Section 13.3 Notes

Example 1: Graphical Repesentation of $f_x(x,y)$

Let $z = f(x,y) = x^2 + 3y^2 + 2$

a) $f_x(x,y) = 2x$ and $f_y(x,y) = 6y$

b) $f_x(1,1) = 2(1) = 2$ and $f_y(1,1) = 6(1) = 6$

c) The intersection of the plane $y = 1$ and $z = x^2 + 3y^2 + 2$ is a curve.

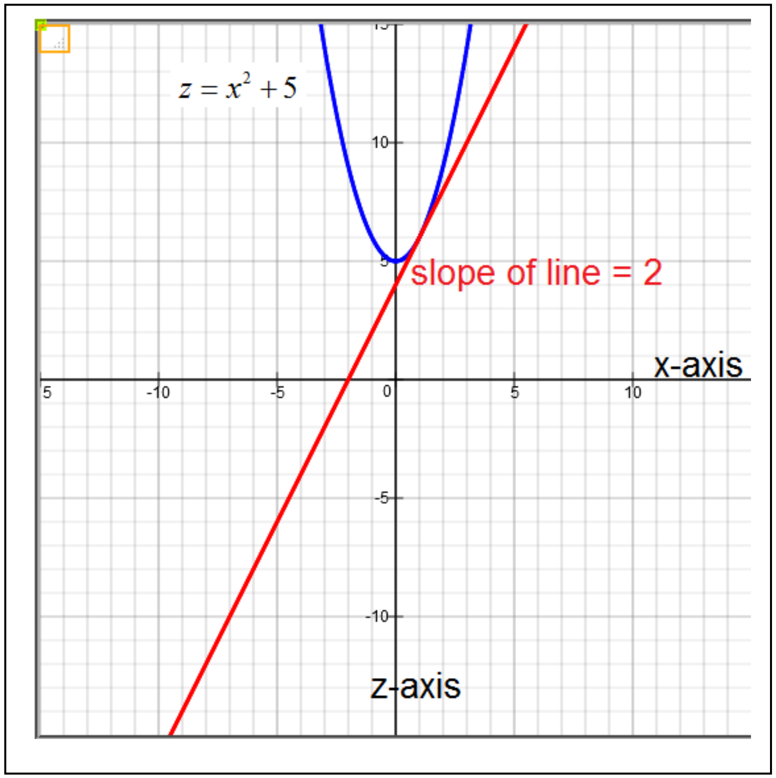
Find a vector-valued function $\mathbf{r}(t)$ representing the intersection.

$z = x^2 + 3y^2 + 2 = x^2 + 3(1)^2 + 2 = x^2 + 5$

Corresponding Parametric Equations for $z = x^2 + 5$: $x = t$ and $z = t^2 + 5$

Intersection of the plane $y = 1$ and $z = x^2 + 3y^2 + 2$ is the curve:

$$\mathbf{r}(t) = \langle x = t,\ y = 1,\ z = t^2 + 5 \rangle = \langle t, 1, t^2 + 5 \rangle$$



d) Graphical representation of $f_x(1,1) = 2(1) = 2$:

When $x = 1$ and $y = 1$, $z = x^2 + 3y^2 + 2 = 6$.

Hence, $(1,1,6)$ is a point on the graph of $z = x^2 + 3y^2 + 2$.

$\mathbf{r}(t) = \langle t, 1, t^2 + 5 \rangle$ and $\mathbf{r}(1) = \langle 1, 1, 6 \rangle$.

$\mathbf{r}'(t) = \langle 1, 0, 2t \rangle$ and $\mathbf{r}'(1) = \langle 1, 0, 2 \rangle$

And $\mathbf{r}(1) + \mathbf{r}'(1) = \langle 1,1,6 \rangle + \langle 1,0,2 \rangle = \langle 2,1,8 \rangle$

$\mathbf{r}'(1)$ is a vector from $(1,1,6)$ to $(2,1,8)$.

The line containing the points $(1,1,6)$ and $(2,1,8)$ is a line in the plane $y = 1$ and has a slope of $2 = f_x(1,1)$.

Example 2: Graphical Repesentation of $f_x(x,y)$

Let $z = f(x,y) = x^2 + 3y^2 + 2$

a) $f_x(x,y) = 2x$ and $f_y(x,y) = 6y$

b) $f_x(1,1) = 2(1) = 2$ and $f_y(1,1) = 6(1) = 6$

c) The intersection of the plane $x = 1$ and $z = x^2 + 3y^2 + 2$ is a curve.

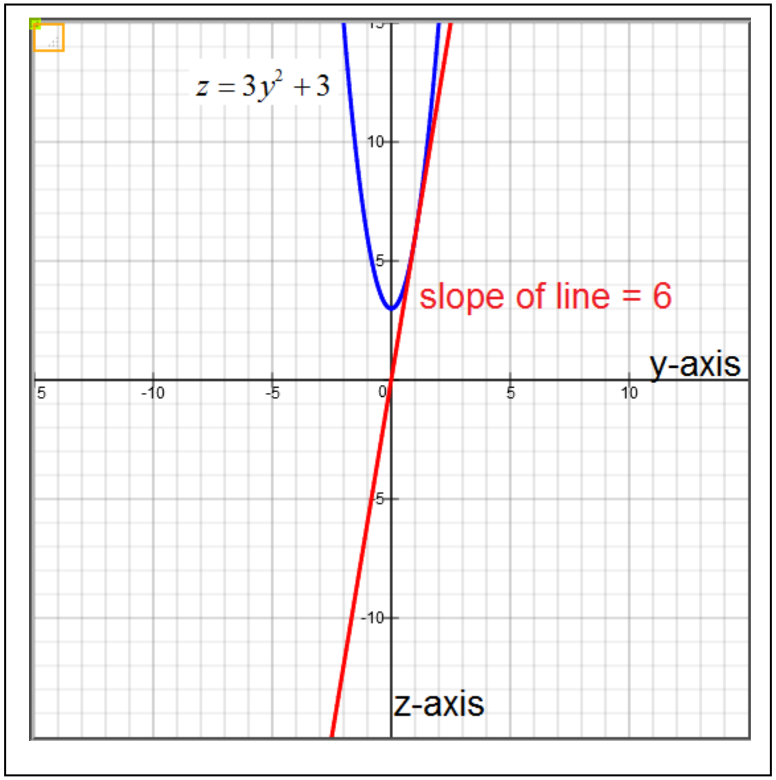
Find a vector-valued function $\mathbf{r}(t)$ representing the intersection.

$z = x^2 + 3y^2 + 2 = 1^2 + 3y^2 + 2 = 3y^2 + 3$

Corresponding Parametric Equations for $z = 3y^2 + 3$: $y = t$ and $z = 3t^2 + 3$

Intersection of the plane $x = 1$ and $z = x^2 + 3y^2 + 2$ is the curve:

$$\mathbf{r}(t) = \langle x = 1,\ y = t,\ z = 3t^2 + 3\rangle = \langle 1, t, 3t^2 + 3\rangle$$



d) Graphical representation of $f_y(1,1) = 6(1) = 6$:

When $x = 1$ and $y = 1$, $z = x^2 + 3y^2 + 2 = 6$.

Hence, (1,1,6) is a point on the graph of $z = x^2 + 3y^2 + 2$.

$\mathbf{r}(t) = \langle 1, t, 3t^2 + 3\rangle$ and $\mathbf{r}(1) = \langle 1, 1, 6\rangle$.

$\mathbf{r}'(t) = \langle 0, 1, 6t\rangle$ and $\mathbf{r}'(1) = \langle 0, 1, 6\rangle$

And $\mathbf{r}(1) + \mathbf{r}'(1) = \langle 1,1,6\rangle + \langle 0,1,6\rangle = \langle 1,2,12\rangle$

$\mathbf{r}'(1)$ is a vector from $(1,1,6)$ to (1,2, 12).

The line containing the points $(1,1,6)$ and (1, 2,12) is a line in the plane $x = 1$ and has a slope of $6 = f_y(1,1)$.

Example 3: Let $z = f(x,y) = \sqrt{6-x^2-y^2}$

When $x=1$ and $y=1$, $z=\sqrt{6-x^2-y^2} = \sqrt{6-1^2-1^2} = 2$; Hence, the point $(1,1,2)$ is on the graph of $z=\sqrt{6-x^2-y^2}$.

$$f_x(x,y) = \frac{1}{2}\left(6-x^2-y^2\right)^{-1/2}(-2x) = \frac{-x}{\left(6-x^2-y^2\right)^{1/2}} = \frac{-x}{\sqrt{6-x^2-y^2}}; \quad f_x(1,1) = \frac{-x}{\sqrt{6-x^2-y^2}} = \frac{-1}{\sqrt{6-1^2-1^2}} = \frac{-1}{2}$$

c) The intersection of the plane $y=1$ and $z=\sqrt{6-x^2-y^2}$ is a curve.

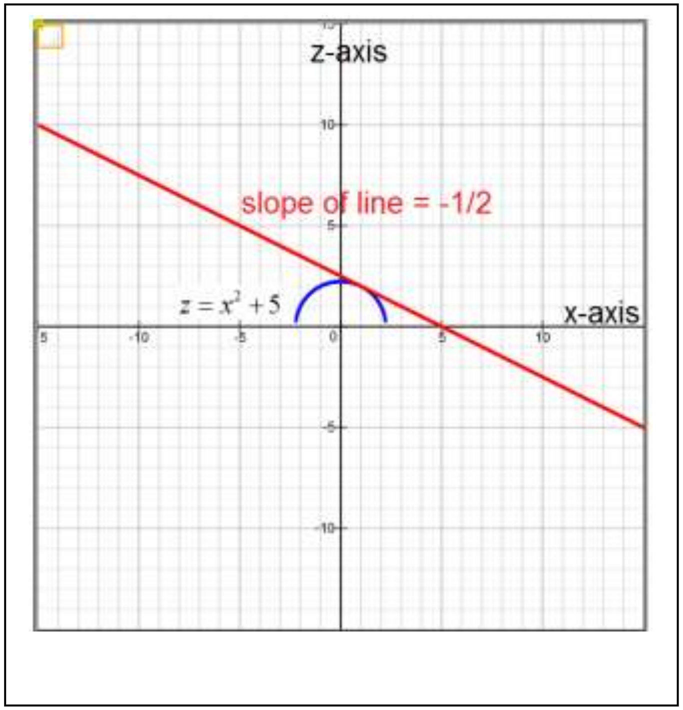
Find a vector-valued function $\mathbf{r}(t)$ representing the intersection.

$$z=\sqrt{6-x^2-y^2} = \sqrt{6-x^2-1^2} = \sqrt{5-x^2}$$

Corresponding Parametric Equations for $z=\sqrt{5-x^2}$: $x=t$ and $z=\sqrt{5-t^2}$

Intersection of the plane $y=1$ and $z=\sqrt{6-x^2-y^2}$ is the curve:

$$\mathbf{r}(t) = \left\langle x=t,\ y=1,\ z=\sqrt{5-t^2}\right\rangle = \left\langle t,1,\sqrt{5-t^2}\right\rangle$$



d) Graphical representation of $f_x(1,1) = \dfrac{-1}{2}$:

When $x=1$ and $y=1$, $z=\sqrt{6-x^2-y^2} = \sqrt{6-1^2-1^2} = 2$

Hence, $(1,1,2)$ is a point on the graph of $z=\sqrt{6-x^2-y^2}$.

$$\mathbf{r}(t) = \left\langle t,1,\sqrt{5-t^2}\right\rangle; \quad \mathbf{r}(1) = \langle 1,1,2\rangle; \quad \mathbf{r}'(t) = \left\langle 1,0,\frac{1}{2}\left(5-t^2\right)^{-1/2}(-2t)\right\rangle; \quad \text{and} \quad \mathbf{r}'(1) = \left\langle 1,0,-\frac{1}{2}\right\rangle$$

And $\mathbf{r}(1) + \mathbf{r}'(1) = \langle 1,1,2\rangle + \left\langle 1,0,\dfrac{-1}{2}\right\rangle = \langle 2,1,1.5\rangle$

$\mathbf{r}'(1)$ is a vector from $(1,1,2)$ to $(2,1,1.5)$.

The line containing the points $(1, 1, 2)$ and $(2, 1, 1.5)$ is a line in the plane $y=1$ and has a slope of -1/2 = $f_x(1,1)$.

Example 4: Let $z = f(x,y) = \sqrt{6 - x^2 - y^2}$

When $x = 1$ and $y = 1$, $z = \sqrt{6 - x^2 - y^2} = \sqrt{6 - 1^2 - 1^2} = 2$; Hence, the point $(1,1,2)$ is on the graph of $z = \sqrt{6 - x^2 - y^2}$.

$$f_y(x,y) = \frac{1}{2}\left(6 - x^2 - y^2\right)^{-1/2}(-2y) = \frac{-y}{\left(6 - x^2 - y^2\right)^{1/2}} = \frac{-y}{\sqrt{6 - x^2 - y^2}}; \quad f_y(1,1) = \frac{-y}{\sqrt{6 - x^2 - y^2}} = \frac{-1}{\sqrt{6 - 1^2 - 1^2}} = \frac{-1}{2}$$

c) The intersection of the plane $x = 1$ and $z = \sqrt{6 - x^2 - y^2}$ is a curve.

Find a vector-valued function $\mathbf{r}(t)$ representing the intersection.

$$z = \sqrt{6 - x^2 - y^2} = \sqrt{6 - 1^2 - y^2} = \sqrt{5 - y^2}$$

Corresponding Parametric Equations for $z = \sqrt{5 - y^2}$: $y = t$ and $z = \sqrt{5 - t^2}$

Intersection of the plane $x = 1$ and $z = \sqrt{6 - x^2 - y^2}$ is the curve:

$$\mathbf{r}(t) = \left\langle x = 1,\ y = t,\ z = \sqrt{5 - t^2} \right\rangle = \left\langle 1, t, \sqrt{5 - t^2} \right\rangle$$

d) Graphical representation of $f_y(1,1) = \frac{-1}{2}$:

When $x = 1$ and $y = 1$, $z = \sqrt{6 - x^2 - y^2} = \sqrt{6 - 1^2 - 1^2} = 2$;

Hence, (1,1,2) is a point on the graph of $z = \sqrt{6 - x^2 - y^2}$.

$$\mathbf{r}(t) = \left\langle 1, t, \sqrt{5 - t^2} \right\rangle; \quad \mathbf{r}(1) = \langle 1,1,2 \rangle; \quad \mathbf{r}'(t) = \left\langle 0, 1, \frac{1}{2}\left(5 - t^2\right)^{-1/2}(-2t) \right\rangle; \quad \text{and} \quad \mathbf{r}'(1) = \left\langle 0, 1, -\frac{1}{2} \right\rangle$$

And $\mathbf{r}(1) + \mathbf{r}'(1) = \langle 1,1,2 \rangle + \left\langle 0, 1, \frac{-1}{2} \right\rangle = \langle 1, 2, 1.5 \rangle$

$\mathbf{r}'(1)$ is a vector from $(1,1,2)$ to (1,2,1.5).

The line containing the points $(1, 1, 2)$ and (1,2, 1.5) is a line in the plane $x = 1$ and has a slope of -1/2 = $f_y(1,1)$.

Example 5: Let $z = f(x,y) = 4x - 15y + 1$

a) $f_x(x,y) = 4 - 0 + 0 = 4$ b) $f_y(x,y) = 0 - 15 + 0 = -15$ c) $f_x(1,1) = 4$ d) $f_y(1,1) = -15$

e) The intersection of the plane $x = 1$ and $z = 4x - 15y + 1$ is a curve.

Find a vector-valued function $\mathbf{r}(t)$ representing the intersection.

$z = 4(1) - 15y + 1 = -15y + 5$

Corresponding Parametric Equations for $z = -15y + 5$: $y = t$ and $z = -15t + 5$

Intersection of the plane $x = 1$ and $z = 4x - 15y + 1$ is the curve:

$$\mathbf{r}(t) = \langle x = 1,\ y = t,\ z = -15t + 5\rangle = \langle 1, t, -15t + 5\rangle$$

f) Graphical representation of $f_y(x,y) = 0 - 15 + 0 = -15$:

When $x = 1$ and $y = 1$, $z = 4x - 15y + 1 = -10$.

$\mathbf{r}(t) = \langle 1, t, -15t + 5\rangle$ and $\mathbf{r}(1) = \langle 1, 1, -10\rangle$

$\mathbf{r}'(t) = \langle 0, 1, -15\rangle$ and $\mathbf{r}'(1) = \langle 0, 1, -15\rangle$

Note: At $(1,1,-10)$, $t = 1$. So $\mathbf{r}(1) = \langle 1, 1, -10\rangle$ and $\mathbf{r}'(1) = \langle 0, 1, -15\rangle$.

And $\mathbf{r}(1) + \mathbf{r}'(1) = \langle 1, 1, -10\rangle + \langle 0, 1, -15\rangle = \langle 1, 2, -25\rangle$

$\mathbf{r}'(1)$ is a vector from $(1,1,-10)$ and (1, 2, -25).

The line containing the points $(1,1,-10)$ and (1, 2, -25) is a line in the plane $x = 1$ and has a slope of -15 = $f_y(1,1)$.

Example 6: Let $z = f(x,y) = 4x - 15y + 1$

a) $f_x(x,y) = 4 - 0 + 0 = 4$; b) $f_y(x,y) = 0 - 15 + 0 = -15$; c) $f_x(1,1) = 4$; d) $f_y(1,1) = -15$

e) The intersection of the plane $y = 1$ and $z = 4x - 15y + 1$ is a curve.

Find a vector-valued function $\mathbf{r}(t)$ representing the intersection.

$z = 4x - 15(1) + 1 = 4x - 14$

Corresponding Parametric Equations for $z = 4x - 14$: $x = t$ and $z = 4t - 14$

Intersection of the plane $y = 1$ and $z = 4x - 15y + 1$ is the curve:

$$\mathbf{r}(t) = \langle x = t,\ y = 1,\ z = 4t - 14 \rangle = \langle t, 1, 4t - 14 \rangle$$

f) Graphical representation of $f_x(x,y) = 4 - 0 + 0 = 4$:

Note when $x = 1$ and $y = 1$, $z = 4x - 15y + 1 = 4(1) - 15(1) + 1 = -10$;

Hence, the point (1,1,-10) is on the graph of $z = 4x - 15y + 1$.

$\mathbf{r}(t) = \langle t, 1, 4t - 14 \rangle$ and $\mathbf{r}(1) = \langle 1, 1, -10 \rangle$

$\mathbf{r}'(t) = \langle 1, 0, 4 \rangle$ and $\mathbf{r}'(1) = \langle 1, 0, 4 \rangle$

And $\mathbf{r}(1) + \mathbf{r}'(1) = \langle 1, 1, -10 \rangle + \langle 1, 0, 4 \rangle = \langle 2, 1, -6 \rangle$

$\mathbf{r}'(1)$ is a vector from $(1,1,-10)$ and $(2,1,-6)$.

The line containing the points $(1,1,-10)$ and $(2,1,-6)$ is a line in the plane $y = 1$ and has a slope of $4 = f_x(1,1)$.

Example 7: Let $z = x^4\sqrt{y} = x^4 y^{1/2}$

a) $f_x(x,y) = \sqrt{y}\left(4x^3\right) = 4\sqrt{y}x^3$

b) $f_y(x,y) = \left(x^4\right)\left(\frac{1}{2}y^{-1/2}\right)$

Example 8: Let $z = 5x^3 e^{xy}$

a) $f_x(x,y) = \left(5x^3\right)\left[\frac{\partial}{\partial x}\left(e^{xy}\right)\right] + \left(e^{xy}\right)\left[\frac{\partial}{\partial x}\left(5x^3\right)\right] = \left(5x^3\right)\left[e^{xy}\cdot y\right] + \left(e^{xy}\right)\left[15x^2\right]$

b) $f_y(x,y) = \left(5x^3\right)\frac{\partial}{\partial x}\left(e^{xy}\right) = \left(5x^3\right)\left[e^{xy}\cdot x\right]$

Example 9: Let $z = \ln\left(x^2 - 6y + 1\right)$

a) $f_x\left(x, y\right) = \dfrac{1}{x^2 - 6y + 1}\left(2x\right) = \dfrac{2x}{x^2 - 6y + 1}$

b) $f_y\left(x, y\right) = \dfrac{1}{x^2 - 6y + 1}\left(-6\right) = \dfrac{-6}{x^2 - 6y + 1}$

Example 10: Let $z = \sqrt{x + 3y^3} = \left(x + 3y^3\right)^{1/2}$

a) $f_x\left(x, y\right) = \dfrac{1}{2}\left(x + 3y^3\right)^{-1/2}\dfrac{\partial}{\partial x}\left(x + 3y^3\right) = \dfrac{1}{2}\left(x + 3y^3\right)^{-1/2}(1) = \dfrac{1}{2}\left(x + 3y^3\right)^{-1/2} = \dfrac{1}{2\left(x + 3y^3\right)^{1/2}}$

b) $f_y\left(x, y\right) = \dfrac{1}{2}\left(x + 3y^3\right)^{-1/2}\dfrac{\partial}{\partial y}\left(x + 3y^3\right) = \dfrac{1}{2}\left(x + 3y^3\right)^{-1/2}\left(9y^2\right) = \dfrac{9y^2}{2\left(x + 3y^3\right)^{1/2}}$

Example 11: $z = \dfrac{x-4y}{x+3y}$

a) $f_x(x,y) = \dfrac{(x+3y)\frac{\partial}{\partial x}(x-4y) - (x-4y)\frac{\partial}{\partial x}(x+3y)}{(x+3y)^2} = \dfrac{(x+3y)(1)-(x-4y)(1)}{(x+3y)^2} = \dfrac{7y}{(x+3y)^2}$

b) $f_y(x,y) = \dfrac{(x+3y)\frac{\partial}{\partial y}(x-4y) - (x-4y)\frac{\partial}{\partial y}(x+3y)}{(x+3y)^2} = \dfrac{(x+3y)(-4)-(x-4y)(3)}{(x+3y)^2} = \dfrac{-7x}{(x+3y)^2}$

Example 12: Let $z = \sin(x+8y)$

a) $f_x(x,y) = \cos(x+8y)\cdot\dfrac{\partial}{\partial x}(x+8y) = \cos(x+8y)\cdot(1) = \cos(x+8y)$

b) $f_y(x,y) = \cos(x+8y)\cdot\dfrac{\partial}{\partial y}(x+8y) = \cos(x+8y)\cdot(8) = 8\cos(x+8y)$

Example 13: Let $z = \tan\left(x+9y\right)$

a) $f_x\left(x,y\right)=\sec^2\left(x+y\right)\cdot\dfrac{\partial}{\partial x}\left(x+9y\right)=\sec^2\left(x+y\right)\cdot(1)=\sec^2\left(x+y\right)$

b) $f_y\left(x,y\right)=\sec^2\left(x+y\right)\cdot\dfrac{\partial}{\partial y}\left(x+9y\right)=\sec^2\left(x+y\right)\cdot(9)=9\sec^2\left(x+y\right)$

Exmaple 14: Let $f(x,y,z)=\cos(x-5y+3z)$

a) $f_x(x,y,z)=-\sin(x-5y+3z)\cdot\frac{\partial}{\partial x}(x-5y+3z)=-\sin(x-5y+3z)\cdot(1)=-\sin(x-5y+3z)$

b) $f_y(x,y,z)=-\sin(x-5y+3z)\cdot\frac{\partial}{\partial y}(x-5y+3z)=-\sin(x-5y+3z)\cdot(-5)=5\sin(x-5y+3z)$

c) $f_z(x,y,z)=-\sin(x-5y+3z)\cdot\frac{\partial}{\partial z}(x-5y+3z)=-\sin(x-5y+3z)\cdot(3)=-3\sin(x-5y+3z)$

Example 15: Let $w=\ln(3x^2-5y^3+z^4)$

a) $\frac{\partial w}{\partial x}=\frac{1}{3x^2-5y^3+z^4}(6x)=\frac{6x}{3x^2-5y^3+z^4}$

b) $\frac{\partial w}{\partial y}=\frac{1}{3x^2-5y^3+z^4}(-15y^2)=\frac{-15y^2}{3x^2-5y^3+z^4}$

c) $\frac{\partial w}{\partial z}=\frac{1}{3x^2-5y^3+z^4}(4z^3)=\frac{4z^3}{3x^2-5y^3+z^4}$

Example 16: Let $z = 4x^2 y^4$

a) $\dfrac{\partial z}{\partial x} = \left(4y^4\right)\left(2x\right) = 8xy^4$

b) $\dfrac{\partial z}{\partial y} = \left(4x^2\right)\left(4y^3\right) = 16x^2 y^3$

c) $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(8xy^4\right) = 8y^4$

d) $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(16x^2 y^3\right) = \left(16x^2\right)\left(3y^2\right) = 48x^2 y^2$

e) $\dfrac{\partial^2 z}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial x}\right) = \dfrac{\partial}{\partial y}\left(8xy^4\right) = \left(8x\right)\left(4y^3\right) = 32xy^3$

f) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(16x^2 y^3\right) = \left(16y^3\right)\left(2x\right) = 32xy^3$

Example 17: Let $z = e^{xy} - 5y$

$$a)\ \frac{\partial z}{\partial x} = \left(e^{xy}\right)(y) = ye^{xy}$$

$$b)\ \frac{\partial z}{\partial y} = \left(e^{xy}\right)(x) = xe^{xy}$$

$$c)\ \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(ye^{xy}\right) = (y)\left(e^{xy} \cdot y\right) = y^2 e^{xy}$$

$$d)\ \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(xe^{xy}\right) = (x)\left(e^{xy} \cdot x\right) = x^2 e^{xy}$$

$$e)\ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(ye^{xy}\right) = (y)\frac{\partial}{\partial y}\left(e^{xy}\right) + \left(e^{xy}\right)\frac{\partial}{\partial y}(y) = (y)\left(e^{xy} \cdot x\right) + \left(e^{xy}\right)(1) = xye^{xy} + e^{xy}$$

$$f)\ \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(xe^{xy3}\right) = (x)\frac{\partial}{\partial y}\left(e^{xy}\right) + \left(e^{xy}\right)\frac{\partial}{\partial y}(x) = (x)\left(e^{xy} \cdot y\right) + \left(e^{xy}\right)(1) = xye^{xy} + e^{xy}$$

18) The temperature at any point (x, y) on a metal plate is $T(x, y) = x^2 + 3y^2 - 5$ degrees.

Assume that x and y are measured in feet.

a) Find the rate at which the temperature changes with respect to distance

if we start at the point (1,4) and move to the right (parallel to x-axis).

$T_x(x, y) = 2x$

$T_x(1, 4) = 2$

Hence, temperature changes at a rate of 2 degrees per foot.

b) Find the rate at which the temperature changes with respect to distance

if we start at the point (1,4) and move up (parallel to y-axis).

$T_y(x, y) = 6x$

$T_x(1, 4) = 24$

Hence, temperature changes at a rate of 24 degrees per foot.