Section 12.5 Notes

# Arc Legth

In 2D space:

Let $\mathbf{r}(t) = \langle x(t), y(t) \rangle$;  hence, $\mathbf{r}'(t) = \langle x'(t), y'(t) \rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2}$

$$s = \text{arc length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|dt$$

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2} = \|\mathbf{r}'(t)\| = \|d\mathbf{r}/dt\|$$

$$ds = \|\mathbf{r}'(t)\|\,dt$$

In 3D space:

Let $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$;  hence, $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|dt$$

$$\frac{ds}{dt} = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} = \|\mathbf{r}'(t)\|$$

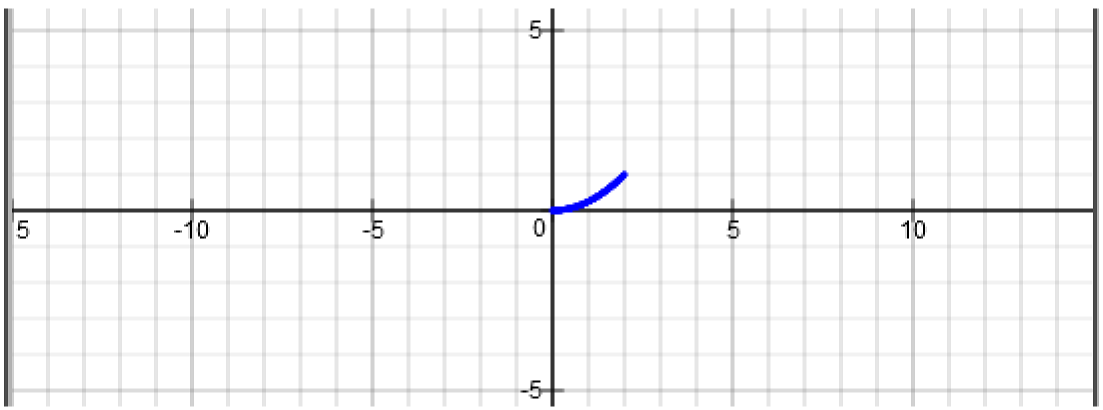
$$ds = \|\mathbf{r}'(t)\|\,dt = \|d\mathbf{r}/dt\|$$

Example 1: Let $\mathbf{r}(t) = \langle 2t, t^2 \rangle \qquad 0 \le t \le 1.$ Find the arc length.

$$s = \text{arc length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|dt$$

$$\|\mathbf{r}'(t)\| = \|\langle 2, 2t\rangle\| = \sqrt{(2)^2 + (2t)^2} = \sqrt{4+4t^2}$$

$$s = \int_a^b \|\mathbf{r}'(t)\|dt = \int_0^1 \sqrt{4+4t^2}\,dt = 2.29$$



Example 2: Let $\mathbf{r}(t) = \left\langle t, \frac{4}{3}t^{3/2}, \frac{1}{2}t^2 \right\rangle$ $\quad 0 \le t \le 2.$ Find the arc length.

$$s = \text{arc length} = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2 + \left(z'(t)\right)^2}\,dt = \int_a^b \|\mathbf{r}'(t)\|dt$$

$$\|\mathbf{r}'(t)\| = \left\|\left\langle 1, 2t^{1/2}, t \right\rangle\right\| = \sqrt{(1)^2 + \left(2t^{1/2}\right)^2 + (t)^2} = \sqrt{1+4t+t^2}$$

$$s = \int_a^b \|\mathbf{r}'(t)\|dt = \int_0^2 \sqrt{1+4t+t^2}\,\sqrt{4+4t^2}\,dt = 4.815$$

# Curvature

$$\frac{ds}{dt} = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} = \|\mathbf{r}'(t)\| = \|d\mathbf{r}/dt\|$$

$$ds = \|\mathbf{r}'(t)\| dt$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\|\mathbf{r}'(t)\|} \cdot \mathbf{r}'(t) = \frac{1}{\|\mathbf{r}'(t)\|} \cdot \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{\|\mathbf{r}'(t)\| dt} = \frac{d\mathbf{r}}{ds}$$

$$\mathbf{T}'(s) = \frac{d}{ds}\left[\mathbf{T}(t)\right] = \frac{d}{ds}\left[\frac{d\mathbf{r}}{ds}\right]$$

$$\mathbf{T}'(s) = \frac{d\mathbf{T}(t)}{ds}$$

Curvature $(K)$ is defined as follows: $K = \left\|\frac{d\mathbf{T}}{ds}\right\| = \|\mathbf{T}'(s)\|$

$K =$ Curvature = measure of how sharply a curve bends at time $t$

Claim: $K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

Proof:

$$K = \left\|\frac{d\mathbf{T}}{ds}\right\| = \left\|\frac{d\mathbf{T}/dt}{ds/dt}\right\| = \left\|\frac{d\mathbf{T}/dt}{\|d\mathbf{r}/dt\|}\right\| = \frac{\|d\mathbf{T}/dt\|}{\|d\mathbf{r}/dt\|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$$

Claim: $K = \dfrac{\left\|\mathbf{r}'(t) \times \mathbf{r}''(t)\right\|}{\left\|\mathbf{r}'(t)\right\|^3}$

Proof:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\left\|\mathbf{r}'(t)\right\|} \quad \Rightarrow \quad \mathbf{r}'(t) = \left\|\mathbf{r}'(t)\right\| \cdot \mathbf{T}(t)$$

$$\mathbf{r}''(t) = \left\|\mathbf{r}'(t)\right\| \cdot \mathbf{T}'(t) + \mathbf{T}(t) \cdot \left\|\mathbf{r}'(t)\right\|'$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\left\|\mathbf{T}'(t)\right\|} \quad \Rightarrow \quad \mathbf{T}'(t) = \left\|\mathbf{T}'(t)\right\| \cdot \mathbf{N}(t)$$

$$K = \frac{\left\|\mathbf{T}'(t)\right\|}{\left\|\mathbf{r}'(t)\right\|} \quad \Rightarrow \quad \left\|\mathbf{T}'(t)\right\| = K\left\|\mathbf{r}'(t)\right\|$$

Hence, $\mathbf{T}'(t) = \left\|\mathbf{T}'(t)\right\| \cdot \mathbf{N}(t) = K\left\|\mathbf{r}'(t)\right\| \cdot \mathbf{N}(t)$

$$\mathbf{r}''(t) = \left\|\mathbf{r}'(t)\right\| \cdot \mathbf{T}'(t) + \mathbf{T}(t) \cdot \left\|\mathbf{r}'(t)\right\|'$$

$$\mathbf{r}''(t) = \left\|\mathbf{r}'(t)\right\| \cdot K\left\|\mathbf{r}'(t)\right\| \cdot \mathbf{N}(t) + \mathbf{T}(t) \cdot \left\|\mathbf{r}'(t)\right\|' = \left\|\mathbf{r}'(t)\right\|^2 \cdot K \cdot \mathbf{N}(t) + \mathbf{T}(t) \cdot \left\|\mathbf{r}'(t)\right\|'$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\left\|\mathbf{r}'(t)\right\| \cdot \mathbf{T}(t)\right) \times \left(\left\|\mathbf{r}'(t)\right\|^2 \cdot K \cdot \mathbf{N}(t) + \mathbf{T}(t) \cdot \left\|\mathbf{r}'(t)\right\|'\right)$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left(\left\|\mathbf{r}'(t)\right\| \cdot \mathbf{T}(t)\right) \times \left(\left\|\mathbf{r}'(t)\right\|^2 \cdot K \cdot \mathbf{N}(t)\right) + \left(\left\|\mathbf{r}'(t)\right\| \cdot \mathbf{T}(t)\right) \times \left(\mathbf{T}(t) \cdot \left\|\mathbf{r}'(t)\right\|'\right)$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|^2 K\left(\mathbf{T}(t)\times\mathbf{N}(t)\right) \quad + \quad \left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|' \quad \left(\mathbf{T}(t)\times\mathbf{T}(t)\right)$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|^2 K\left(\mathbf{T}(t)\times\mathbf{N}(t)\right) \quad + \quad \left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|' \quad \left(\mathbf{0}\right) \qquad \text{Note: } \mathbf{T}(t)\times\mathbf{T}(t) = \mathbf{0}$$

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|^2 K\left(\mathbf{T}(t)\times\mathbf{N}(t)\right)$$

$$\left\|\mathbf{r}'(t)\times\mathbf{r}''(t)\right\| = \left\|\left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|^2 K\left(\mathbf{T}(t)\times\mathbf{N}(t)\right)\right\| = \left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|^2 K\left\|\left(\mathbf{T}(t)\times\mathbf{N}(t)\right)\right\|$$

$$\left\|\mathbf{r}'(t)\times\mathbf{r}''(t)\right\| = \left\|\mathbf{r}'(t)\right\|\cdot\left\|\mathbf{r}'(t)\right\|^2 K\cdot 1 \qquad \text{Note: } \left\|\left(\mathbf{T}(t)\times\mathbf{N}(t)\right)\right\| = 1$$

$$K = \frac{\left\|\mathbf{r}'(t)\times\mathbf{r}''(t)\right\|}{\left\|\mathbf{r}'(t)\right\|^3}$$

Example 3: Let $\mathbf{r}(t) = \left\langle -t, 4t^2, 3t^2 \right\rangle$. Find the curvature at $t = 2$.

Solution:

At $t = 2$, $\mathbf{r}(2) = \left\langle -t, 4t^2, 3t^2 \right\rangle = \left\langle -2, 16, 12 \right\rangle$

$\mathbf{r}'(t) = \left\langle -1, 8t, 6t \right\rangle$ and $\mathbf{r}'(2) = \left\langle -1, 16, 12 \right\rangle$

$\mathbf{r}''(t) = \left\langle 0, 8, 6 \right\rangle$ and $\mathbf{r}''(2) = \left\langle 0, 8, 6 \right\rangle$

$$\left\|\mathbf{r}'(2)\right\| = \left\|\left\langle -2, 16, 12 \right\rangle\right\| = \sqrt{(-2)^2 + (16)^2 + (12)^2} = \sqrt{401}$$

$$\mathbf{r}'(2) \times \mathbf{r}''(2) = \left\langle -1, 16, 12 \right\rangle \times \left\langle 0, 8, 6 \right\rangle = \left\langle 0, 6, -8 \right\rangle$$

$$\left\|\mathbf{r}'(2) \times \mathbf{r}''(2)\right\| = \left\|\left\langle 0, 6, -8 \right\rangle\right\| = \sqrt{(0)^2 + (6)^2 + (-8)^2} = 10$$

$$K = \frac{\left\|\mathbf{r}'(t) \times \mathbf{r}''(t)\right\|}{\left\|\mathbf{r}'(t)\right\|^3} = \frac{\left\|\mathbf{r}'(2) \times \mathbf{r}''(2)\right\|}{\left\|\mathbf{r}'(2)\right\|^3} = \frac{10}{\left(\sqrt{401}\right)^3} = 0.00124$$

# Curvature and Acceleration

$$\frac{ds}{dt} = \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} = \|\mathbf{r}'(t)\| = \|d\mathbf{r}/dt\| = \|\mathbf{v}(t)\|$$

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \quad \Rightarrow \quad \|\mathbf{T}'(t)\| = K\|\mathbf{r}'(t)\| = K\|\mathbf{v}(t)\| = K \cdot \frac{ds}{dt}$$

$$\mathrm{a}_{\mathrm{T}} = \frac{d}{dt}\left[\|\mathbf{v}(t)\|\right] \quad \text{and} \quad \mathrm{a}_{\mathrm{N}} = \|\mathbf{v}(t)\|\|\mathbf{T}'(t)\|$$

$$\mathbf{a}(t) = \frac{d}{dt}\left[\|\mathbf{v}(t)\|\right] \cdot \mathbf{T}(t) \;+\; \|\mathbf{v}(t)\| \cdot \|\mathbf{T}'(t)\| \cdot \mathbf{N}$$

$$\mathbf{a}(t) = \frac{d}{dt}\left[\frac{ds}{dt}\right] \cdot \mathbf{T}(t) \;+\; \frac{ds}{dt} \cdot K \cdot \frac{ds}{dt} \cdot \mathbf{N}$$

$$\mathbf{a}(t) = \frac{d^2 s}{dt^2} \cdot \mathbf{T}(t) \;+\; K \cdot \left(\frac{ds}{dt}\right)^2 \cdot \mathbf{N}\left(t\right)$$

Therefore:

$\mathrm{a}_{\mathrm{T}}$ = tangential component of acceleration = $\dfrac{d^2 s}{dt^2}$

$\mathrm{a}_{\mathrm{N}}$ = normal component of acceleration = $K \cdot \left(\dfrac{ds}{dt}\right)^2$

Example 4:

A cannon ball is fired 10 feet from the ground level with speed of 100 feet/sec and

angle of elevation is $\theta = 45^\circ = \pi/4$ rad.

Parametric equations for the cannon ball's path:

$$\mathbf{r}(\mathrm{t}) = \left(v_0 \cos\theta\right)t\mathbf{i} + \left(h + \left(v_0 \sin\theta\right)t - \frac{1}{2}gt^2\right)\mathbf{j}.$$

$$\mathbf{r}(\mathrm{t}) = \left(100\cos\left(\pi/4\right)\right)t\mathbf{i} + \left(10 + \left(100\sin\left(\pi/4\right)\right)t - \frac{1}{2}\cdot 32t^2\right)\mathbf{j} = 50\sqrt{2}t\mathbf{i} + \left(10 + 50\sqrt{2}t - 16t^2\right)\mathbf{j}$$

Find the curvature at $t = 1$.

$$\mathbf{r}(1) = 50\sqrt{2}\left(1\right)\mathbf{i} + \left(10 + 50\sqrt{2}\left(1\right) - 16\left(1\right)^2\right)\mathbf{j} = \left\langle 50\sqrt{2},\ -6+50\sqrt{2}\right\rangle = \left\langle 70.71067811865476,\ \ 64.71067811865476\right\rangle$$

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j} = \left\langle 50\sqrt{2},\ 50\sqrt{2} - 32t\right\rangle$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = \left(-32\right)\mathbf{j} = \left\langle 0, -32\right\rangle$$

$$\left\|\mathbf{v}(t)\right\| = \sqrt{\left(50\sqrt{2}\right)^2 + \left(50\sqrt{2} - 32t\right)^2} = \sqrt{5000 + \left(50\sqrt{2} - 32t\right)^2}$$

At $t=1$:

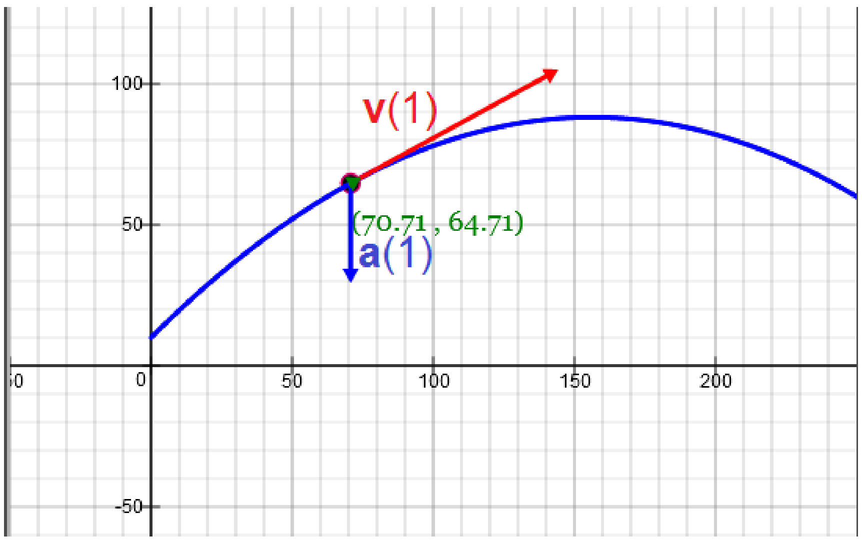
$$\mathbf{v}(1)=\mathbf{r}'(1)=\left\langle 50\sqrt{2},\ 50\sqrt{2}-32t\right\rangle=\left\langle 50\sqrt{2},\ 50\sqrt{2}-32\right\rangle==\langle 70.71,\ 38.71\rangle \quad \text{and} \quad \|\mathbf{v}(1)\|=\sqrt{(70.71)^2+(38.71)^2}=80.61$$

$$\mathbf{T}(1)=\frac{\mathbf{v}(1)}{\|\mathbf{v}(1)\|}=\frac{\langle 70.71,\ 38.71\rangle}{80.61}=\langle 0.87718,\ 0.48021\rangle$$

$$\mathbf{a}(1)=\ \langle 0,-32\rangle \quad \text{and} \quad \|\mathbf{a}(t)\|=\sqrt{0^2+(-32)^2}=32$$

$$\mathrm{a_T}(1)=\frac{-1600+1024t}{\sqrt{5000+(50-32t)^2}}=\frac{-1600\sqrt{2}+1024(1)}{\sqrt{5000+(50-32(1))^2}}=\frac{-1238.7416997969522}{72.9657453878188}=\ -16.977030704105612$$

$$\mathrm{a_N}(1)=\sqrt{\|\mathbf{a}(1)\|^2-(\mathrm{a_T})^2}=\sqrt{(32)^2-(\ -16.977030704105612)^2}=27.12527287371568$$



At $t = 1$:

$$\mathbf{v}(1) = \mathbf{r}'(1) = \left\langle 50\sqrt{2},\ 50\sqrt{2} - 32t \right\rangle = \left\langle 50\sqrt{2},\ 50\sqrt{2} - 32 \right\rangle = = \langle 70.71,\ 38.71 \rangle$$

$$\|\mathbf{v}(1)\| = \|\mathbf{r}'(1)\| = \sqrt{(70.71)^2 + (38.71)^2} = 80.61$$

$$\mathbf{r}''(1) = \mathbf{a}(1) = \ \langle 0, -32 \rangle \quad \text{and} \quad \|\mathbf{a}(1)\| = \sqrt{0^2 + (-32)^2} = 32$$

$$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 70.71,\ 38.71 \rangle \times \langle 0, -32 \rangle = \langle 70.71,\ 38.71,\ 0 \rangle \times \langle 0, -32, 0 \rangle = \langle 0\ ,\ 0\ ,\ \text{-}2262.72 \rangle$$

$$\|\mathbf{r}'(1) \times \mathbf{r}''(1)\| = \|\langle 0\ ,\ 0\ ,\ \text{-}2262.72 \rangle\| = \sqrt{0^2 + 0^2 + (\text{-}2262.72)^2} = 2262.72$$

At $t = 1$:

$$K = \ \frac{\|\mathbf{r}'(1) \times \mathbf{r}''(1)\|}{\|\mathbf{r}'(1)\|^3} = \frac{2262.72}{(80.61)^3} = 0.004319804097865602$$

Find when cannon ball reached maximum height.

$$\mathbf{r}(\mathrm{t}) = 50\sqrt{2}t\mathbf{i} + \left(10 + 50\sqrt{2}t - 16t^2\right)\mathbf{j}$$

Height of cannon ball at time $t$: $y = 10 + 50\sqrt{2}t - 16t^2$

$$y' = 50\sqrt{2} - 32t$$

Set $y' = 0 \quad \Rightarrow \quad 50\sqrt{2} - 32t = 0 \quad \Rightarrow \quad t = 50\sqrt{2}/32 = 2.20970869120796$ sec

Hence, cannon ball reached maximum height at $t = 2.20970869120796$.

Find curvature at $t = 2.20970869120796$.

$$\mathbf{v}(2.20970869120796) = \left\langle 50\sqrt{2},\ 50\sqrt{2} - 32\left(2.20970869120796\right)\right\rangle = \left\langle 70.71,\ 0\right\rangle$$

$$\left\|\mathbf{v}(2.20970869120796)\right\| = \sqrt{\left(70.71\right)^2 + \left(0\right)^2} = 70.71$$

$\mathbf{a}(2.20970869120796) = \left\langle 0, -32\right\rangle$ and $\left\|\mathbf{a}(2.20970869120796)\right\| = \sqrt{0^2 + \left(-32\right)^2} = 32$

$$\mathbf{v}(2.20970869120796) \times \mathbf{a}(2.20970869120796) = \left\langle 70.71,\ 0\right\rangle \times \left\langle 0, -32\right\rangle = \left\langle 70.71,\ 0,\ 0\right\rangle \times \left\langle 0, -32, 0\right\rangle = \left\langle 0\ ,\ 0\ ,\ \text{-}2262.72\right\rangle$$

$$\left\|\mathbf{v}(2.20970869120796) \times \mathbf{a}(2.20970869120796)\right\| = \left\|\left\langle 0\ ,\ 0\ ,\ \text{-}2262.72\right\rangle\right\| = \sqrt{0^2 + 0^2 + \left(\text{-}2262.72\right)^2} = 2262.72$$

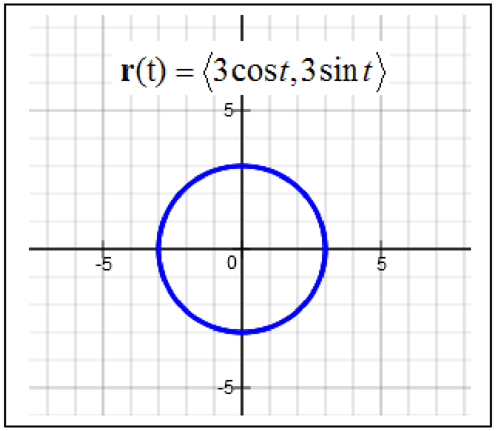
$$K = \frac{\left\|\mathbf{v}(2.20970869120796) \times \mathbf{a}(2.20970869120796)\right\|}{\left\|\mathbf{v}(2.20970869120796)\right\|^3} = \frac{2262.72}{\left(70.71\right)^3} = 0.00640012275435443$$

Example 5: Circular Motion and Curvature

Let $\mathbf{r}(\mathrm{t}) = \langle 3\cos t, 3\sin t \rangle$ represent path of particle with mass $m$ traveling in circular motion

$$\mathbf{v}(t) = \mathbf{r}'(\mathrm{t}) = \langle -3\sin t, 3\cos t \rangle \quad \text{and} \quad \|\mathbf{v}(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2} = 3$$

$$\mathbf{a}(t) = \mathbf{r}''(\mathrm{t}) = \langle -3\cos t, -3\sin t \rangle \quad \text{and} \quad \|\mathbf{a}(t)\| = \sqrt{(-3\cos t)^2 + (-3\sin t)^2} = 3$$




$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{\|\mathbf{v}(t)\|} = \frac{\langle -3\sin t, 3\cos t \rangle}{3} = \langle -\sin t, \cos t \rangle$$

$$\mathbf{T}'(t) = \langle -\cos t, -\sin t \rangle \quad \text{and} \quad \|\mathbf{T}'(t)\| = \sqrt{(-\cos t)^2 + (-\sin t)^2} = 1$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\langle -\cos t, -\sin t \rangle}{1} = \langle -\cos t, -\sin t \rangle$$

$$\mathbf{T}(t) \cdot \mathbf{N}(t) = \langle -\sin t, \cos t \rangle \langle -\cos t, -\sin t \rangle = \sin t \cdot \cos t - \sin t \cdot \cos t = 0$$

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2} = \frac{\langle -3\cos t, -3\sin t \rangle \cdot \langle -\cos t, -\sin t \rangle}{3^2} = \frac{3\cos^2 t + 3\sin^2 t}{9} = \frac{3}{9} = \frac{1}{3}$$

In general, $K = \dfrac{1}{radius}$ for circular motion with constant speed.