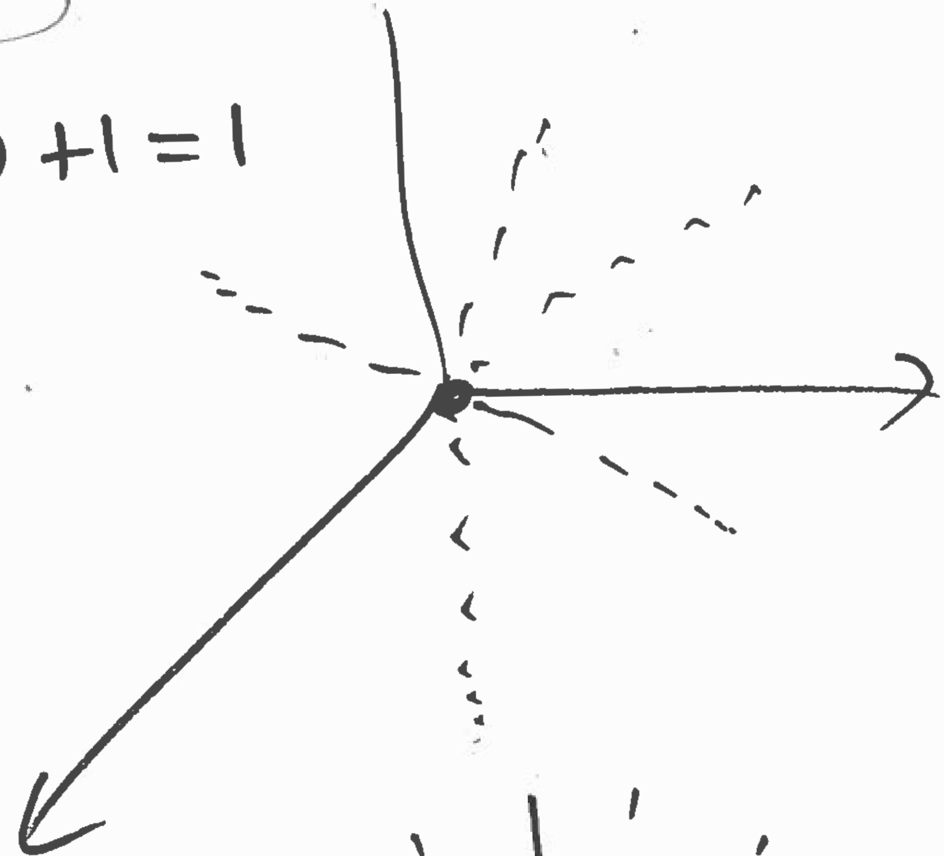
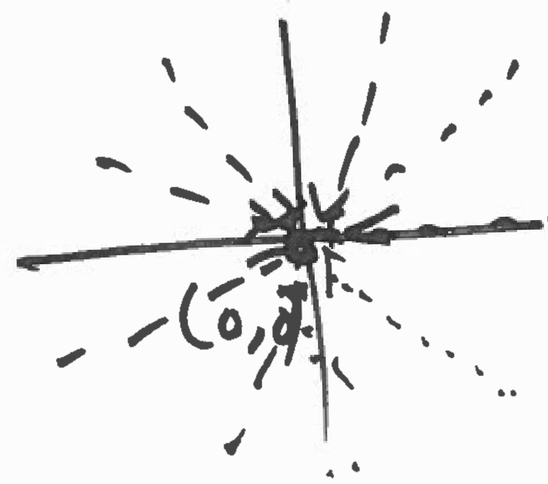
13.2 $f(x,y) = x + 4y + 1$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 + 4(0) + 1 = 1$$

As $x \to 0$ and $y \to 0$,

$z \to 1$





$$f(x,y) = \frac{x + y}{x^2 + 1}$$

$$\lim_{(x,y)\to(2,4)} f(x,y) = \frac{(2) + (4)}{(2)^2 + 1} = 6/5$$

As $x \to 2$ and $y \to 4$, $z \to 6/5$

$$\lim_{(x,y)\to(1,2)} f(x,y) = \frac{1 + 2}{1 - 2} = -3$$

$$f(x,y) = \frac{x + y}{x - y}$$

$$f(x,y) = \frac{1}{x^2 y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{\to 0} = \infty = \text{DNE}$$

$$\lim_{(x,y)\to(2,1)} f(x,y) = \cancel{\frac{2-1-1}{\sqrt{2-1}-1}} = \cancel{\frac{0}{0}}$$

$$= \lim_{(x,y)\to(2,1)} \left(\sqrt{x-y} + 1\right)$$

$$= \sqrt{2-1} + 1$$

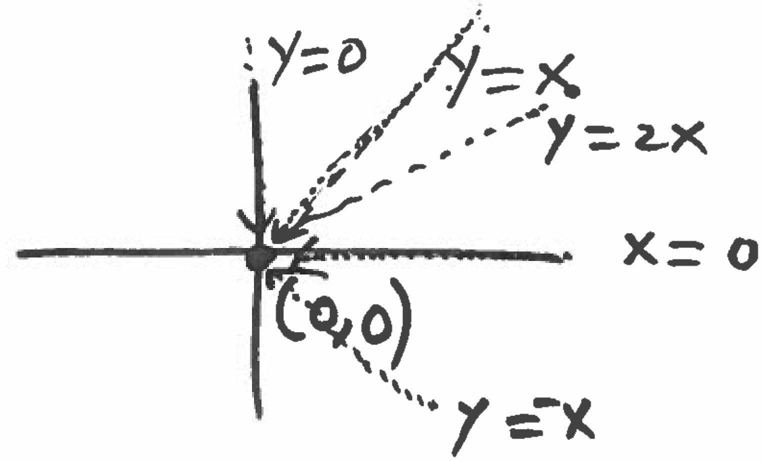
$$= 2$$

$$f(x,y) = \frac{(x-y-1)}{(\sqrt{x-y}-1)} \cdot \frac{(\sqrt{x-y}+1)}{(\sqrt{x-y}+1)}$$

$$= \frac{(x-y-1)(\sqrt{x-y}+1)}{(x-y) - 1^2}$$

$$= \frac{(x-y-1)(\sqrt{x-y}+1)}{(x-y-1)}$$

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2-y^2}$$



Prove that limit does not exist.

Along the path $y=x$

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2-x^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{x}{0}$$

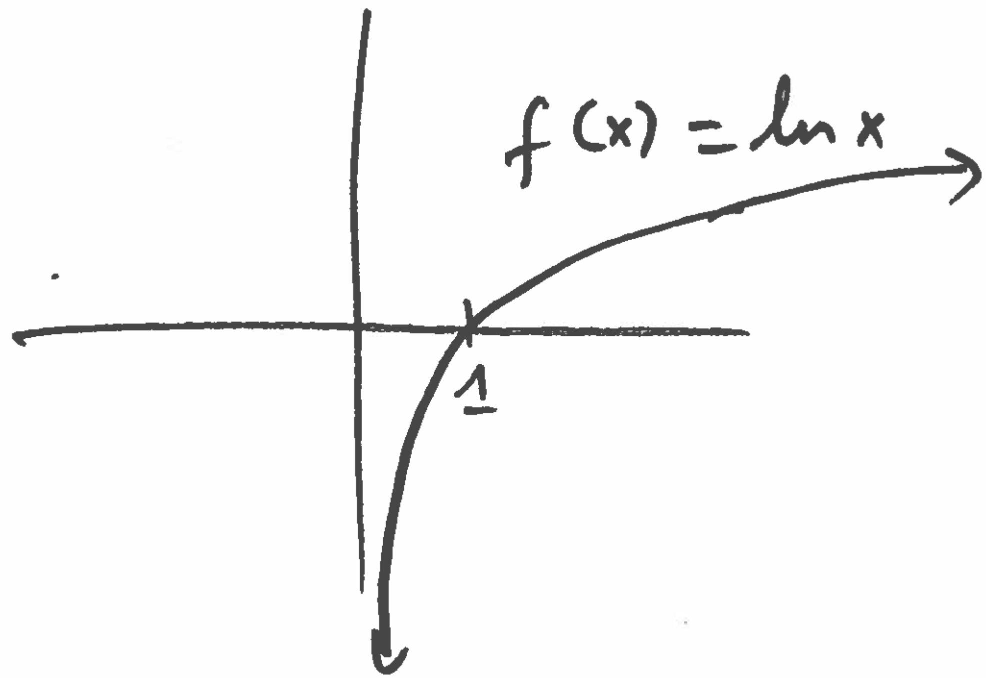
$$= \lim_{(x,y)\to(0,0)} \text{undefined}$$

Along the path $y=2x$

$$\lim_{(x,y)\to(0,0)} \frac{x}{x^2-4x^2}$$

$$= \lim_{(x,y)\to(0,0)} -\frac{1}{3x} = -\infty$$

$$\lim_{(x,y)\to(0,0)} \ln(x^2+y^2) = \ln(\to 0) = -\infty = \text{DNE}$$



Find $\lim_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2}$.

Note: Since $x^2+y^2=r^2, (x,y)\to(0,0)$ means that $r\to 0$.

Also, $x=r\cos\theta$ and $y=r\sin\theta$

Hence, $$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{(r\cos\theta)^3+(r\sin\theta)^3}{(r\cos\theta)^2+(r\sin\theta)^2} = \lim_{r\to 0} \frac{r^3\left[(\cos\theta)^3+(\sin\theta)^3\right]}{r^2\left[(\cos\theta)^2+(\sin\theta)^2\right]}$$

$$= \lim_{r\to 0} \frac{r\left[(\cos\theta)^3+(\sin\theta)^3\right]}{[1]} = \lim_{r\to 0}\left(r\left[(\cos\theta)^3+(\sin\theta)^3\right]\right)$$

$$= 0\left[(\cos\theta)^3+(\sin\theta)^3\right] = 0$$

Note that $-2 \le (\cos\theta)^3+(\sin\theta)^3 \le 2$.

Find $\displaystyle \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}}$.

Note: Since $x^2 + y^2 = r^2, (x, y) \to (0,0)$ means that $r \to 0$.

Also, $x = r\cos\theta$ and $y = r\sin\theta$

Hence, $$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{\sqrt{x^2 + y^2}} = \lim_{r\to 0} \frac{(r\cos\theta)^2 - (r\sin\theta)^2}{\sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}} = \lim_{r\to 0} \frac{r^2\left[(\cos\theta)^2 - (\sin\theta)^2\right]}{r\left[\sqrt{(\cos\theta)^2 + (\sin\theta)^2}\right]}$$

$$= \lim_{r\to 0} \frac{r\left[(\cos\theta)^2 - (\sin\theta)^2\right]}{[1]} = \lim_{r\to 0}\left(r\left[(\cos\theta)^2 - (\sin\theta)^2\right]\right)$$

$$= 0\left[(\cos\theta)^2 - (\sin\theta)^2\right] = 0$$

Note that $-1 \le (\cos\theta)^2 - (\sin\theta)^2 \le 1.$

Find $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2}$.

Note: Since $x^2+y^2=r^2, (x,y)\to(0,0)$ means that $r\to 0$.

Also, $x=r\cos\theta$ and $y=r\sin\theta$

Hence, $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim\limits_{r\to 0}\dfrac{\sin\left(r^2\right)}{r^2} = \dfrac{0}{0}$ We can apply L'Hopital's Rule

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(x^2+y^2\right)}{x^2+y^2} = \lim_{r\to 0}\frac{\cos\left(r^2\right)\cdot(2r)}{2r} = \lim_{r\to 0}\cos\left(r^2\right) = \cos(0) = 1$$

$$f(x, y) = 2x + y^2$$

$$f_x = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$$

$$f_x = \lim_{\Delta x \to 0} \left(2 \right) = 2$$

Partial Derivative with respect to x.

$$f_y = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$$

$$f_y = \lim_{\Delta y \to 0} \left(2y + \Delta y \right) = 2y.$$

$$\cancel{f(x,y) =}$$

$$f(x,y) = 3x^2 - 2y$$

$$f_x = \lim_{\Delta x \to 0} \left[\frac{\cancel{f(x+\Delta x)} \quad f(x+\Delta x, y) - f(x,y)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to} \left[6x + 3 \cdot \Delta x \right] = 6x$$

$$f_y = \lim_{\Delta y \to 0} \left[\frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \right].$$