Tangent and Normal Vectors

Example 1: Let $\mathbf{r}(\mathrm{t}) = t\mathbf{i} + t^2\mathbf{j}$. Find $\mathbf{T}(1)$ and $\mathbf{N}(1)$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle 1, 2t \rangle$$

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle 1, 2t \rangle\| = \sqrt{1+4t^2}$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}} = \frac{\langle 1, 2 \rangle}{\sqrt{1+4}} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$\mathbf{T}(t) = \frac{\langle 1, 2t \rangle}{\sqrt{1+4t^2}} = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle = \left\langle \left(1+4t^2\right)^{-1/2}, 2t \cdot \left(1+4t^2\right)^{-1/2} \right\rangle$$

$$\mathbf{T}'(t) = \left\langle \frac{-1}{2}\left(1+4t^2\right)^{-3/2}(8t), 2t \cdot \left[\frac{-1}{2}\left(1+4t^2\right)^{-3/2}(8t)\right] + \left(1+4t^2\right)^{-1/2} \cdot [2] \right\rangle$$

$$\mathbf{T}'(1) = \left\langle \frac{-1}{2}(5)^{-3/2}(8), 2 \cdot \left[\frac{-1}{2}(5)^{-3/2}(8)\right] + (5)^{-1/2} \cdot [2] \right\rangle = \left\langle -4(5)^{-3/2}(8), -8(5)^{-3/2} + 2(5)^{-1/2} \right\rangle$$

$$\|\mathbf{T}'(1)\| = \sqrt{\left[-4(5)^{-3/2}(8)\right]^2 + \left[-8(5)^{-3/2} + 2(5)^{-1/2}\right]^2}$$

$$\mathbf{N}(1) = \frac{\mathbf{T}'(1)}{\|\mathbf{T}'(1)\|} = \frac{\left\langle -4(5)^{-3/2}(8), -8(5)^{-3/2} + 2(5)^{-1/2} \right\rangle}{\sqrt{\left[-4(5)^{-3/2}(8)\right]^2 + \left[-8(5)^{-3/2} + 2(5)^{-1/2}\right]^2}}$$

Example 1: Let $\mathbf{r}(\mathrm{t}) = t^3\mathbf{i} + 2t^2\mathbf{j}$. Find $\mathbf{T}(1)$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \left\langle 3t^2, 4t \right\rangle$$

$$\left\| \mathbf{v}(t) \right\| = \left\| \mathbf{r}'(\mathrm{t}) \right\| = \left\| \left\langle 3t^2, 4t \right\rangle \right\| = \sqrt{9t^4 + 16t^2}$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\left\| \mathbf{v}(1) \right\|} = \frac{\mathbf{r}'(1)}{\left\| \mathbf{r}'(1) \right\|} = \frac{\left\langle 3t^2, 4t \right\rangle}{\sqrt{9t^4 + 16t^2}} = \frac{\left\langle 3, 4 \right\rangle}{5} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

Example 1: Let $\mathbf{r}(\mathrm{t}) = e^t \cos t\,\mathbf{i} + e^t\,\mathbf{j}$. Find $\mathbf{T}(0)$.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \left\langle e^t\left(-\sin t\right) + e^t \cos t, e^t \right\rangle$$

$$\mathbf{v}(0) = \mathbf{r}'(0) = \left\langle e^t\left(-\sin t\right) + e^t \cos t, e^t \right\rangle = \left\langle e^0\left(-\sin 0\right) + e^0 \cos 0, e^0 \right\rangle = \left\langle 1,1 \right\rangle$$

$$\left\| \mathbf{v}(0) \right\| = \left\| \mathbf{r}'(0) \right\| = \left\| \left\langle 1,1 \right\rangle \right\| = \sqrt{1+1} = \sqrt{2}$$

$$\mathbf{T}(0) = \frac{\mathbf{v}(0)}{\left\| \mathbf{v}(0) \right\|} = \frac{\mathbf{r}'(0)}{\left\| \mathbf{r}'(0) \right\|} = \frac{\left\langle 1,1 \right\rangle}{\sqrt{2}} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

Example 1: Let $\mathbf{r}(\mathrm{t}) = \left\langle t^2, t, \frac{4}{3} \right\rangle$.

Find the tangent line passing through the point P(1,1,4/3).

Note: At the point P(1,1,4/3), $t^2 = 1$ and $t = 1$.

The vectors $\mathbf{r}'(1)$ and $\mathbf{T}(1)$ are tangent vectors at the point P(1,1,4/3);

and $\mathbf{r}'(1)$ and $\mathbf{T}(1)$ are parallel to the tangent line passing through P(1,1,4/3).

Hence, the direction vector $\langle a,b,c \rangle$ for the tangent line passing through

the point P(1,1,4/3) can be the vector $\mathbf{r}'(1)$ or $\mathbf{T}(1)$.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle 2t, t, 0 \rangle$$

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle 2t, t, 0 \rangle\| = \sqrt{4t^2 + t^2}$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 2t, t, 0 \rangle}{\sqrt{4t^2 + t^2}} = \frac{\langle 2,1,0 \rangle}{5} = \left\langle \frac{2}{5}, \frac{1}{5}, 0 \right\rangle$$

So direction vector $\langle a,b,c \rangle$ can be $\langle 2,1,0 \rangle$ or $\left\langle \frac{2}{5}, \frac{1}{5}, 0 \right\rangle$.

Parametric equations for the tangent line at P(1,1,4/3):

$x = x_1 + at$; $\quad y = y_1 + bt$; $\quad z = z_1 + ct$;

$x = 1 + 2t$; $\quad y = 1 + t$; $\quad z = 4/3$

Example 1: Let $\mathbf{r}(\mathrm{t}) = \langle \cos t, \sin t, 2t \rangle$.

Find a_T = tangential component of acceleration.

Find a_N = normal component of acceleration.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle -\sin t, \cos t, 2 \rangle$$

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle -\sin t, \cos t, 2 \rangle\| = \sqrt{\sin^2 t + \cos^2 t + 4} = \sqrt{5}$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = \langle -\cos t, -\sin t, 0 \rangle$$

$$\|\mathbf{a}(t)\| = \|\mathbf{r}''(\mathrm{t})\| = \|\langle -\cos t, -\sin t, 0 \rangle\| = \sqrt{\sin^2 t + \cos^2 t + 0} = 1$$

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\mathbf{r}'(\mathrm{t})}{\|\mathbf{r}'(\mathrm{t})\|} = \frac{\langle -\sin t, \cos t, 2 \rangle}{\sqrt{5}} = \left\langle \frac{-\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$a_{\mathrm{T}} = \mathbf{a}(t) \cdot \mathbf{T}(t) = \langle -\cos t, -\sin t, 0 \rangle \left\langle \frac{-\sin t}{\sqrt{5}}, \frac{\cos t}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle =$$

$$= (-\cos t)\left(\frac{-\sin t}{\sqrt{5}}\right) + (-\sin t)\left(\frac{\cos t}{\sqrt{5}}\right) + (0)\left(\frac{2}{\sqrt{5}}\right) = 0$$

$$a_N = \sqrt{\|\mathbf{a}(t)\|^2 - \left(a_{\mathrm{T}}\right)^2} = \sqrt{1-0} = 1$$

Example 1: Let $\mathbf{r}(\mathrm{t}) = \left\langle t, t^2, t^2/2 \right\rangle$. Find $\mathrm{a_T}$ and $\mathrm{a_N}$ at $t = 1$.

$\mathrm{a_T}$ = tangential component of acceleration.

$\mathrm{a_N}$ = normal component of acceleration.

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}; \qquad \mathbf{a}(t) = \mathbf{r}''(\mathrm{t})$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle 1, 2t, t \rangle$$

$$\|\mathbf{v}(t)\| = \|\mathbf{r}'(\mathrm{t})\| = \|\langle 1, 2t, t \rangle\| = \sqrt{1+4t^2+t^2} = \sqrt{1+5t^2}$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = \langle 0, 2, 1 \rangle$$

$$\|\mathbf{a}(t)\| = \|\mathbf{r}''(\mathrm{t})\| = \|\langle 0, 2, 1 \rangle\| = \sqrt{0+4+1} = \sqrt{5}$$

$$\mathbf{T}(1) = \frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|} = \frac{\mathbf{r}'(1)}{\|\mathbf{r}'(1)\|} = \frac{\langle 1, 2, 1 \rangle}{\sqrt{6}} = \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$a_{\mathrm{T}} = \mathbf{a}(1) \cdot \mathbf{T}(1) = \langle 0, 2, 1 \rangle \left\langle \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right\rangle$$

$$= (0)\left(\frac{1}{\sqrt{6}}\right) + (2)\left(\frac{2}{\sqrt{6}}\right) + (1)\left(\frac{1}{\sqrt{6}}\right) = \frac{5}{\sqrt{6}}$$

$$a_N = \sqrt{\left(\|\mathbf{a}(1)\|\right)^2 - \left(a_{\mathrm{T}}\right)^2} = \sqrt{\left(\sqrt{5}\right)^2 - \left(\frac{5}{\sqrt{6}}\right)^2} = \sqrt{5 - \frac{25}{6}}$$