Velocity and Acceleration Vectors

Example 1: Let $\mathbf{r}(\mathrm{t}) = t\mathbf{i} + \left(4 - t^2\right)\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \dfrac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}; \quad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$

$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = 1\mathbf{i} + \left(-2t\right)\mathbf{j} = \langle 1, -2t \rangle$

speed = $\|\mathbf{v}(t)\| = \sqrt{(1)^2 + \left(-2t\right)^2} = \sqrt{1 + 4t^2}$

$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(\mathrm{t}) = 0\mathbf{i} + \left(-2\right)\mathbf{j} = \langle 0, -2 \rangle$

Graphing $\mathbf{r}(\mathrm{t}) = t\mathbf{i} + \left(4 - t^2\right)\mathbf{j}$:

Use parametric equations: $x(t) = t; \quad y(t) = 4 - t^2$

Example 2: Let $\mathbf{r}(\mathrm{t}) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$

$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \dfrac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}; \quad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$

$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \left(-3\sin t\right)\mathbf{i} + \left(2\cos t\right)\mathbf{j} = \langle -3\sin t, 2\cos t \rangle$

speed = $\|\mathbf{v}(t)\| = \sqrt{\left(-3\sin t\right)^2 + \left(2\cos t\right)^2} = \sqrt{9\sin^2 t + 4\cos^2 t}$

$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(\mathrm{t}) = \left(-3\cos t\right)\mathbf{i} + \left(-2\sin t\right)\mathbf{j} = \langle -3\cos t, -2\sin t \rangle$

Graphing $\mathbf{r}(\mathrm{t}) = 3\cos t\mathbf{i} + 2\sin t\mathbf{j}$:

Use parametric equations: $x(t) = 3\cos t; \quad y(t) = 2\sin t$

Example 3: Let $\mathbf{r}(\mathrm{t}) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$

Note: Graph of $\mathbf{r}(\mathrm{t}) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$ is a circle.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} = \langle -4\sin t, 4\cos t \rangle$$

$$\text{speed} = \|\mathbf{v}(t)\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2} = \sqrt{16\sin^2 t + 16\cos^2 t} == \sqrt{16(\sin^2 t + \cos^2 t)} = \sqrt{4(1)} = 2$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(\mathrm{t}) = (-4\cos t)\mathbf{i} + (-4\sin t)\mathbf{j} = \langle -4\cos t, -4\sin t \rangle$$

$$\mathbf{v}(t)\cdot\mathbf{a}(t) = \langle -4\sin t, 4\cos t \rangle \cdot \langle -4\cos t, -4\sin t \rangle = 16\sin t\cos t - 16\sin t\cos t = 0$$

For circular motion: $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are orthogonal.

Example 4: Let $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k} = \langle 2,0,3\rangle$; $\mathbf{v}(0) = 4\mathbf{j} = \langle 0,4,0\rangle$; $\mathbf{r}(0) = \mathbf{0} = \langle 0,0,0\rangle$.

Find $\mathbf{r}$(t).

$$\mathbf{r}'(\mathrm{t}) = \int \mathbf{a}(t)dt = \int \mathbf{r}''(t)dt = \int \langle 2,0,3\rangle dt = \langle 2t + C_1, 0 + C_2, 3t + C_3\rangle$$

$$\mathbf{r}'(0)=\mathbf{v}(0) = \langle 2t + C_1, 0 + C_2, 3t + C_3\rangle = \langle C_1, C_2, C_3\rangle$$

$$\langle 0,4,0\rangle = \langle C_1, C_2, C_3\rangle$$

$$C_1 = 0; \quad C_2 = 4; \quad C_3 = 0$$

So $\mathbf{r}'(\mathrm{t}) = \int \mathbf{a}(t)dt = \int \mathbf{r}''(t)dt = \int \langle 2,0,3\rangle dt = \langle 2t + 0, 0 + 4, 3t + 0\rangle = \langle 2t, 4, 3t\rangle$

$$\mathbf{r}(\mathrm{t}) = \int \mathbf{v}(t)dt = \int \mathbf{r}'(t)dt = \int \langle 2t, 4, 3t\rangle dt = \left\langle t^2 + C_4, 4t + C_5, \frac{3t^2}{2} + C_6\right\rangle$$

$$\mathbf{r}(0) = \left\langle t^2 + C_4, 4t + C_5, \frac{3t^2}{2} + C_6\right\rangle = \langle 0 + C_4, 0 + C_5, 0 + C_6\rangle = \langle C_4, C_5, C_6\rangle$$

$$\langle 0,0,0\rangle = \langle C_4, C_5, C_6\rangle$$

$$\mathbf{r}(\mathrm{t}) = \int \mathbf{v}(t)dt = \int \mathbf{r}'(t)dt = \int \langle 2t, 4, 3t\rangle dt = \left\langle t^2 + 0, 4t + 0, \frac{3t^2}{2} + 0\right\rangle = \left\langle t^2, 4t, \frac{3t^2}{2}\right\rangle$$