# Section 12.4 Notes

Tangential and Normal Components of Acceleration

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} \qquad \Rightarrow \qquad \mathbf{v}(t) = \mathbf{T}(t)\cdot\|\mathbf{v}(t)\|$$

$$\mathbf{N}(t) = \frac{\mathbf{T}(t)}{\|\mathbf{T}(t)\|}$$

Note: $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthogonal; hence, $\mathbf{T}(t)\cdot\mathbf{N}(t) = 0$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \frac{d}{dt}\Big[\|\mathbf{v}(t)\|\Big]\cdot\mathbf{T}(t)\cdot+\|\mathbf{v}(t)\|\cdot\frac{d}{dt}\big[\mathbf{T}(t)\big]$$

Using Product Rule for Derivative

$$\mathbf{a}(t) = \frac{d}{dt}\Big[\|\mathbf{v}(t)\|\Big]\cdot\mathbf{T}(t) \;+\; \|\mathbf{v}(t)\|\cdot\mathbf{T}'(t)$$

$$\mathbf{a}(t) = \frac{d}{dt}\Big[\|\mathbf{v}(t)\|\Big]\cdot\mathbf{T}(t) \;+\; \|\mathbf{v}(t)\|\cdot\mathbf{T}'(t)\cdot\frac{\|\mathbf{T}(t)\|}{\|\mathbf{T}(t)\|}$$

$$\mathbf{a}(t) = \frac{d}{dt}\Big[\|\mathbf{v}(t)\|\Big]\cdot\mathbf{T}(t) \;+\; \|\mathbf{v}(t)\|\cdot\|\mathbf{T}'(t)\|\cdot\frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{d}{dt}\Big[\|\mathbf{v}(t)\|\Big]\cdot\mathbf{T}(t) \;+\; \|\mathbf{v}(t)\|\cdot\|\mathbf{T}'(t)\|\cdot\mathbf{N}(t)$$

Note: $\mathbf{a}(t)$ is a linear combination of $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

$\mathrm{a}_{\mathrm{T}} = \frac{d}{dt}\Big[\;\|\mathbf{v}(t)\|\;\Big] =$ Tangential Component of Acceleration

$\mathrm{a}_{\mathrm{N}} = \|\mathbf{v}(t)\|\cdot\|\mathbf{T}'(t)\| =$ Normal Component of Acceleration

Tangential Component of Acceleration

Claim: $a_T = \frac{d}{dt}\left[\|\mathbf{v}(t)\|\right] = \mathbf{a}(t)\cdot\mathbf{T}(t) = = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{\|\mathbf{v}(t)\|}$

Proof:

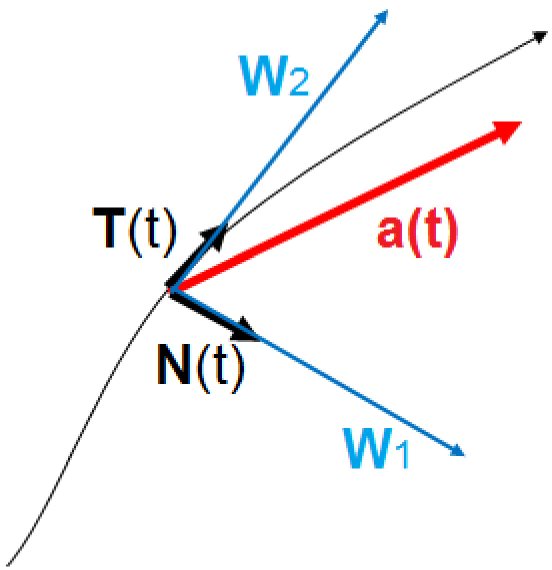
$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} \quad \text{and} \quad \mathbf{N}(t) = \frac{\mathbf{T}(t)}{\|\mathbf{T}(t)\|}$$

Note the following:

1) $\mathbf{T}(t)$ is a vector tangent to the curve at time $t$.

2) $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are orthogonal; hence, $\mathbf{T}(t)\cdot\mathbf{N}(t) = 0$

3) $\mathbf{T}(t)$ and $\mathbf{N}(t)$ are unit vectors; hence, $\|\mathbf{T}(t)\| = 1$ and $\|\mathbf{N}(t)\| = 1$



$\mathbf{W}_1$ = projection of $\mathbf{a}(t)$ onto $\mathbf{N}(t) = \dfrac{(\mathbf{a}\cdot\mathbf{N})}{\|\mathbf{N}\|^2}\cdot\mathbf{N} = \dfrac{(\mathbf{a}\cdot\mathbf{N})}{1}\cdot\mathbf{N} = (\mathbf{a}\cdot\mathbf{N})\mathbf{N}$

$\mathbf{W}_2$ = projection of $\mathbf{a}(t)$ onto $\mathbf{T}(t) = \dfrac{(\mathbf{a}\cdot\mathbf{T})}{\|\mathbf{T}\|^2}\cdot\mathbf{T} = (\mathbf{a}\cdot\mathbf{T})\mathbf{T}$

Hence, $\mathbf{a} = \mathbf{W}_2 + \mathbf{W}_1 = (\mathbf{a}\cdot\mathbf{T})\mathbf{T} + (\mathbf{a}\cdot\mathbf{N})\mathbf{N}$

We shown that: $\mathbf{a}(t) = \dfrac{d}{dt}\left[\|\mathbf{v}(t)\|\right]\cdot\mathbf{T}(t) \;+\; \|\mathbf{v}(t)\|\cdot\|\mathbf{T}'(t)\|\cdot\mathbf{N}(t)$

Therefore, $\mathrm{a}_{\mathrm{T}} = \dfrac{d}{dt}\left[\|\mathbf{v}(t)\|\right] = (\mathbf{a}\cdot\mathbf{T})$ and $\mathrm{a}_{\mathrm{N}} = \|\mathbf{v}(t)\|\cdot\|\mathbf{T}'(t)\| = (\mathbf{a}\cdot\mathbf{N})$

$$\mathrm{a}_{\mathrm{T}} = \frac{d}{dt}\left[\|\mathbf{v}(t)\|\right] = (\mathbf{a}\cdot\mathbf{T}) = \mathbf{T}\cdot\mathbf{a} = \frac{\mathbf{v}}{\|\mathbf{v}\|}\cdot\mathbf{a} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|}$$

Claim: $a_N = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}$

Proof;

Note the following:

1) **T** and **N** are orthogonal so $\mathbf{T}\cdot\mathbf{N} = 0$.

2) $\|\mathbf{T}\| = 1, \quad \|\mathbf{N}\| = 1,$ Parallelogram formed by **T** and **N** will be a square and has area of 1.

Hence, $\|\mathbf{T}\times\mathbf{N}\|$ = Area of parallelogram formed by **T** and $\mathbf{N} = \mathbf{1}$

3) **T** and **T** are parallel and so $\mathbf{T}\times\mathbf{T} = \mathbf{0}$

4) Recall: $\mathbf{T} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} \quad \Rightarrow \quad \mathbf{v} = \mathbf{T}\cdot\|\mathbf{v}\|$

$$\mathbf{a} = (\mathbf{a}\cdot\mathbf{T})\mathbf{T} + (\mathbf{a}\cdot\mathbf{N})\mathbf{N}$$

$$\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$$

$$\mathbf{v}\times\mathbf{a} = \mathbf{T}\cdot\|\mathbf{v}\|\times(a_T\mathbf{T} + a_N\mathbf{N}) = \mathbf{T}\cdot\|\mathbf{v}\|\times a_T\mathbf{T} \quad + \quad \mathbf{T}\cdot\|\mathbf{v}\|\times a_N\mathbf{N} \qquad \text{by property of Cross Product}$$

$$\mathbf{v}\times\mathbf{a} = \|\mathbf{v}\|a_T(\mathbf{T}\times\mathbf{T}) \quad + \quad \|\mathbf{v}\|a_N(\mathbf{T}\times\mathbf{N}) \quad = \quad \mathbf{0} + \|\mathbf{v}\|a_N(\mathbf{T}\times\mathbf{N}) \quad = \quad \|\mathbf{v}\|a_N(\mathbf{T}\times\mathbf{N}) \qquad \text{Note: } \|\mathbf{v}\|,\ a_T,\ a_N \text{ are scalars.}$$

$$\|\mathbf{v}\times\mathbf{a}\| \quad = \quad \big\|\|\mathbf{v}\|a_N(\mathbf{T}\times\mathbf{N})\big\| \quad = \quad \|\mathbf{v}\|a_N\cdot\|(\mathbf{T}\times\mathbf{N})\| \quad = \quad \|\mathbf{v}\|a_N\cdot 1 \quad = \quad \|\mathbf{v}\|a_N$$

So, $a_N = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|}$

Claim: $a_N = \sqrt{\|\mathbf{a}\|^2 - (a_T)^2}$

Proof:

$$\mathbf{a}\cdot\mathbf{a} = (a_T\mathbf{T} + a_N\mathbf{N})\cdot(a_T\mathbf{T} + a_N\mathbf{N})$$

$$\mathbf{a}\cdot\mathbf{a} = a_T\mathbf{T}\cdot a_T\mathbf{T} \;+\; a_T\mathbf{T}\cdot a_N\mathbf{N} \;+\; a_N\mathbf{N}\cdot a_T\mathbf{T} \;+\; a_N\mathbf{N}\cdot a_N\mathbf{N}$$

$$\mathbf{a}\cdot\mathbf{a} = (a_T)^2\,\mathbf{T}\cdot\mathbf{T} \;+\; a_T a_N\mathbf{T}\cdot\mathbf{N} \;+\; a_N a_T\mathbf{N}\cdot\mathbf{T} \;+\; a_N a_N\mathbf{N}\cdot\mathbf{N}$$

$$\mathbf{a}\cdot\mathbf{a} = (a_T)^2\,\mathbf{T}\cdot\mathbf{T} \;+\; a_T a_N\mathbf{T}\cdot\mathbf{N} \;+\; a_N a_T\mathbf{N}\cdot\mathbf{T} \;+\; a_N a_N\mathbf{N}\cdot\mathbf{N}$$

$$\|\mathbf{a}\|^2 = (a_T)^2\|\mathbf{T}\|^2 \;+\; a_T a_N\cdot 0 \;+\; a_N a_T\cdot 0 \;+\; (a_N)^2\|\mathbf{N}\|^2$$

$$\|\mathbf{a}\|^2 = (a_T)^2\|\mathbf{T}\|^2 \;+\; (a_N)^2\|\mathbf{N}\|^2$$

$$\|\mathbf{a}\|^2 = (a_T)^2\cdot 1 \;+\; (a_N)^2\cdot 1$$

$$(a_N)^2 = \|\mathbf{a}\|^2 - (a_T)^2$$

Hence, $a_N = \sqrt{\|\mathbf{a}\|^2 - (a_T)^2}$

Therefore, $a_N = \dfrac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - (a_T)^2}$

Example 1: Circular Motion

Let $\mathbf{r}(\mathrm{t}) = \langle 3\cos t, 3\sin t \rangle$ represent path of particle with mass $m$ traveling in circular motion

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle -3\sin t, 3\cos t \rangle \quad \text{and} \quad \|\mathbf{v}(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2} = 3$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = \langle -3\cos t, -3\sin t \rangle \quad \text{and} \quad \|\mathbf{a}(t)\| = \sqrt{(-3\cos t)^2 + (-3\sin t)^2} = 3$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle -3\sin t, 3\cos t \rangle}{3} = \langle -\sin t, \cos t \rangle$$

$$\mathbf{T}'(t) = \langle -\cos t, -\sin t \rangle \quad \text{and} \quad \|\mathbf{T}'(t)\| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\langle -\cos t, -\sin t \rangle}{1} = \langle -\cos t, -\sin t \rangle$$

$$\mathbf{T}(t) \cdot \mathbf{N}(t) = \langle -\sin t, \cos t \rangle \langle -\cos t, -\sin t \rangle = \sin t \cdot \cos t - \sin t \cdot \cos t = 0$$

Circular Motion Example

$\mathrm{a}_{\mathrm{T}} =$ tangential component of acceleration = $\mathbf{a}(t)\cdot\mathbf{T}(t)$

$$=\langle -3\cos t, -3\sin t\rangle\langle -\sin t, \cos t\rangle = 3\sin t\cdot\cos t - 3\sin t\cdot\cos t = 0$$

$\mathrm{a}_{\mathrm{N}} =$ normal (or centripetal) component of acceleration = $\mathbf{a}(t)\cdot\mathbf{N}(t)$

$$=\langle -3\cos t, -3\sin t\rangle\langle -\cos t, -\sin t\rangle = 3\cos^2 t + 3\sin^2 t = 3$$

Centripetal Force needed to keep particle in circular motion = mass times normal acceleration

$$F = ma = (mass)(\mathrm{a}_{\mathrm{N}}) = (mass)(3)$$

$$K = Curvature = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{1}{3}$$ for circular motion with uniform speed, $K = \dfrac{1}{\text{radius}}$

Exampe 2: A cannon ball is fired 10 feet from the ground level with speed of 100 feet/sec and angle of elevation is $\theta = 45^\circ = \pi/4$ rad.

Parametric equations for the cannon ball's path:

$$\mathbf{r}(\mathrm{t}) = (v_0 \cos\theta)t\mathbf{i} + \left(h + (v_0 \sin\theta)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

$$\mathbf{r}(\mathrm{t}) = (100\cos(\pi/4))t\mathbf{i} + \left(10 + (100\sin(\pi/4))t - \frac{1}{2}\cdot 32t^2\right)\mathbf{j} = 50\sqrt{2}t\mathbf{i} + \left(10 + 50\sqrt{2}t - 16t^2\right)\mathbf{j}$$

Find tangential and normal components of acceleration at time $t = 1$.

$$\mathbf{r}(1) = 50\sqrt{2}(1)\mathbf{i} + \left(10 + 50\sqrt{2}(1) - 16(1)^2\right)\mathbf{j} = \left\langle 50\sqrt{2},\ -6+50\sqrt{2}\right\rangle = \langle 70.71067811865476,\ \ 64.71067811865476\rangle$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j} = \left\langle 50\sqrt{2},\ 50\sqrt{2} - 32t\right\rangle$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = (-32)\,\mathbf{j} = \langle 0, -32\rangle$$

$$\|\mathbf{v}(t)\| = \sqrt{\left(50\sqrt{2}\right)^2 + \left(50\sqrt{2} - 32t\right)^2} = \sqrt{5000 + \left(50\sqrt{2} - 32t\right)^2}$$

$$\mathrm{a}_{\mathrm{T}} = \text{ tangential component of acceleration } = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{\|\mathbf{v}(t)\|} = \frac{0\cdot 50\sqrt{2} + -32\left(50\sqrt{2} - 32t\right)}{\sqrt{5000 + (50 - 32t)^2}} = \frac{-1600\sqrt{2} + 1024t}{\sqrt{5000 + (50 - 32t)^2}}$$

$$\mathrm{a}_{\mathrm{N}} = \text{normal component of acceleration } = \frac{\|\mathbf{v}(t)\cdot\mathbf{a}(t)\|}{\|\mathbf{v}(t)\|} = \sqrt{\|\mathbf{a}(t)\|^2 - (\mathrm{a}_{\mathrm{T}})^2} = \sqrt{(1024)^2 - \left(\frac{-1600\sqrt{2} + 1024t}{\sqrt{5000 + (50 - 32t)^2}}\right)^2}$$

At $t = 1$:

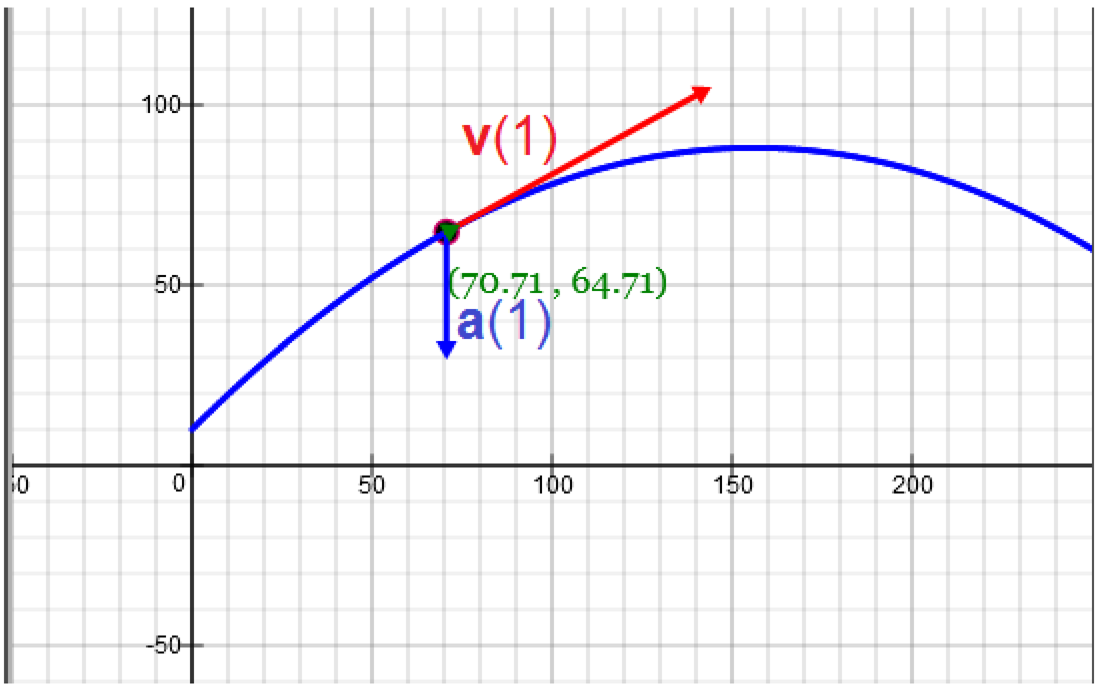
$$\mathbf{v}(1) = \mathbf{r}'(1) = \left\langle 50\sqrt{2},\ 50\sqrt{2} - 32t \right\rangle = \left\langle 50\sqrt{2},\ 50\sqrt{2} - 32 \right\rangle = = \langle 70.71,\ 38.71 \rangle \quad \text{and} \quad \|\mathbf{v}(1)\| = \sqrt{(70.71)^2 + (38.71)^2} = 80.61$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\langle 70.71,\ 38.71 \rangle}{80.61} = \langle 0.87718,\ 0.48021 \rangle$$

$$\mathbf{a}(1) = \ \langle 0, -32 \rangle \quad \text{and} \quad \|\mathbf{a}(t)\| = \sqrt{0^2 + (-32)^2} = 32$$

$$\mathrm{a}_{\mathrm{T}}(1) = \frac{-1600 + 1024t}{\sqrt{5000 + (50 - 32t)^2}} = \frac{-1600\sqrt{2} + 1024(1)}{\sqrt{5000 + (50 - 32(1))^2}} = \frac{-1238.7416997969522}{72.9657453878188} = \ \text{-16.977030704105612}$$

$$\mathrm{a}_{\mathrm{N}}(1) = \sqrt{\|\mathbf{a}(1)\|^2 - (\mathrm{a}_{\mathrm{T}})^2} = \sqrt{(32)^2 - (\ \text{-16.977030704105612})^2} = 27.12527287371568$$



# Tangent and Normal Vectors

Example 3: Let $\mathbf{r}(\mathrm{t}) = t\mathbf{i} + t^2\mathbf{j}$. Find $\mathbf{T}(1)$ and $\mathbf{N}(1)$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle 1, 2t \rangle \quad \text{and} \quad \|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle 1, 2t \rangle\| = \sqrt{1+4t^2}$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}} = \frac{\langle 1, 2 \rangle}{\sqrt{1+4}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$\mathbf{T}(t) = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}} = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle = \left\langle \left(1+4t^2\right)^{-1/2}, 2t \cdot \left(1+4t^2\right)^{-1/2} \right\rangle$$

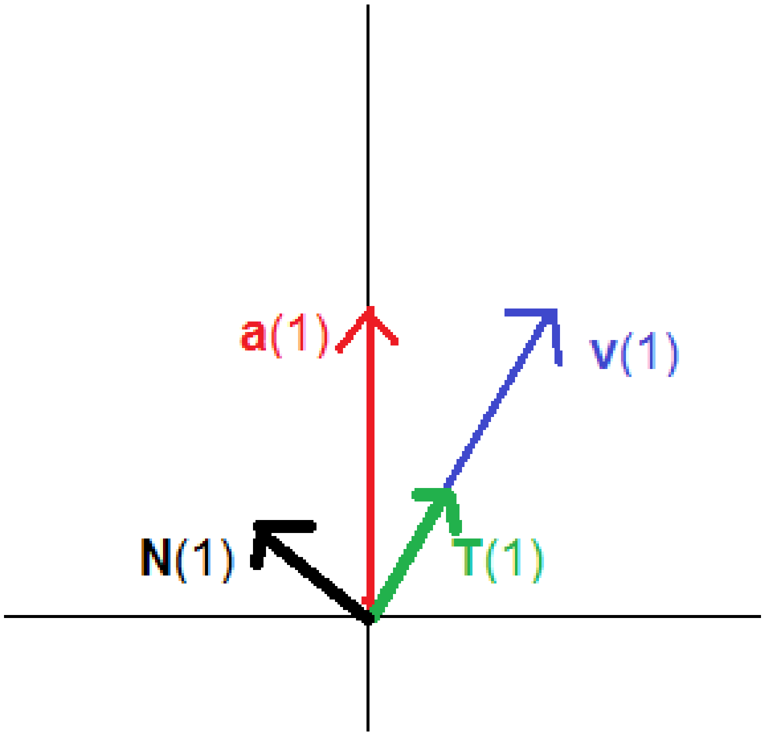
$$\mathbf{T}'(t) = \left\langle \frac{-1}{2}\left(1+4t^2\right)^{-3/2}(8t), 2t \cdot \left[\frac{-1}{2}\left(1+4t^2\right)^{-3/2}(8t)\right] + \left(1+4t^2\right)^{-1/2} \cdot [2] \right\rangle$$

$$\mathbf{T}'(1) = \left\langle \frac{-1}{2}(5)^{-3/2}(8), 2 \cdot \left[\frac{-1}{2}(5)^{-3/2}(8)\right] + (5)^{-1/2} \cdot [2] \right\rangle = \left\langle -4(5)^{-3/2}, -8(5)^{-3/2} + 2(5)^{-1/2} \right\rangle$$

$$\|\mathbf{T}'(1)\| = \sqrt{\left[-4(5)^{-3/2}(8)\right]^2 + \left[-8(5)^{-3/2} + 2(5)^{-1/2}\right]^2}$$

$$\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|} = \frac{\left\langle -4(5)^{-3/2}, -8(5)^{-3/2} + 2(5)^{-1/2} \right\rangle}{\sqrt{\left[-4(5)^{-3/2}\right]^2 + \left[-8(5)^{-3/2} + 2(5)^{-1/2}\right]^2}} = \frac{\langle \text{-0.35777087639996635, 0.17888543819998315} \rangle}{\sqrt{0.16}}$$

$$\mathbf{N}(1) = \langle \text{-0.8944, 0.4472135} \rangle$$




Note: $\mathbf{a}(1) = a_T(1) \cdot \mathbf{T}(1) + a_N(1) \cdot \mathbf{N}(1)$

where $a_T(1) =$ Tangential component of acceleration at $t = 1$

where $a_N(1) =$ Normal (or centripetal) component of acceleration at $t = 1$

Also, $\mathbf{T}(1)$ and $\mathbf{N}(1)$ are orthogonal.

Example 4: Let $\mathbf{r}(\mathrm{t}) = t^3\mathbf{i} + 2t^2\mathbf{j}$. Find $\mathbf{T}(1)$ and $\mathbf{N}(1)$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

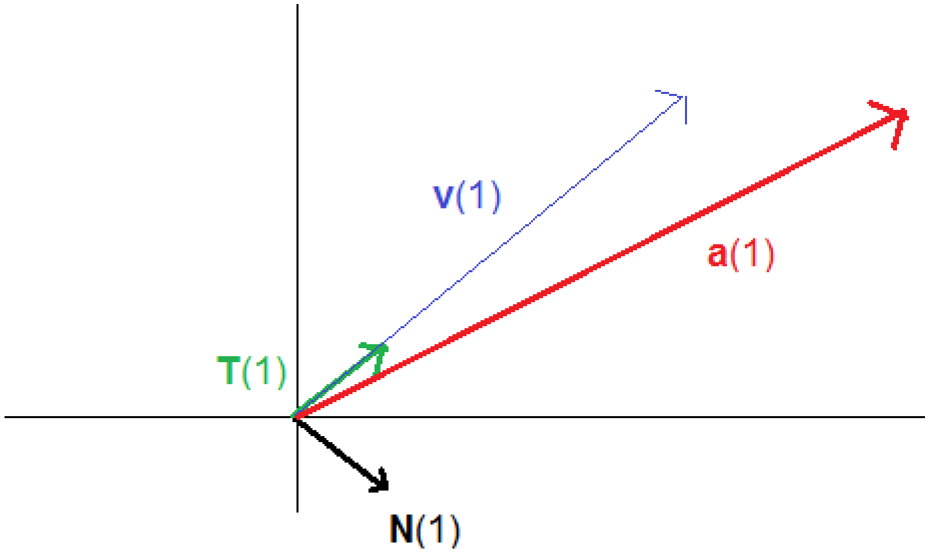
$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \left\langle 3t^2, 4t \right\rangle \quad \text{and} \quad \|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \left\| \left\langle 3t^2, 4t \right\rangle \right\| = \sqrt{9t^4 + 16t^2} \quad \text{and} \quad \mathbf{v}(1) = \left\langle 3, 4 \right\rangle$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\left\langle 3t^2, 4t \right\rangle}{\sqrt{9t^4 + 16t^2}} = \left\langle \frac{3t^2}{\sqrt{9t^4 + 16t^2}}, \frac{4t}{\sqrt{9t^4 + 16t^2}} \right\rangle$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\left\langle 3t^2, 4t \right\rangle}{\sqrt{9t^4 + 16t^2}} = \frac{\left\langle 3, 4 \right\rangle}{5} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{48t^3}{\left(9t^4 + 16t^2\right)^{3/2}}, -\frac{36t^2}{\left(9t^4 + 16t^2\right)^{3/2}} \right\rangle$$

$$\mathbf{N}(t) = \frac{\left\langle \frac{48t^3}{\left(9t^4 + 16t^2\right)^{3/2}}, -\frac{36t^2}{\left(9t^4 + 16t^2\right)^{3/2}} \right\rangle}{\sqrt{\left(\frac{48t^3}{\left(9t^4 + 16t^2\right)^{3/2}}\right)^2 + \left(-\frac{36t^2}{\left(9t^4 + 16t^2\right)^{3/2}}\right)^2}} \quad \text{and} \quad \mathbf{N}(1) = \frac{\left\langle \frac{48}{(25)^{3/2}}, -\frac{36}{(25)^{3/2}} \right\rangle}{\sqrt{\left(\frac{48}{(25)^{3/2}}\right)^2 + \left(-\frac{36}{(25)^{3/2}}\right)^2}} = \left\langle 0.8, -0.6 \right\rangle$$




Note: $\mathbf{a}(1) = a_T(1) \cdot \mathbf{T}(1) + a_N(1) \cdot \mathbf{N}(1)$

where $a_T(1) =$ Tangential component of acceleration at $t = 1$

where $a_N(1) =$ Normal (or centripetal) component of acceleration at $t = 1$

Also, $\mathbf{T}(1)$ and $\mathbf{N}(1)$ are orthogonal.

Example 5: Let $\mathbf{r}(\mathrm{t}) = \left\langle t^2, t, \frac{4}{3} \right\rangle$.

Find the tangent line passing through the point P(1,1,4/3). Note: At the point P(1,1,4/3), $t^2 = 1$ and $t = 1$.

The vectors $\mathbf{r}'(1)$ and $\mathbf{T}(1)$ are tangent vectors at the point P(1,1,4/3);

and $\mathbf{r}'(1)$ and $\mathbf{T}(1)$ are parallel to the tangent line passing through P(1,1,4/3).

Hence, the direction vector $\langle a, b, c \rangle$ for the tangent line passing through

the point P(1,1,4/3) can be the vector $\mathbf{r}'(1)$ or $\mathbf{T}(1)$.

$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle 2t, t, 0 \rangle$ and $\|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle 2t, t, 0 \rangle\| = \sqrt{4t^2 + t^2}$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 2t, t, 0 \rangle}{\sqrt{4t^2 + t^2}} = \frac{\langle 2,1,0 \rangle}{5} = \left\langle \frac{2}{5}, \frac{1}{5}, 0 \right\rangle$$

So direction vector $\langle a, b, c \rangle$ can be $\langle 2,1,0 \rangle$ or $\left\langle \frac{2}{5}, \frac{1}{5}, 0 \right\rangle$.

Parametric equations for the tangent line at P(1,1,4/3):

$x = x_1 + at$; $y = y_1 + bt$; $z = z_1 + ct$;

$x = 1 + 2t$; $y = 1 + t$; $z = 4/3$

Example 6: Let $\mathbf{r}(\mathrm{t}) = \langle \cos t, \sin t, 2t \rangle$.

Find $\mathrm{a}_{\mathrm{T}}$ = tangential component of acceleration and $\mathrm{a}_{\mathrm{N}}$ = normal component of acceleration.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle -\sin t, \cos t, 2 \rangle$$

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle -\sin t, \cos t, 2 \rangle\| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = \langle -\cos t, -\sin t, 0 \rangle$$

$$\|\mathbf{a}(t)\| = \|\mathbf{r}''(\mathrm{t})\| = \|\langle -\cos t, -\sin t, 0 \rangle\| = \sqrt{\sin^2 t + \cos^2 t + 0} = 1$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\mathbf{r}'(\mathrm{t})}{\|\mathbf{r}'(\mathrm{t})\|} = \frac{\langle -\sin t, \cos t, 2 \rangle}{\sqrt{5}} = \left\langle \frac{-\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$a_{\mathrm{T}} = \mathbf{a}(t) \cdot \mathbf{T}(t) = \langle -\cos t, -\sin t, 0 \rangle \left\langle \frac{-\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle =$$

$$= (-\cos t)\left(\frac{-\sin t}{\sqrt{5}}\right) + (-\sin t)\left(\frac{\cos t}{\sqrt{5}}\right) + (0)\left(\frac{2}{\sqrt{5}}\right) = 0$$

$$a_N = \sqrt{\|\mathbf{a}(t)\|^2 - (a_{\mathrm{T}})^2} = \sqrt{1-0} = 1$$

Example 7: Let $\mathbf{r}(\mathrm{t}) = \left\langle t, t^2, t^2/2 \right\rangle$. Find $\mathrm{a}_{\mathrm{T}}$ and $\mathrm{a}_{\mathrm{N}}$ at $t = 1$.

$\mathrm{a}_{\mathrm{T}}$ = tangential component of acceleration and $\mathrm{a}_{\mathrm{N}}$ = normal component of acceleration.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle 1, 2t, t \rangle \quad \text{and} \quad \|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle 1, 2t, t \rangle\| = \sqrt{1 + 4t^2 + t^2} = \sqrt{1 + 5t^2}$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = \langle 0, 2, 1 \rangle$$

$$\|\mathbf{a}(t)\| = \|\mathbf{r}''(\mathrm{t})\| = \|\langle 0, 2, 1 \rangle\| = \sqrt{0 + 4 + 1} = \sqrt{5}$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 1, 2, 1 \rangle}{\sqrt{6}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$a_{\mathrm{T}} = \mathbf{a}(1) \cdot \mathbf{T}(1) = \langle 0, 2, 1 \rangle \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$= (0)\left(\frac{1}{\sqrt{6}}\right) + (2)\left(\frac{2}{\sqrt{6}}\right) + (1)\left(\frac{1}{\sqrt{6}}\right) = \frac{5}{\sqrt{6}}$$

$$a_N = \sqrt{\left(\|\mathbf{a}(1)\|\right)^2 - \left(a_{\mathrm{T}}\right)^2} = \sqrt{\left(\sqrt{5}\right)^2 - \left(\frac{5}{\sqrt{6}}\right)^2} = \sqrt{5 - \frac{25}{6}}$$