13.3 Partial Derivatives

Example 1

$$f(x) = \frac{1}{x+y} = (x+y)^{-1} \quad \text{Find } f_x \quad f_y$$

$$f_x = \cancel{(x+y)^{-1}} = -1(x+y)^{-2}(1+0)$$

$$= \frac{-1}{(x+y)^2}$$

$$f_y = -1(x+y)^{-2}(0+1) = -1(x+y)^{-2}$$

$$= \frac{-1}{(x+y)^2}$$

Example 2

$$\cancel{f(x)}\ z = f(x,y) = \underset{F}{(x)} \cdot \underset{S}{\left(e^{x^2y}\right)}$$

$$f_x = \frac{\partial}{\partial x} f(x,y) = z_x = \frac{\partial z}{\partial x}$$

$$f_x = (x)\left(e^{x^2y} \cdot 2xy\right) + e^{x^2 \cdot y} \cdot (1)$$

$$= 2x^2ye^{x^2y} + e^{x^2y}$$

$\cancel{f_x(1,}$

$$f_y = x \cdot e^{x^2 \cdot y} \cdot x^2 = x^3 \cdot e^{x^2y}$$

When $x = 1$, $y = \ln 2$, $z = 2$

$$z = f(1, \ln 2) = (1)\, e^{(1)^2 \cdot \ln 2} = e^{\ln 2} = 2$$

$$f_x(1, \ln 2) = 2x^2 e^{x^2 y} + e^{x^2 y}$$

$$= 2 \cdot (1)^2 e^{(1)^2 \ln 2} + e^{(1)^2 \ln 2}$$

$$= 2 \cdot 2 + 2 = 6$$

$$f_y(1, \ln 2) = x^3 \cdot e^{x^2 y} = (1)^3 \cdot e^{1^2 \ln 2} =$$

$$= e^{\ln 2} = 2$$

Example 3

$$f(x,y) = 3\cdot x - x^2\cdot y^2 + 2\cdot x^3\cdot y$$

$$f_x = 3 - (y^2)(2x) + (2y)\cdot(3x^2)$$

$$= 3 - 2xy^2 + 6x^2y \, .$$

$$f_y = 0 - (x^2)\cdot(2y) + (2x^3)(1)$$

$$= -2x^2y + 2x^3$$

Example 4

$$z = f(x,y) = \frac{-x^2}{2} - y^2 + \frac{25}{8}$$

when $x = \frac{1}{2}$, $y = 1$, $z = 2$

$$z = \frac{-(+1/2)^2}{2} - (1) + \frac{25}{8} = 2$$

$$f_x = -\frac{1}{2}(2x) = -x$$

$$f_y = -2y$$

$$f_x\left(\frac{1}{2}, 1\right) = -\frac{1}{2}\left(2\left(\frac{1}{2}\right)\right) = -\frac{1}{2}$$

$$f_y\left(\frac{1}{2}, 1\right) = -2(1) = -2$$

Example 5

$$f(x,y) = 1 - (x-1)^2 - (y-2)^2$$

At $x=1,\ y=2$ $\quad f(1,2) = 1$

$$f_x(x,y) = -2(x-1)$$

$$f_y(x,y) = -2(y-2)$$

$$f_x(1,2) = -2(1-1) = 0$$

$$f_y(1,2) = -2(2-2) = 0$$

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

L to R   inside first   R to L

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

Example 6

$$f(x,y) = 3xy^2 - 2y + 5x^2y^2$$

$$f_x = (3y^2)(1) - 0 + (5y^2)\cdot(2x)$$

$$f_x = 3y^2 + 10xy^2$$

$$f_{xx} = 0 + (10y^2)(1) = 10y^2$$

$$f_{xy} = 6y + (10x)(2y) = 6y + 20xy$$

$$f_y = (3x)(2y) - 2 + 0 = 6xy - 2$$

$$f_{yy} = 6x$$

$$f_{yx} = 6y$$

Example 7

$$f(x,y,z) = y\cdot e^x + x\cdot \ln z$$

$$f_x = (y)\cdot e^x + (\ln z)\cdot(1) = ye^x + \ln z$$

$$f_{xz} = 0 + \frac{1}{z} = \frac{1}{z}$$

$$f_{xzz} = -1/z^2$$

Example 8

$$f(x,y) = x^2 - xy + y^2 - 5x + y$$

Find x, y such that $f_x = 0$ and $f_y = 0$

$$f_x = 2x - y - 5$$

$$f_y = -x + 2y + 1$$

Set $f_x = 0$ $\qquad$ $2x - y - 5 = 0$ ①

$\qquad f_y = 0$ $\qquad$ $-x + 2y + 1 = 0$ ②

$$
\begin{array}{rrrrl}
① & 4x & -2y & -10 & = 0 \\
② & -x & +2y & +1 & = 0 \\
\hline
 & 3x & & -9 & = 0
\end{array}
$$

$$x = 3$$

① $2x - y - 5 = 0$

$2(3) - y - 5 = 0$

$y = 1$

When $x = 3, y = 1$

$z = (3)^2 - (3)(1) + (1)^2 - 5(3) + 1$

$z = -7$

$(3, 1, -7)$

Example 9

Laplace's Equation:

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0 \iff f_{xx} + f_{yy} = 0$$

$$z = e^x \cdot \sin y$$

Does equation satisfy Laplace's equation?

$f_x = (\sin y) \cdot e^x$ $\quad$ $f_y = e^x \cdot (\cos y)$

$f_{xx} = (\sin y) e^x$ $\quad$ $f_{yy} = e^x \cdot (-\sin y)$

$$f_{xx} + f_{yy} = (\sin y) e^x + (-\sin y) e^x = 0$$

Example 10

Wave Equation :

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2}\right) \Leftrightarrow z_{tt} = c^2 z_{xx}$$

$z = \cos(4x + 4ct)$ Does this ~~equ~~ function satisfy the wave equation?

$$\frac{\partial z}{\partial t} = z_t = -\sin(4x+4ct)(4c)$$
$$= -4c \cdot \sin(4x+4ct)$$

$$\frac{\partial^2 z}{\partial t^2} = z_{tt} = (-4c)\left(\cos(4x+4ct)\right)(4c)$$
$$= c^2\left(-16\cos(4x+4ct)\right)$$

$$\frac{\partial z}{\partial x} = z_x = -\sin(4x + 4ct)(4)$$

$$z_{xx} = -4\cos(4x + 4ct)(4) = -16\cos(4x+4ct)$$

$$z_{tt} = c^2 \cdot z_{xx}$$