Section 13.2 Notes

Exmaple 1: Let $f(x,y)=xy$

a) Find $\lim\limits_{(x,y)\to(1,3)} f(x,y)= (1)(3)=3$

b) Discuss continuity of $f(x,y)$.

$f(x,y)$ is continuous everywhere.

Example 2: Let $f(x,y)=5e^{xy}$

a) Find $\lim\limits_{(x,y)\to(1,-2)} f(x,y)=5e^{(1)(-2)}=5e^{-2}$

b) Discuss continuity of $f(x,y)$.

$f(x,y)$ is continuous everywhere.

Example 3: $f(x,y)=\dfrac{5x+y}{xy}$

a) Find $\lim\limits_{(x,y)\to(1,9)} f(x,y)=\dfrac{5(1)+9}{(1)(9)}=\dfrac{14}{9}$

b) Discuss continuity of $f(x,y)$.

Note: $f(x,y)$ is undefined when $xy=0$ and $f(x,y)$ is defined when $xy\neq 0$.

So $f(x,y)$ is continuous when $xy\neq 0$.

4) $f(x,y)=\dfrac{x-y}{\sqrt{y-x}}$

a) Find $\lim\limits_{(x,y)\to(1,4)} f(x,y)=\dfrac{1-4}{\sqrt{4-1}}=\dfrac{-3}{\sqrt{3}}$

b) Discuss continuity of $f(x,y)$.

Note: $f(x,y)$ is undefined when $y-x\le 0$ and $f(x,y)$ is defined when $y-x>0$.

So $f(x,y)$ is continuous when $y-x>0$.

Example 5: Let $f(x,y)=\dfrac{4x+5y}{x-y}$. Show that $\lim\limits_{(x,y)\to(1,1)} f(x,y)$ does not exist.

Solution:

We will choose three lines on the *xy*-plane that contain the point $(1,\ 1)$.

These three lines are called paths containing $(1,\ 1)$. We will show that these paths will lead to different limit values.

a) Find $\lim\limits_{(x,y)\to(1,1)} f(x,y)$ along the line $x=1$

$$\lim_{(x,y)\to(1,1)} f(x,y)=\lim_{(x,y)\to(1,1)}\frac{4x+5y}{x-y}=\lim_{(x,y)\to(1,1)}\frac{4(1)+5y}{(1)-y}=\lim_{(x,y)\to(0,0)}\frac{4+5y}{1-y}=\frac{4+0}{1-0}=4$$

b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $y=2x-1$.

$$\lim_{(x,y)\to(1,1)} f(x,y)=\lim_{(x,y)\to(1,1)}\frac{4x+5y}{x-y}=\lim_{(x,y)\to(1,1)}\frac{4x+5(2x-1)}{x-(2x-1)}=\lim_{(x,y)\to(0,0)}\frac{14x-5}{-x+1}=\frac{-5}{1}=-5 \quad \text{Note: } x\to 0$$

c) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $y=x$

$$\lim_{(x,y)\to(1,1)} f(x,y)=\lim_{(x,y)\to(1,1)}\frac{4x+5y}{x-y}=\lim_{(x,y)\to(1,1)}\frac{4x+5x}{x-x}=\lim_{(x,y)\to(0,0)}\frac{9x}{0}=\text{undefined}$$

Therefore, $\lim\limits_{(x,y)\to(1,1)} f(x,y)$ does not exist.

Exmaple 6: Let $f(x,y)=\dfrac{x+y}{x^2-y}$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution:

We will choose three lines on the *xy*-plane that contain the point $(0,0)$.

These three lines are called paths containing $(0,0)$. We will show that these paths will lead to different limit values.

a) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $x=0$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x+y}{x^2-y}=\lim_{(x,y)\to(0,0)}\frac{0+y}{0-y}=\lim_{(x,y)\to(0,0)} -1= -1$$

b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $y=x$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x+y}{x^2-y}=\lim_{(x,y)\to(0,0)}\frac{x+x}{x^2-x}=\lim_{(x,y)\to(0,0)}\frac{2x}{x(x-1)}=\lim_{(x,y)\to(0,0)}\frac{2}{(x-1)}=\frac{2}{-1}=-2$$

c) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $y=0$.

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x+y}{x^2-y}=\lim_{(x,y)\to(0,0)}\frac{x+0}{x^2-0}==\lim_{(x,y)\to(0,0)}\frac{1}{x}=\infty \quad \text{Note: } x\to 0$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example 7: Let $f(x,y)=\dfrac{x}{x^2+y^2}$

Exmaple 6: Let $f(x,y)=\dfrac{x}{x^2+y^2}$. Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Solution:

We will choose three lines on the *xy*-plane that contain the point $(0,0)$.

These three lines are called paths containing $(0,0)$. We will show that these paths will lead to different limit values.

a) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $x=0$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{0}{0^2+y^2}=\lim_{(x,y)\to(0,0)} 0=0$$

b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $y=x$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{x}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{x}{2x^2}=\lim_{(x,y)\to(0,0)}\frac{1}{2x}=\infty \quad \text{Note: } x\to 0$$

c) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ along the line $y=0$.

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x}{x^2+y^2}=\lim_{(x,y)\to(0,0)}\frac{x}{x^2+0^2}==\lim_{(x,y)\to(0,0)}\frac{1}{x}=\infty \quad \text{Note: } x\to 0$$

Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Exmaple 8: Let $f(x,y)=\dfrac{4x^3}{x^2+y^2}$. Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

Use polar coordinates to find the limit. Hint: $x=r\cos\theta$; $y=r\sin\theta$; $x^2+y^2=r^2$

Note: As $(x,y)\to(0,0)$, $r=\sqrt{x^2+y^2}\to 0$

a) Express $\dfrac{4x^3}{x^2+y^2}$ in terms of r and/or θ:

$$\frac{4x^3}{x^2+y^2}=\frac{4(r\cos\theta)^3}{r^2}=4r\cos^3\theta$$

b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{r\to 0} f(r,\theta)=\lim\limits_{r\to 0}\left(4r\cos^3\theta\right)=\lim\limits_{r\to 0}\left(4(0)\cos^3\theta\right)=\lim\limits_{r\to 0}\left(0\right)=0$

Exmaple 9: Let $f(x,y)=\sin\sqrt{x^2+y^2}$. Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

Use polar coordinates to find the limit. $x=r\cos\theta;\quad y=r\sin\theta;\quad x^2+y^2=r^2$

Note: As $(x,y)\to(0,0),\quad r=\sqrt{x^2+y^2}\to 0$

a) Express $\sin\sqrt{x^2+y^2}$ in terms of r and/or θ:

$\sin\sqrt{x^2+y^2} = \sin r$

b) Find $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{r\to 0} f(r,\theta)=\lim\limits_{r\to 0}\sin r=0$

Exmaple 10: Let $f(x,y)=x^2+y$

a) $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\left[(x+\Delta x)^2+y\right]-\left[x^2+y\right]}{\Delta x}=\lim_{\Delta x\to 0}\frac{\left[x^2+2x\Delta x+(\Delta x)^2+y\right]-\left[x^2+y\right]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left[2x\Delta x+(\Delta x)^2\right]}{\Delta x}=\lim_{\Delta x\to 0}\left(2x+\Delta x\right)=2x$$

b) Find $$\lim_{\Delta x\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=\lim_{\Delta y\to 0}\frac{\left[x^2+\left(y+\Delta y\right)\right]-\left[x^2+y\right]}{\Delta y}=\lim_{\Delta y\to 0}\frac{\Delta y}{\Delta y}=1$$

Example 11: Let $f(x,y)=x+y^2-2xy$

a) $$\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y)-f(x,y)}{\Delta x}=\lim_{\Delta x\to 0}\frac{\left[x+\Delta x+y^2-2(x+\Delta x)y\right]-\left[x+y^2-2xy\right]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left[x+\Delta x+y^2-2xy-2\Delta xy\right]-\left[x+y^2-2xy\right]}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{\left[\Delta x-2\Delta xy\right]}{\Delta x}=\lim_{\Delta x\to 0}\left(1-2y\right)=1-2y$$

b) $$\lim_{\Delta x\to 0}\frac{f(x,y+\Delta y)-f(x,y)}{\Delta y}=\lim_{\Delta x\to 0}\frac{\left[x+\left(y+\Delta y\right)^2-2x\left(y+\Delta y\right)\right]-\left[x+y^2-2xy\right]}{\Delta y}$$

$$=\lim_{\Delta x\to 0}\frac{\left[x+y^2+2y\Delta y+\left(\Delta y\right)^2-2xy-2x\Delta y\right]-\left[x+y^2-2xy\right]}{\Delta y}$$

$$=\lim_{\Delta x\to 0}\frac{\left[2y\Delta y+\left(\Delta y\right)^2-2x\Delta y\right]}{\Delta y}=\lim_{\Delta y\to 0}\left(2y+\Delta y-2x\right)=2y-2x$$

Example 12: Let $f(x,y) = 4x^2 + y^2$

a) Find $\lim\limits_{\Delta x \to 0} \dfrac{f(x+\Delta x, y) - f(x,y)}{\Delta x} = \lim\limits_{\Delta x \to 0} \dfrac{\left[4(x+\Delta x)^2 + y^2\right] - \left[4x^2 + y^2\right]}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{\left[4x^2 + 8x\Delta x + 4(\Delta x)^2 + y^2\right] - \left[4x^2 + y^2\right]}{\Delta x} = = \lim_{\Delta x \to 0} \frac{\left[8x\Delta x + 4(\Delta x)^2\right]}{\Delta x} = = \lim_{\Delta x \to 0} \left(8x + 4\Delta x\right) = 8x$$

b) Find $\lim\limits_{\Delta y \to 0} \dfrac{f(x, y+\Delta y) - f(x,y)}{\Delta y} = \lim\limits_{\Delta y \to 0} \dfrac{\left[4x^2 + (y+\Delta y)^2\right] - \left[4x^2 + y^2\right]}{\Delta x}$

$$= \lim_{\Delta y \to 0} \frac{\left[4x^2 + y^2 + 2y\Delta y + (\Delta y)^2\right] - \left[4x^2 + y^2\right]}{\Delta y} = \lim_{\Delta y \to 0} \frac{\left[2y\Delta y + (\Delta y)^2\right]}{\Delta y} = \lim_{\Delta y \to 0} \left(2y + \Delta y\right) = 2y$$