Calculus III Section 13.10 Notes

Lagrange Multiplier Method

Example 1: A manufacturer of walking and running shoes finds that the total cost $C(x)$ of producing x units of walking shoes and y units of running shoes is given by

$$C(x) = 100 + 5x - .01x^2 + 5y - .02y^2.$$

Find x and y so that the cost $C(x)$ is minimized.

$C_x = 5 - 0.02x$	$C_y = 5 - 0.04y$
Set $C_x = 0$	Set $C_y = 0$
$5 - 0.02x = 0$	$5 - 0.04y = 0$
$x = 250$	$y = 125$

Therefore, cost is minized when $x = 250$ and $y = 125$

Example 2: A manufacturer of walking and running shoes finds that the total cost $C(x)$ of producing x units of walking shoes and y units of running shoes is given by

$$C(x) = 100 + 5x - .01x^2 + 5y - .02y^2.$$

Due to budget constraint, the manufacturer can only produce a total of 100 pairs of shoes.

Find x and y so that the cost $C(x)$ is minimized.

$$C(x) = 100 + 5x - .01x^2 + 5y - .02y^2.$$

Constraint: $x + y = 100$

Let $g(x, y) = x + y - 100 = 0$

$$C_x = 5 - 0.02x \qquad C_y = 5 - 0.04y$$

$$\nabla C(x,y) = \left\langle C_x, C_y \right\rangle = \left\langle 5 - 0.02x, 5 - 0.04y \right\rangle$$

$$g_x = 1 \qquad g_y = 1$$

$$\nabla g(x,y) = \left\langle g_x, g_y \right\rangle = \left\langle 1,1 \right\rangle$$

Lagrange Equations: $\nabla C(x,y) = \lambda \nabla g(x,y)$

$$\left\langle 5 - 0.02x, 5 - 0.04y \right\rangle = \lambda \left\langle 1,1 \right\rangle$$

$$\left\langle 5 - 0.02x, 5 - 0.04y \right\rangle = \left\langle \lambda, \lambda \right\rangle$$

$$5 - 0.02x = \lambda \qquad 5 - 0.04y = \lambda$$

$$x = \frac{5 - \lambda}{0.02} \qquad y = \frac{5 - \lambda}{0.04}$$

Constraint: $x + y = 100$

$$\frac{5-\lambda}{0.02} + \frac{5-\lambda}{0.04} = 100$$

$$0.04\left(\frac{5-\lambda}{0.02}\right) + 0.04\left(\frac{5-\lambda}{0.04}\right) = 0.04(100)$$

$$10 - 2\lambda + 5 - \lambda = 4$$

$$-3\lambda = -11$$

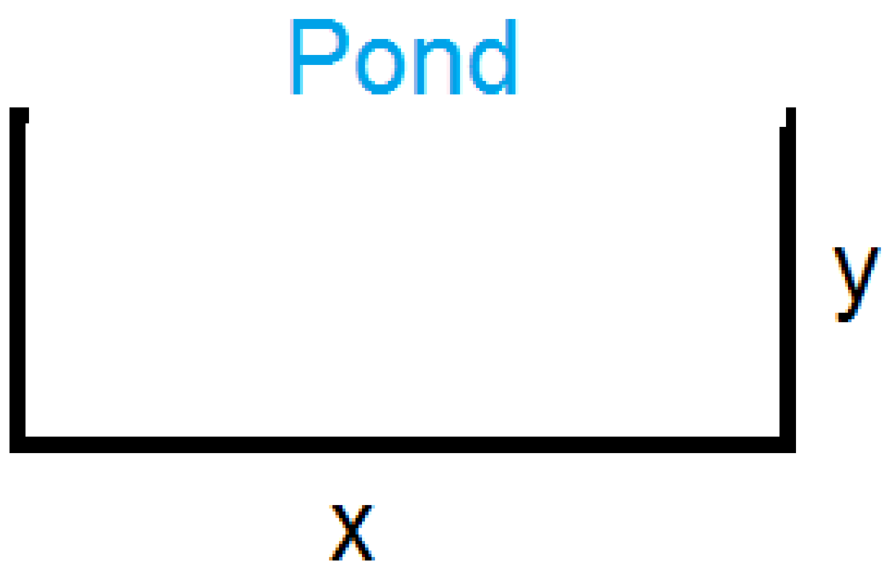
$$\lambda = 11/3$$

$$x = \frac{5-\lambda}{0.02} = \frac{5-11/3}{0.02} \approx 66.7 = 67$$

$$y = 100 - 67 = 33$$

Therefore, cost is minimized when $x = 67$ and $y = 33$.

Example 3: A rancher is planning to build an enclosed area along a pond. It is to be rectangular with an area of 3,000 square feet and is to be fenced on the three sides not adjacent to the pond. Find the dimensions (x and y) such that the least amount of fencing that will be needed to construct the enclosed area?



f = Amount of fencing = = $x+2y$

Area of enclosed area = $xy = 3000$

$f(x, y) = x + 2y$

Constraint: $g(x, y) = \; xy - 3000 = 0$

$f_x = 1 \qquad\qquad f_y = 2$

$\nabla f(x, y) = \langle 1, 2 \rangle$

$g_x = 1 \qquad\qquad g_y = 1$

$\nabla g(x, y) = \langle y, x \rangle$

Lagrange Equations: $\nabla f(x, y) = \lambda \nabla g(x, y)$

$$\langle 1,2\rangle = \lambda\langle y,x\rangle$$

$$\langle 1,2\rangle = \langle \lambda y,\lambda x\rangle$$

$$1=\lambda x \qquad\qquad 2=\lambda y$$

$$x=\frac{1}{\lambda} \qquad\qquad y=\frac{2}{\lambda}$$

Constraint: $g(x,y) = \ xy - 3000 = 0$

$$\left(\frac{1}{\lambda}\right)\left(\frac{2}{\lambda}\right) - 3000 = 0$$

$$2 = 3000\lambda^2 \quad \Leftrightarrow \quad \lambda^2 = \frac{2}{3000} \quad \Leftrightarrow \quad \lambda = \sqrt{\frac{2}{3000}}$$

$$x = \frac{1}{\lambda} = \frac{1}{\sqrt{\frac{2}{3000}}} = 38.72983346$$

$$x = \frac{2}{\lambda} = \frac{2}{\sqrt{\frac{2}{3000}}} = 77.45966692$$

Example 4: Minimize $f(x, y) = 2x + y$

subject to the constraint: $xy = 16$

$$g(x, y) = xy - 16 = 0$$

$$\nabla f = \langle f_x, f_y \rangle = \langle 2,1 \rangle \qquad \nabla g = \langle g_x, g_y \rangle = \langle y, x \rangle$$

Solve the system of equations: $\nabla f = \lambda \nabla g$; $xy - 16 = 0$

$$\nabla f = \lambda \nabla g \quad \Leftrightarrow \quad \langle 2,1 \rangle = \lambda \langle y, x \rangle$$

$$\Rightarrow \quad 2 = \lambda y; \qquad 1 = \lambda x$$

$$y = 2/\lambda \qquad x = 1/\lambda$$

Minimize $f(x,y)=2x+y$; Constraint: $xy=16$

$$xy-16=0$$

$$(2/\lambda)(1/\lambda)-16=0$$

$$\frac{2}{\lambda^2}=16$$

$$16\lambda^2=2$$

$$\lambda^2=1/8$$

$$\sqrt{\lambda^2}=\sqrt{1/8}$$

$$\lambda=\frac{1}{\sqrt{8}}=\frac{1}{2\sqrt{2}}$$

Minimize $f(x, y) = 2x + y;$ Constraint: $xy = 16$

Hence, $y = 2/\lambda$ $x = 1/\lambda$

$$y = \frac{2}{\frac{1}{2\sqrt{2}}} = 4\sqrt{2} \qquad x = \frac{1}{\frac{1}{2\sqrt{2}}} = 2\sqrt{2}$$

Therefore, $f(x, y) = 2x + y$ is minimized when

$x = 2\sqrt{2}$ and $y = 4\sqrt{2}$

$$f(x, y) = 2x + y = 2\left(2\sqrt{2}\right) + \left(4\sqrt{2}\right) = 8\sqrt{2}$$

Example 5: Maximize $f(x,y)=x^2-y^2$

subject to the constraint: $4y-x^2=0$

$$g(x,y)=4y-x^2=0$$

$$\nabla f=\left\langle f_x, f_y\right\rangle=\left\langle 2x,-2y\right\rangle \qquad \nabla g=\left\langle g_x, g_y\right\rangle=\left\langle -2x,4\right\rangle$$

Solve the system of equations: $\nabla f=\lambda\nabla g;\ 4y-x^2=0$

$$\nabla f=\lambda\nabla g \quad \Leftrightarrow \quad \left\langle 2x,-2y\right\rangle=\lambda\left\langle -2x,4\right\rangle$$

$$2x=\lambda(-2x)$$

$$2x+\lambda(2x)=0$$

$$\Rightarrow\ (2x)(1+2\lambda)=0$$

$$(2x)=0; \quad (1+2\lambda)=0$$

$$x=0; \quad \lambda=-1/2$$

$$-2y=\lambda(4)$$

$$-2y=(-1/2)(4)$$

$$-2y=-2$$

$$y=1$$

Maximize $f(x,y)=x^2-y^2$ subject to the constraint: $4y-x^2=0$

Case 1: $\lambda=-1/2 \quad \Rightarrow \quad y=1$

Constraint: $4y-x^2=0$

$$4(1)-x^2=0$$

$$x^2=4$$

$$x=\pm 2$$

Hence, $\Rightarrow \quad \lambda=-1/2 \quad \Rightarrow \quad y=1 \quad x=\pm 2$

$$f(x,y)=x^2-y^2=(\pm 2)^2-(1)^2=1$$

Maximize $f(x,y)=x^2-y^2$ subject to the constraint: $4y-x^2=0$

Case 2: $x=0$

Constraint: $4y-x^2=0$

$$4y-0=0$$

$$y=0$$

$$f(x,y)=x^2-y^2=(0)^2-(0)^2=0$$

Therefore, maximum occurs at $(\pm 2,1)$

Example 6: Minimize: $f(x, y) = \sqrt{16 - x^2 - y^2}$

Constraint: $g(x, y) = x + y - 4 = 0$

Note: Minimizing $\sqrt{16 - x^2 - y^2}$ is the same as

minimizing $16 - x^2 - y^2$

So let $h(x, y) = 16 - x^2 - y^2$

$\nabla h(x, y, z) = \langle -2x, -2y \rangle$ and $\nabla g(x, y, z) = \langle 1, 1 \rangle$

Lagrange equations $\nabla f(x, y) = \lambda \nabla g(x, y)$:

$\langle -2x, -2y \rangle = \lambda \langle 1, 1 \rangle$

$-2x = \lambda \quad -2x = \lambda \quad \Rightarrow \quad x = \dfrac{\lambda}{-2} \qquad y = \dfrac{\lambda}{-2}$

Constraint: $g(x, y) = x + y - 4 = 0$

$$\frac{\lambda}{-2} + \frac{\lambda}{-2} - 4 = 0$$

$$(-2)\left(\frac{\lambda}{-2}\right) + (-2)\left(\frac{\lambda}{-2}\right) - (-2)(4) = 0$$

$$\lambda + \lambda + 8 = 0$$

$$2\lambda = -8$$

$$\lambda = -4$$

$$x = \frac{\lambda}{-2} = \frac{-4}{-2} = 2 \qquad y = \frac{\lambda}{-2} = \frac{-4}{-2} = 2$$

$f(x, y) = \sqrt{16 - x^2 - y^2}$ is minimized when $x = 2$ and $y = 2$.

Example 7: Minimize: $f(x, y, z) = x^2 + y^2 + z^2$

Constraint: $g(x, y, z) = x + y + z - 4 = 0$

a) $\nabla f(x, y, z) = \langle 2x, 2y, 2z \rangle$

d) $\nabla g(x, y, z) = \langle 1,1,1 \rangle$

c) Lagrange Equations $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$:

$$\langle 2x, 2y, 2z \rangle = \lambda \langle 1,1,1 \rangle$$

$$2x = \lambda \quad 2y = \lambda \quad 2z = \lambda$$

$$x = \frac{\lambda}{2} \quad y = \frac{\lambda}{2} \quad z = \frac{\lambda}{2}$$

Constraint: $g(x, y, z) = x + y + z - 4 = 0$

$$\frac{\lambda}{2} + \frac{\lambda}{2} + \frac{\lambda}{2} - 4 = 0$$

$$\lambda + \lambda + \lambda - 8 = 0$$

$$3\lambda = 8$$

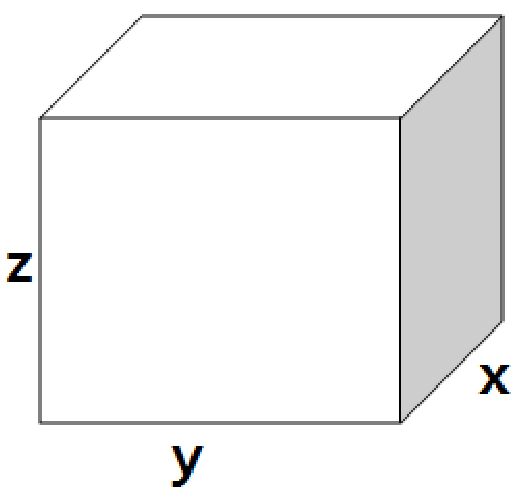
$$\lambda = 8/3$$

$$x = \frac{\lambda}{2} = \frac{8/3}{2} = 4/3 \qquad y = \frac{\lambda}{2} = \frac{8/3}{2} = 4/3$$

$$z = \frac{\lambda}{2} = \frac{8/3}{2} = 4/3$$

Example 8: A box in the shape of a rectangular solid has a volume of 40 cubic feet. The top and bottom of the box cost \$10 per square foot to construct. The sides of the box cost \$8 per square foot to construct. Find the dimensions of the box so that the cost of constructing is minimized.

Let $x =$ Length $y =$ width $z =$ height $C =$ cost of material



$C(x, y, z) =$ cost function $= 20xy + 16xz + 16yz$

$C_x = 20y + 16z \qquad C_y = 20x + 16z \qquad C_y = 16x + 16y$

Constraint Function: $g(x, y, z) = xyz - 40 = 0$

$g_x = yz; \quad g_y = xz; \quad g_z = xy$

Lagrange Equations: $\nabla C = \lambda \nabla g$

$20y + 16z = \lambda yz \quad (1)$

$20x + 16z = \lambda xz \quad (2)$

$16x + 16y = \lambda xy \quad (3)$

$xyz = 40 \qquad (4)$

This is a system of four equations with four unknowns

From equation (4): $z = \dfrac{40}{xy}$

Hence,

$$20y + 16z = \lambda yz \quad \Rightarrow \quad 20y + 16\left(\frac{40}{xy}\right) = \lambda y\left(\frac{40}{xy}\right) \qquad (1)$$

$$20x + 16z = \lambda xz \quad \Rightarrow \quad 20x + 16\left(\frac{40}{xy}\right) = \lambda x\left(\frac{40}{xy}\right) \qquad (2)$$

$$16x + 16y = \lambda xy \quad \Rightarrow \quad 16x + 16y = \lambda xy \qquad (3)$$

Multiply (1) and (2) by xy:

$$20y(xy)+16\left(\frac{40}{xy}\right)(xy)=\lambda y\left(\frac{40}{xy}\right)(xy) \quad (1)$$

$$20x(xy)+16\left(\frac{40}{xy}\right)(xy)=\lambda x\left(\frac{40}{xy}\right)(xy) \quad (2)$$

Hence,

$$20xy^2+640=40y\lambda \quad (1)$$

$$20x^2y+640=40x\lambda \quad (2)$$

$$16x+16y=\lambda xy \quad (3)$$

Solve equation (3) for λ and substitute result into (1) and (2):

$$\lambda = \frac{16x + 16y}{xy} \qquad (3)$$

$$20xy^2 + 640 = 40y\left(\frac{16x+16y}{xy}\right) \;\Rightarrow\; 20xy^2 + 640 = 40\left(\frac{16x+16y}{x}\right) \qquad (1)$$

$$20x^2y + 640 = 40x\left(\frac{16x+16y}{xy}\right) \;\Rightarrow\; 20x^2y + 640 = 40\left(\frac{16x+16y}{y}\right) \qquad (2)$$

Multiply (1) by x: $20x^2y^2 + 640x = 40(16x + 16y)$

Multiply (2) by y: $20x^2y^2 + 640y = 40(16x + 16y)$

Subtract (2) from (1): $640x - 640y = 0$

$$640x = 640y$$

$$x = y$$

From (1) above: $20xy^2 + 640 = 40\left(\dfrac{16x + 16y}{x}\right)$

$$20(y)y^2 + 640 = 40\left(\frac{16(y) + 16(y)}{(y)}\right) \qquad \text{Note: } x = y$$

$$20(y)y^2 + 640 = 40(32)$$

$$20y^3 + 640 = 1280$$

$$20y^3 = 640$$

$$y^3 = 32$$

$$y = \sqrt[3]{32} = 3.1748021039363623$$

Hence, $x = y = \sqrt[3]{32} = 3.1748021039363623$

$$z = \frac{40}{\sqrt[3]{32}\sqrt[3]{32}} = 3.9685026299205903$$

Therefore, cost is minimized when $x = 3.1748021039363623$,

$y = 3.1748021039363623$ and $z = 3.9685026299205903$