12.5 Arc Length and Curvature

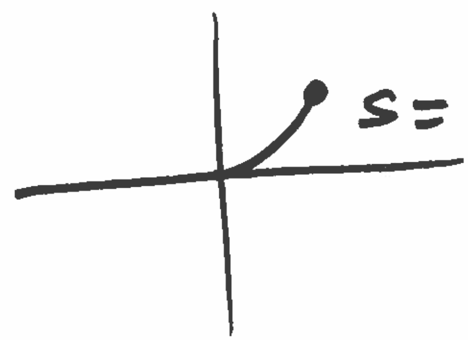
$\underset{\sim}{r}(t) = \langle 2t, t^2 \rangle \qquad 0 \le t \le 1$

$s = \text{arc length} = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$

$x(t) = 2t \qquad y(t) = t^2$

$x'(t) = 2 \qquad y'(t) = 2t$

$s = \int_0^1 \sqrt{(2)^2 + (2t)^2}\,dt = \int_0^1 \sqrt{4 + 4t^2}\,dt = 2.29$



$s = 2.29$

$$\underset{\sim}{r}(t) = \left\langle t,\ \frac{4}{3}t^{3/2},\ \frac{1}{2}t^2 \right\rangle \qquad 0 \le t \le 2$$

$$s = \text{arc length} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$x(t) = t$	$y(t) = \frac{4}{3}t^{3/2}$	$z(t) = \frac{1}{2}t^2$
$x'(t) = 1$	$y'(t) = 2t^{1/2}$	$z'(t) = t$

$$s = \int_0^2 \sqrt{(1)^2 + (2t^{1/2})^2 + (t)^2}\, dt = \int_0^2 \sqrt{1 + 4t + t^2}\, dt$$

$$= 4.815$$

$$\underset{\sim}{r}(t) = \langle x(t),\ y(t),\ z(t) \rangle$$

$$\underset{\sim}{r}'(t) = \langle x'(t),\ y'(t),\ z'(t) \rangle$$

$$\| \underset{\sim}{r}'(t) \| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$$s = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$$

$$s(t) = \int_a^b \| \underset{\sim}{r}'(t) \| \, dt \Rightarrow$$

$$s(t) = \int_a^b \| r'(t) \| \, du$$

$$s(t) = \int_a^t \| r'(u) \| \, du = \text{the length arc from } t = a \text{ to some } t.$$

Look at Example 1:

$$\underset{\sim}{r}(t) = \langle 2t, t^2 \rangle$$

$$s(1) = s = \int_0^1 \|r'(t)\| dt = 2.29$$

$$s(2) = s = \int_0^2 \|r'(t)\| dt = \ldots$$

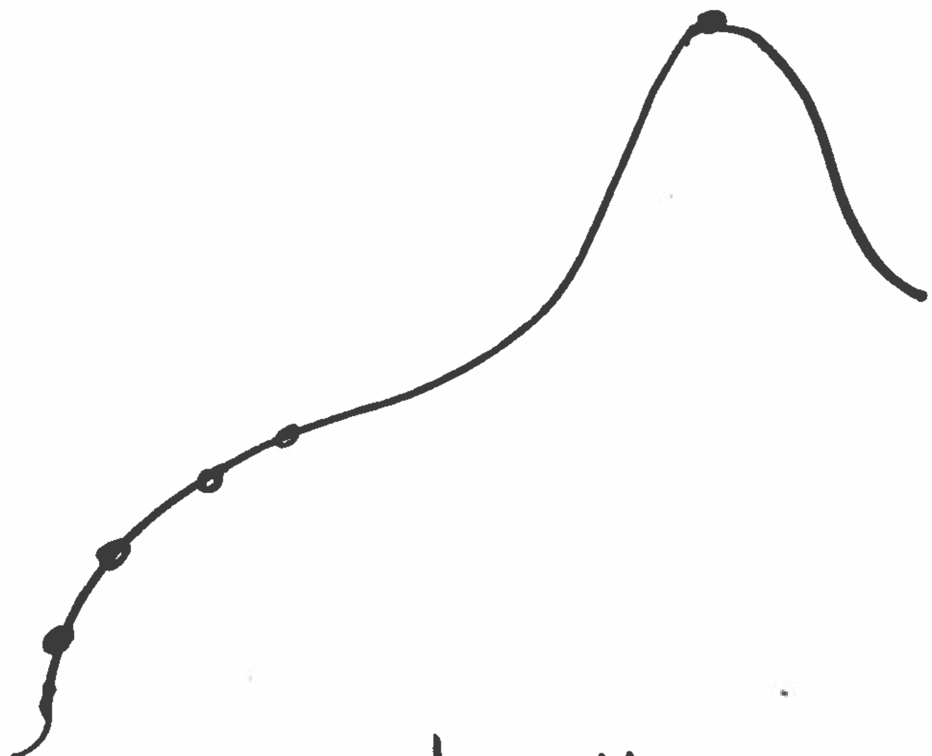
$$s(3) = s = \int_0^3 \|r'(t)\| dt = \ldots$$

In general: $$s(t) = \int_a^t \|\underset{\sim}{r}(u)\| du$$

$$s'(t) = \|\underset{\sim}{r}(u)\|$$

$$\frac{ds}{dt} = \|\underset{\sim}{r}(u)\|$$

K = curvature = measures how sharply a curve bends at time t.



$$K = \left\| \frac{dT}{ds} \right\| = \|T'(s)\|$$

$r(t) = \langle -t, \ 4t^2, \ 3t^2 \rangle$

Find K at $t = 2$

Note: At $t = 2$, $\underset{\sim}{r}(2) = \langle -2, \cancel{8}\,16, \cancel{4}\,12 \rangle$

$$K = \frac{\| r'(t) \times r''(t) \|}{\| r'(t) \|^3}$$

$$\begin{array}{rccc} & i & j & k \\ \underset{\sim}{r}'(t) = \langle -1, 8t, \cancel{3}\,6t \rangle = & \langle -1, & 8t, & 6t \rangle \\ r''(t) = & \langle 0, & 8, & 6 \rangle \end{array}$$

$$\begin{aligned} r'(t) \times r''(t) = \ & (48t - 48t)\,\underset{\sim}{i} \\ & - (-6 - 0)\,\underset{\sim}{j} \\ & + (-8 - 0)\,\underset{\sim}{k} \\ = \ & \langle 0, 6, -8 \rangle \end{aligned}$$

$$\|r'(t) \times r''(t)\| = \sqrt{0 + (6)^2 + (-8)^2} = 10$$

$$r'(t) = \langle -1, 8t, 6t \rangle$$

$$r'(2) = \langle -1, 16, 12 \rangle$$

$$\|r'(2)\| = \sqrt{401}$$

$$K = \text{curvature at } t=2 = \frac{10}{(\sqrt{401})^3} = 0.00124$$