Derivative of Vector-Valued Functions

$$\mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

Example 1: Let $\mathbf{r}(\mathrm{t}) = 3t\mathbf{i} + t^2\mathbf{j}$

$$\mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\left[3\left(t+\Delta t\right)\mathbf{i} + \left(t+\Delta t\right)^2 \mathbf{j}\right]\mathbf{r}(t+\Delta t) - \left[3t\mathbf{i} + t^2\mathbf{j}\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[3t\mathbf{i} + 3\Delta t\mathbf{i} + t^2\mathbf{j} + 2t\Delta t\mathbf{j} + \left(\Delta t\right)^2 \mathbf{j}\right] - \left[3t\mathbf{i} + t^2\mathbf{j}\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\left[3\Delta t\mathbf{i} + 2t\Delta t\mathbf{j} + \left(\Delta t\right)^2 \mathbf{j}\right]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \left[3\mathbf{i} + 2t\mathbf{j} + \left(\Delta t\right)\mathbf{j}\right]$$

$$= 3\mathbf{i} + 2t\mathbf{j}$$

$$\mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

Example 1: Let $\mathbf{r}(\mathrm{t}) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}$. Find $\mathbf{r}'(\mathrm{t})$ and $\mathbf{r}'(2)$.

$$\mathbf{r}'(\mathrm{t}) = D_t\left(t\right)\mathbf{i} + D_t\left(t^2\right)\mathbf{j} + D_t\left(\frac{3}{2}\right)\mathbf{k}$$

$$\mathbf{r}'(\mathrm{t}) = 1\mathbf{i} + 2t\mathbf{j} + 0\mathbf{k} = \langle 1, 2t, 0\rangle$$

$$\mathbf{r}'(2) = \langle 1, 2t, 0\rangle = \langle 1, 2(2), 0\rangle = \langle 1, 4, 0\rangle$$

Example 2: Let $\mathbf{r}(\mathrm{t}) = 4\sqrt{t}\,\mathbf{i} + t^2\mathbf{j} + \ln t\,\mathbf{k}$. Find $\mathbf{r}'(\mathrm{t})$ and $\mathbf{r}'(2)$.

$$\mathbf{r}'(\mathrm{t}) = D_t\left(4\sqrt{t}\right)\mathbf{i} + D_t\left(t^2\right)\mathbf{j} + D_t\left(\ln t\right)\mathbf{k}$$

$$\mathbf{r}'(\mathrm{t}) = 2t^{-1/2}\mathbf{i} + 2t\mathbf{j} + \frac{1}{t}\mathbf{k} = \left\langle 2t^{-1/2}, 2t, \frac{1}{t}\right\rangle$$

$$\mathbf{r}'(2) = \left\langle 2t^{-1/2}, 2t, \frac{1}{t}\right\rangle = \left\langle 2(2)^{-1/2}, 4, \frac{1}{2}\right\rangle$$

Example 3: Let $\mathbf{r}(\mathrm{t}) = \left(t^2+1\right)\mathbf{i} + \left(t^2-1\right)\mathbf{j}$. Find $\mathbf{r}'(\mathrm{t})$, $\mathbf{r}''(\mathrm{t})$, $\mathbf{r}'(\mathrm{t})\cdot\mathbf{r}''(\mathrm{t})$.

$$\mathbf{r}'(\mathrm{t}) = D_t\left(t^2+1\right)\mathbf{i} + D_t\left(t^2-1\right)\mathbf{j}$$

$$\mathbf{r}'(\mathrm{t}) = 2t\mathbf{i} + 2t\mathbf{j} = \langle 2t, 2t\rangle$$

$$\mathbf{r}''(\mathrm{t}) = 2\mathbf{i} + 2\mathbf{j} = \langle 2, 2\rangle$$

$$\mathbf{r}'(\mathrm{t})\cdot\mathbf{r}''(\mathrm{t}) = 2t\cdot 2 + 2t\cdot 2 = 8t$$

Example 4: Let $\mathbf{r}(\mathrm{t}) = t^3\mathbf{i} + \left(2t^2+3\right)\mathbf{j} + \left(3t-5\right)\mathbf{k}$.

Find $\mathbf{r}'(\mathrm{t})$, $\mathbf{r}''(\mathrm{t})$, $\mathbf{r}'(\mathrm{t})\cdot\mathbf{r}''(\mathrm{t})$, $\mathbf{r}'(\mathrm{t})\times\mathbf{r}''(\mathrm{t})$.

$$\mathbf{r}'(\mathrm{t}) = D_t\left(t^3\right)\mathbf{i} + D_t\left(2t^2+3\right)\mathbf{j} + D_t\left(3t-5\right)\mathbf{k}$$

$$\mathbf{r}'(\mathrm{t}) = 3t^2\mathbf{i} + 4t\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{r}''(\mathrm{t}) = 6t\mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$$

$$\mathbf{r}'(\mathrm{t})\cdot\mathbf{r}''(\mathrm{t}) = 3t^2\cdot 6t + 4t\cdot 4 + 3\cdot 0 = 18t^3 + 16t$$

$$\mathbf{r}'(\mathrm{t})\times\mathbf{r}''(\mathrm{t}) = \left[(4t)(0)-(3)(4)\right]\mathbf{i} - \left[\left(3t^2\right)(0)-(3)(6t)\right]\mathbf{j} + \left[\left(3t^2\right)(4)-(4t)(6t)\right]\mathbf{k}$$

$$\mathbf{r}'(\mathrm{t})\times\mathbf{r}''(\mathrm{t}) = [-12]\mathbf{i} - [-18t]\mathbf{j} + \left[-12t^2\right]\mathbf{k}$$

Example 5: Let $\mathbf{r}(\mathrm{t}) = e^{-t}\mathbf{i} + t^2\mathbf{j} + \tan t\mathbf{k}$.

Find $\mathbf{r}'(\mathrm{t})$, $\mathbf{r}''(\mathrm{t})$, $\mathbf{r}'(\mathrm{t}) \cdot \mathbf{r}''(\mathrm{t})$, $\mathbf{r}'(\mathrm{t}) \times \mathbf{r}''(\mathrm{t})$.

$$\mathbf{r}'(\mathrm{t}) = D_t\left(e^{-t}\right)\mathbf{i} + D_t\left(t^2\right)\mathbf{j} + D_t\left(\tan t\right)\mathbf{k}$$

$$\mathbf{r}'(\mathrm{t}) = -e^{-t}\mathbf{i} + 2t\mathbf{j} + \sec^2 t\mathbf{k}$$

$$\mathbf{r}''(\mathrm{t}) = e^{-t}\mathbf{i} + 2\mathbf{j} + \left(2\sec^2 t\tan t\right)\mathbf{k}$$

$$\mathbf{r}'(\mathrm{t}) \cdot \mathbf{r}''(\mathrm{t}) = \left(-e^{-t}\right)\left(e^{-t}\right) + (2t)(2) + \left(\sec^2 t\right)\left(2\sec^2 t\tan t\right) = -e^{-2t} + 4t + 2\sec^3 t\tan t$$

$$\mathbf{r}'(\mathrm{t}) \times \mathbf{r}''(\mathrm{t}) = \left[(2t)\left(2\sec^2 t\tan t\right) - \left(\sec^2 t\right)(2)\right]\mathbf{i}$$

$$-\left[\left(-e^{-t}\right)\left(2\sec^2 t\tan t\right) - \left(\sec^2 t\right)\left(e^{-t}\right)\right]\mathbf{j}$$

$$+\left[\left(-e^{-t}\right)(2) - (2t)\left(e^{-t}\right)\right]\mathbf{k}$$

Example 6: Find $\int\left[\left(t^2+1\right)\mathbf{i} + \left(t^2-1\right)\mathbf{j} + (\cos t)\mathbf{k}\right]dt$

$$\int\left[\left(t^2+1\right)\mathbf{i} + \left(t^2-1\right)\mathbf{j} + (\cos t)\mathbf{k}\right]dt = \left[\int\left(t^2+1\right)dt\right]\mathbf{i} + \left[\int\left(t^2-1\right)dt\right]\mathbf{j} + \left[\int(\cos t)\,dt\right]\mathbf{k}$$

$$= \left[\frac{t^3}{3} + t + C_1\right]\mathbf{i} + \left[\frac{t^3}{3} + t + C_2\right]\mathbf{j} + \left[\sin t + C_3\right]\mathbf{k}$$

Example 7: Find $\int\left[\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right]dt$

$$\int\left[\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right]dt = \left[\int(\ln t)\,dt\right]\mathbf{i} + \left[\int\left(\frac{1}{t}\right)dt\right]\mathbf{j} + \left[\int 1dt\right]\mathbf{k}$$

$$= \left[\frac{1}{t} + C_1\right]\mathbf{i} + \left[\frac{-1}{t^2} + C_2\right]\mathbf{j} + \left[t + C_3\right]\mathbf{k}$$

Example 8: Find $\int \left[e^t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k} \right] dt$

$$\int \left[e^t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k} \right] dt = \left[\int \left(e^t \right) dt \right] \mathbf{i} + \left[\int \left(\sin t \right) dt \right] \mathbf{j} + \left[\int \left(\cos t \right) dt \right] \mathbf{k}$$

$$= \left[e^t + C_1 \right] \mathbf{i} + \left[-\cos t + C_2 \right] \mathbf{j} + \left[\sin t + C_3 \right] \mathbf{k}$$

Example 9: Find $\int_{-1}^{1} \left[t\mathbf{i} + t^3 \mathbf{j} + \sqrt[3]{t} \mathbf{k} \right] dt$

$$\int_{-1}^{1} \left[t\mathbf{i} + t^3 \mathbf{j} + \sqrt[3]{t} \mathbf{k} \right] dt = \left[\int (t) \, dt \right] \mathbf{i} + \left[\int \left(t^3 \right) dt \right] \mathbf{j} + \left[\int \left(t^{1/3} \right) dt \right] \mathbf{k}$$

$$= \left(\left[\frac{t^2}{2} \right] \mathbf{i} + \left[\frac{t^4}{4} \right] \mathbf{j} + \left[\frac{t^{4/3}}{4/3} \right] \mathbf{k} \right) \Bigg|_{-1}^{1}$$

$$= \left(\left[\frac{1}{2} \right] \mathbf{i} + \left[\frac{1}{4} \right] \mathbf{j} + \left[\frac{3}{4} \right] \mathbf{k} \right) - \left(\left[\frac{1}{2} \right] \mathbf{i} + \left[\frac{1}{4} \right] \mathbf{j} + \left[\frac{3}{4} \right] \mathbf{k} \right) = \mathbf{0} = \langle 0, 0, 0 \rangle$$

Example 10: Let $\mathbf{r}'(\mathrm{t}) = 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}$; and $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$.

Find $\mathbf{r}(\mathrm{t})$.

$$\mathbf{r}(\mathrm{t}) = \int \mathbf{r}'(\mathrm{t})dt = \left[\int 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}\right]dt = \left[\int 0\mathbf{i} + 3t^2\mathbf{j} + 6\sqrt{t}\mathbf{k}\right]dt$$

$$= \left[\int 0dt\right]\mathbf{i} + \left[\int 3t^2dt\right]\mathbf{j} + \left[\int 6\sqrt{t}dt\right]\mathbf{k}$$

$$= \left[0 + C_1\right]\mathbf{i} + \left[t^3 + C_2\right]\mathbf{j} + \left[4t^{3/2} + C_3\right]\mathbf{k}$$

$$\mathbf{r}(\mathrm{t}) = = \left[0 + C_1\right]\mathbf{i} + \left[t^3 + C_2\right]\mathbf{j} + \left[4t^{3/2} + C_3\right]\mathbf{k}$$

$$\mathbf{r}(0) = \left[C_1\right]\mathbf{i} + \left[C_2\right]\mathbf{j} + \left[C_3\right]\mathbf{k}$$

$$\mathbf{i} + 2\mathbf{j} = \left[C_1\right]\mathbf{i} + \left[C_2\right]\mathbf{j} + \left[C_3\right]\mathbf{k}$$

$C_1 = 1$; $\quad C_2 = 2$; $\quad C_3 = 0$

$$\mathbf{r}(\mathrm{t}) = = \left[0 + C_1\right]\mathbf{i} + \left[t^3 + C_2\right]\mathbf{j} + \left[4t^{3/2} + C_3\right]\mathbf{k}$$

$$\mathbf{r}(\mathrm{t}) = = \left[0 + 1\right]\mathbf{i} + \left[t^3 + 2\right]\mathbf{j} + \left[4t^{3/2} + 0\right]\mathbf{k}$$

$$\mathbf{r}(\mathrm{t}) = \mathbf{i} + \left[t^3 + 2\right]\mathbf{j} + \left[4t^{3/2}\right]\mathbf{k}$$

Example 11: Let $\mathbf{r}''(\mathrm{t}) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}$; and $\mathbf{r}(0) = 4\mathbf{j}$; and $\mathbf{r}'(0) = 3\mathbf{k}$.

Find $\mathbf{r}(\mathrm{t})$.

$$\mathbf{r}'(\mathrm{t}) = \int \mathbf{r}''(\mathrm{t})dt = \left[\int -4\cos t\mathbf{j} - 3\sin t\mathbf{k}\right]dt = \left[\int 0\mathbf{i} - 4\cos t\mathbf{j} - 3\sin t\mathbf{k}\right]dt$$

$$= \left(0 + C_1\right)\mathbf{i} + \left(-4\sin t + C_2\right)\mathbf{j} + \left(3\cos t + C_3\right)\mathbf{k}$$ Note: $\int \cos t dt = \sin t$; $\int \sin t dt = -\cos t$;

$$\mathbf{r}'(\mathrm{t}) = \left(0 + C_1\right)\mathbf{i} + \left(-4\sin t + C_2\right)\mathbf{j} + \left(3\cos t + C_3\right)\mathbf{k}$$

$$\mathbf{r}'(0) = \left(0 + C_1\right)\mathbf{i} + \left(0 + C_2\right)\mathbf{j} + \left(3 + C_3\right)\mathbf{k}$$

$$3\mathbf{k} = \left(0 + C_1\right)\mathbf{i} + \left(0 + C_2\right)\mathbf{j} + \left(3 + C_3\right)\mathbf{k}$$

$$C_1 = 0; \quad C_2 = 0; \quad 3 + C_3 = 3 \Leftrightarrow C_3 = 0$$

$$\mathbf{r}'(\mathrm{t}) = (0)\mathbf{i} + \left(-4\sin t\right)\mathbf{j} + \left(3\cos t\right)\mathbf{k}$$

$$\mathbf{r}(\mathrm{t}) = \int \mathbf{r}'(\mathrm{t})dt = \left[\int \left(-4\sin t\right)\mathbf{j} + \left(3\cos t\right)\mathbf{k}\right]dt = \left[\int 0\mathbf{i} + \left(-4\sin t\right)\mathbf{j} + \left(3\cos t\right)\mathbf{k}\right]dt$$

$$= \left[0 + C_4\right]\mathbf{i} + \left[4\cos t + C_5\right]\mathbf{j} + \left[3\sin t + C_6\right]\mathbf{k}$$

$$\mathbf{r}(\mathrm{t}) = \left[0 + C_4\right]\mathbf{i} + \left[4\cos t + C_5\right]\mathbf{j} + \left[3\sin t + C_6\right]\mathbf{k}$$

$$\mathbf{r}(0) = \left[C_4\right]\mathbf{i} + \left[4 + C_5\right]\mathbf{j} + \left[C_6\right]\mathbf{k}$$

$$4\mathbf{j} = \left[C_4\right]\mathbf{i} + \left[4 + C_5\right]\mathbf{j} + \left[C_6\right]\mathbf{k}$$

$$C_4 = 0; \qquad C_5 = 0; \qquad C_6 = 0$$

$$\mathbf{r}(\mathrm{t}) = \left[0 + C_4\right]\mathbf{i} + \left[4\cos t + C_5\right]\mathbf{j} + \left[3\sin t + C_6\right]\mathbf{k}$$

$$\mathbf{r}(\mathrm{t}) = [0]\mathbf{i} + [4\cos t]\mathbf{j} + [3\sin t]\mathbf{k} = \langle 0, 4\cos t, 3\sin t\rangle$$