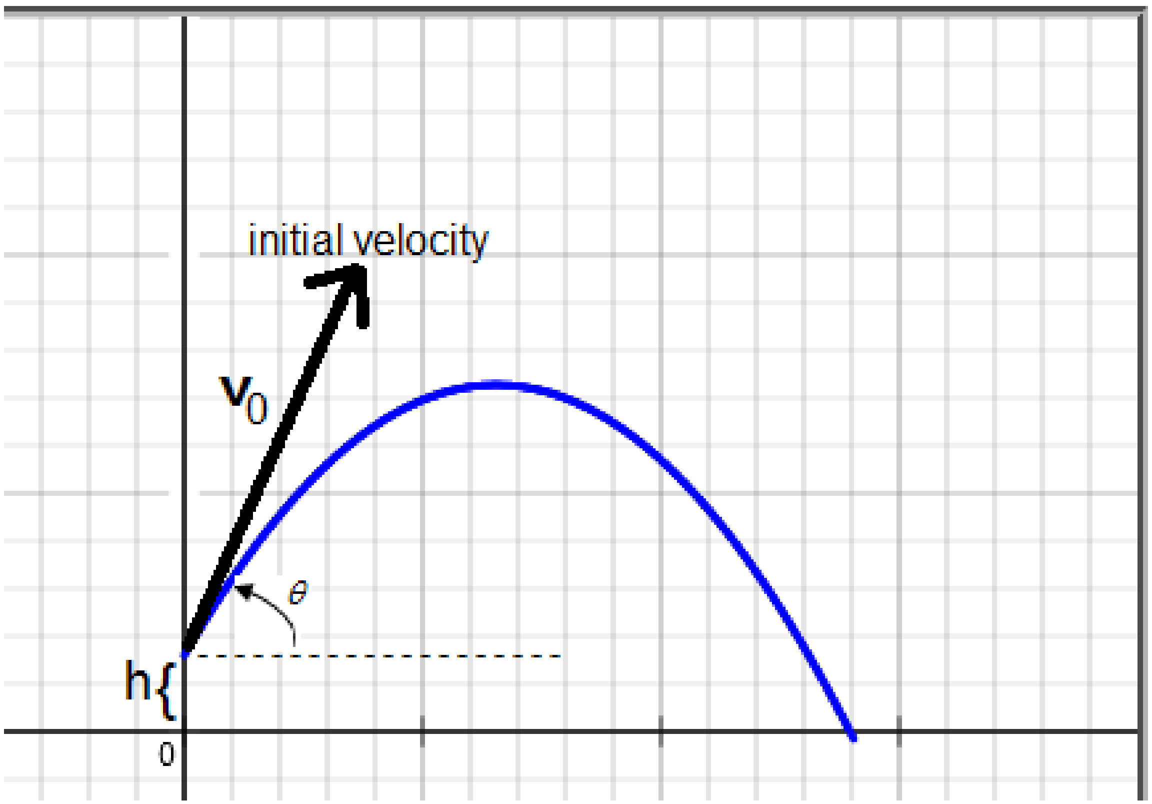
Section 12.3 Velocity and Acceleration

Position Vector for a Projectile: $\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left(h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right)\mathbf{j}.$

where: θ = angle of elevation, v_0 = initial speed, h = initial height, g = acceleration due to gravity (32ft/sec^2)



Example 1: A cannon ball is fired 10 feet from the ground level with speed of 100 feet/sec and angle of elevation is $\theta = 45^\circ = \pi/4$ rad.

a) Find parametric equations for the cannon ball's path:

$$\mathbf{r}(\mathrm{t}) = (v_0 \cos\theta)t\mathbf{i} + \left(h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

$$\mathbf{r}(\mathrm{t}) = (100\cos(\pi/4))t\mathbf{i} + \left(10 + (100\sin(\pi/4))t - \frac{1}{2}\cdot 32t^2\right)\mathbf{j} = 50\sqrt{2}t\mathbf{i} + \left(10 + 50\sqrt{2}t - 16t^2\right)\mathbf{j}$$

b) Find the maximum height reached by the cannon ball.

vertical displacement of cannon ball is $y = 10 + (100\sin(\pi/4))t - \frac{1}{2}\cdot 32t^2$

$$y' = 100\cdot\frac{\sqrt{2}}{2} - 16t = 50\sqrt{2}\text{-}16t$$

Set $y' = 50\sqrt{2}\text{-}16t = 0$

$$t = \frac{50\sqrt{2}}{16} = 2.21 \text{ sec.}$$

Hence, cannon ball reached maximum height at $t = 2.21$ sec.

To find maximum height we use $y = 10 + (100\sin(\pi/4))t - \frac{1}{2}\cdot 32t^2$

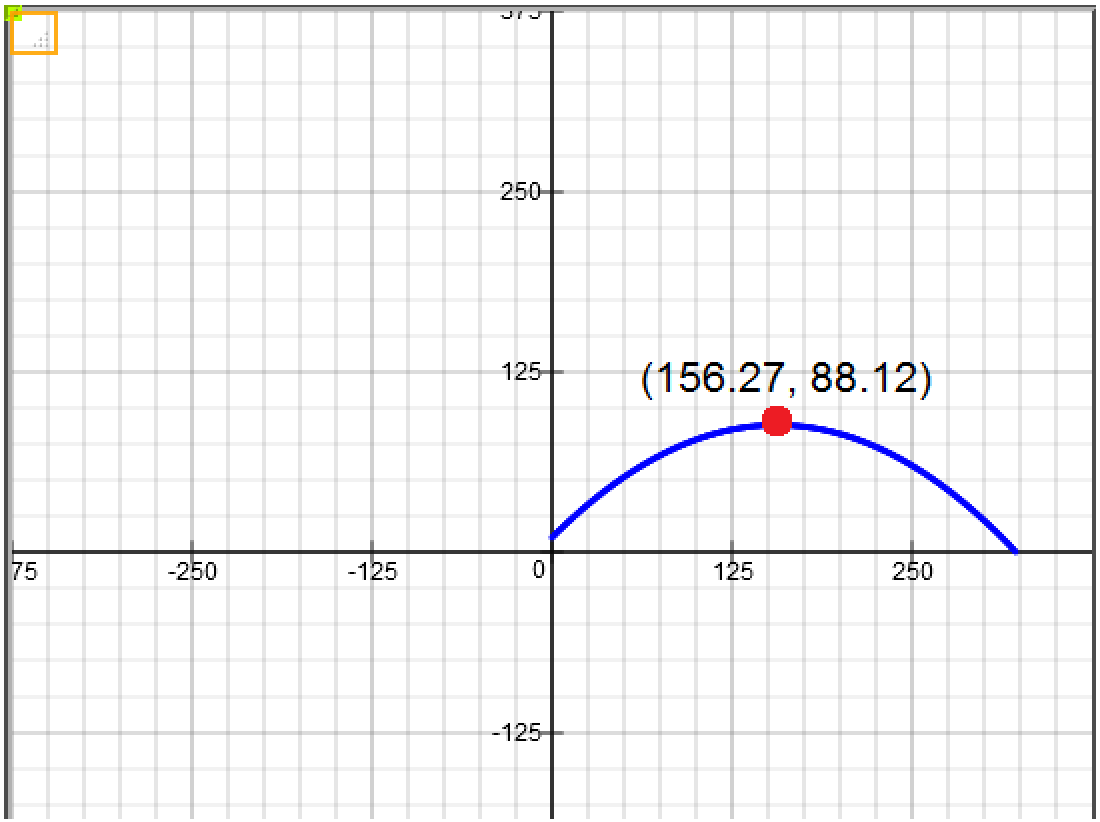
$$y = 10 + (100\sin(\pi/4))t - \frac{1}{2}\cdot 32t^2 = 10 + (100\sin(\pi/4))(2.21) - \frac{1}{2}\cdot 32(2.21)^2 = 88.1249 \text{ feet}$$

At $t = 2.21,$ cannon ball's horizontal displacement is $x = (100\cos(\pi/4))t = 50\sqrt{2}t = 50\sqrt{2}(2.21) = 156.2705986$

c) Graphing cannon ball's path:

$x = 50\sqrt{2}t$

$y = 10 + 50\sqrt{2}t - 16t^2$



d) Determine when the cannon ball reached a horizontal distance of 250 feet.

cannon ball's horizontal displacement is $x = \left(100\cos\left(\pi/4\right)\right)t = 50\sqrt{2}t$

When $x = 250$ feet, we have $250 = 50\sqrt{2}t \quad \Rightarrow \quad t = 3.5355$ sec

At $t = 3.5355$, corresponding height is $y = 10 + 50\sqrt{2}t - 16t^2 = 10 + 50\sqrt{2}\left(3.5355\right) - 16\left(3.5355\right)^2 = 60$ feet

e) Will the cannon ball clear a 50-foot-high wall located 250 feet in front of the cannon?

Yes, because when the cannon is 250 feet from the cannon the corresponding height is 60 feet.

f) What was the speed of the cannon ball when it hit the ground?

When the cannon ball hit the ground, its vertical displacement y is 0.

Hence, $y = 10 + 50\sqrt{2}t - 16t^2 = 10 + 70.71067811865476t - 16t^2 = 0$

Solving for t by using Quadratic Formula: $t = 4.556581610648$ sec

$\mathbf{r}(\mathrm{t}) = 50\sqrt{2}t\mathbf{i} + \left(10 + 50\sqrt{2}t - 16t^2\right)\mathbf{j}$

$\mathbf{r}'(\mathrm{t}) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}$

$\mathbf{r}'(4.556581610648) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32\left(4.556581610648\right)\right)\mathbf{j} = \left\langle 50\sqrt{2},\ \text{-}75.09993342208125 \right\rangle$

Speed when cannon ball hit the ground = $\left\|\mathbf{r}'(4.556581610648)\right\| = \sqrt{\left(50\sqrt{2}\right)^2 + \left(\text{-}75.09993342208125\right)^2} = 103.15$ ft/sec

Example 2: A baseball is hit 3 feet from the ground level with speed of 120 feet/sec and angle of elevation is $\theta = 30^\circ = \pi/6$ rad.

Parametric equations for the cannon ball's path:

$$\mathbf{r}(t) = (v_0 \cos\theta)t\mathbf{i} + \left(h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

$$\mathbf{r}(t) = (120\cos(\pi/6))t\mathbf{i} + \left(3 + (120\sin(\pi/6))t - \frac{1}{2}\cdot 32t^2\right)\mathbf{j} = 120\cdot\frac{\sqrt{3}}{2}t\mathbf{i} + \left(3 + 120\cdot\frac{1}{2}t - 16t^2\right)\mathbf{j} = 60\sqrt{3}t\mathbf{i} + \left(3 + 60t - 16t^2\right)\mathbf{j}$$

a) Graph $\mathbf{r}(t) = 60\sqrt{3}t\mathbf{i} + \left(3 + 60t - 16t^2\right)\mathbf{j}$

b) Find the maximum height reached by the ball.

$y = 3 + 60t - 16t^2$

$y' = 60 - 32t$

Set $y' = 0$; $\quad 60 - 32t = 0$; $\quad t = 60/32 = 1.875$ sec.

Hence, ball reached maximum height when $t = 1.875$ sec.

At $t = 1.875$, corresponding height is $y = 3 + 60t - 16t^2 = 3 + 60(1.875) - 16(1.875)^2 = 59.25$ feet

c) Find the horizontal distance traveled by the ball.

When the ball hit the ground, $y = 0$.

$y = 3 + 60t - 16t^2 = 0$

Using quadratic formula to solve this equation: $t = 3.799$ sec

Hence, total time traveled by the ball is 3.799 sec.

Total horizontal distance is $x = 60\sqrt{3}t = 60\sqrt{3}(3.799) = 394.8036$ feet.

d) Determine when the ball reached a horizontal distance of 350 feet.

Horizontal distance: $x = 60\sqrt{3}t$

When ball reached a horizontal distance of 350 feet we have:

$$350 = 60\sqrt{3}t \quad \text{and} \quad t = \frac{350}{60\sqrt{3}} = 3.367876 \text{ sec.}$$

Corresponding height is $y = 3 + 60t - 16t^2 = 3 + 60(3.367876) - 16(3.367876)^2 =$ 23.59 feet.

Hence, when the ball was at a horizontal distance of 350 feet, it was 23.58 feet high in the air.

e) Will the ball clear a 20-foot-high wall located 350 feet in front of first base? Yes.

Example 3: Let $\mathbf{r}(\mathrm{t}) = t\mathbf{i} + \left(4 - t^2\right)\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = 1\mathbf{i} + \left(-2t\right)\mathbf{j} = \langle 1, -2t \rangle$$

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{(1)^2 + \left(-2t\right)^2} = \sqrt{1 + 4t^2}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(\mathrm{t}) = 0\mathbf{i} + \left(-2\right)\mathbf{j} = \langle 0, -2 \rangle$$

Graphing $\mathbf{r}(\mathrm{t}) = t\mathbf{i} + \left(4 - t^2\right)\mathbf{j}$:

Use parametric equations: $x(t) = t; \quad y(t) = 4 - t^2$

Example 4: Let $\mathbf{r}(\mathrm{t}) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \left(-3\sin t\right)\mathbf{i} + \left(2\cos t\right)\mathbf{j} = \langle -3\sin t, 2\cos t \rangle$$

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{\left(-3\sin t\right)^2 + \left(2\cos t\right)^2} = \sqrt{9\sin^2 t + 4\cos^2 t}$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(\mathrm{t}) = \left(-3\cos t\right)\mathbf{i} + \left(-2\sin t\right)\mathbf{j} = \langle -3\cos t, -2\sin t \rangle$$

Graphing $\mathbf{r}(\mathrm{t}) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$:

Use parametric equations: $x(t) = 3\cos t$; $y(t) = 2\sin t$

Example 5: Let $\mathbf{r}(\mathrm{t}) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

Note: Graph of $\mathbf{r}(\mathrm{t}) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$ is a circle.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} = \langle -4\sin t, 4\cos t \rangle$$

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = \sqrt{16\sin^2 t + 16\cos^2 t} == \sqrt{16(\sin^2 t + \cos^2 t)} = \sqrt{4(1)} = 2$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(\mathrm{t}) = (-4\cos t)\mathbf{i} + (-4\sin t)\mathbf{j} = \langle -4\cos t, -4\sin t \rangle$$

$$\mathbf{v}(t) \cdot \mathbf{a}(t) = \langle -4\sin t, 4\cos t \rangle \cdot \langle -4\cos t, -4\sin t \rangle = 16\sin t\cos t - 16\sin t\cos t = 0$$

For circular motion: $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal.

Example 6: Let $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k} = \langle 2,0,3\rangle$; $\mathbf{v}(0) = 4\mathbf{j} = \langle 0,4,0\rangle$; $\mathbf{r}(0) = \mathbf{0} = \langle 0,0,0\rangle$.

Find **r**(t).

$$\mathbf{r}'(\mathrm{t}) = \int \mathbf{a}(t)dt = \int \mathbf{r}''(t)dt = \int \langle 2,0,3\rangle dt = \langle 2t + C_1, 0 + C_2, 3t + C_3\rangle$$

$$\mathbf{r}'(0) = \mathbf{v}(0) = \langle 2t + C_1, 0 + C_2, 3t + C_3\rangle = \langle C_1, C_2, C_3\rangle$$

$$\langle 0,4,0\rangle = \langle C_1, C_2, C_3\rangle$$

$$C_1 = 0; \quad C_2 = 4; \quad C_3 = 0$$

So $\mathbf{r}'(\mathrm{t}) = \int \mathbf{a}(t)dt = \int \mathbf{r}''(t)dt = \int \langle 2,0,3\rangle dt = \langle 2t + 0, 0 + 4, 3t + 0\rangle = \langle 2t, 4, 3t\rangle$

$$\mathbf{r}(\mathrm{t}) = \int \mathbf{v}(t)dt = \int \mathbf{r}'(t)dt = \int \langle 2t, 4, 3t\rangle dt = \left\langle t^2 + C_4, 4t + C_5, \frac{3t^2}{2} + C_6\right\rangle$$

$$\mathbf{r}(0) = \left\langle t^2 + C_4, 4t + C_5, \frac{3t^2}{2} + C_6\right\rangle = \langle 0 + C_4, 0 + C_5, 0 + C_6\rangle = \langle C_4, C_5, C_6\rangle$$

$$\langle 0,0,0\rangle = \langle C_4, C_5, C_6\rangle$$

$$\mathbf{r}(\mathrm{t}) = \int \mathbf{v}(t)dt = \int \mathbf{r}'(t)dt = \int \langle 2t, 4, 3t\rangle dt = \left\langle t^2 + 0, 4t + 0, \frac{3t^2}{2} + 0\right\rangle = \left\langle t^2, 4t, \frac{3t^2}{2}\right\rangle$$