C_5]\mathbf{j} + [3\sin t + C_6]\mathbf{k}\end{aligned}$$

$$\mathbf{r}(t) = [0 + C_4]\mathbf{i} + [4\cos t + C_5]\mathbf{j} + [3\sin t + C_6]\mathbf{k}$$

$$\mathbf{r}(0) = [C_4]\mathbf{i} + [4 + C_5]\mathbf{j} + [C_6]\mathbf{k}$$

$$4\mathbf{j} = [C_4]\mathbf{i} + [4 + C_5]\mathbf{j} + [C_6]\mathbf{k}$$

$$C_4 = 0; \quad C_5 = 0; \quad C_6 = 0$$

$$\mathbf{r}(t) = [0 + C_4]\mathbf{i} + [4\cos t + C_5]\mathbf{j} + [3\sin t + C_6]\mathbf{k}$$

$$\mathbf{r}(t) = [0]\mathbf{i} + [4\cos t]\mathbf{j} + [3\sin t]\mathbf{k} = \langle 0, 4\cos t, 3\sin t \rangle$$