Calculus III   Section 13.4   Differentials

Example 1: Let $z = 2x^4y - 8x^2y^3$

$$\frac{\partial z}{\partial x} = (2y)(4x^3) - (8y^3)(2x) = 8x^3y - 16xy^3$$

$$\frac{\partial z}{\partial y} = (2x^4)(1) - (8x^2)(3y^2) = 2x^4 - 24x^2y^2$$

$$dz = \frac{\partial z}{\partial x}\cdot dx + \frac{\partial z}{\partial y}\cdot dy$$

$$dz = (8x^3y - 16xy^3)dx + (2x^4 - 24x^2y^2)dy$$

Example 2: Let $w=\frac{x+y}{z-3y}$

$$\frac{\partial w}{\partial x}=\frac{(z-3y)\frac{\partial}{\partial x}(x+y)-(x+y)\frac{\partial}{\partial x}(z-3y)}{(z-3y)^2}=\frac{(z-3y)(1)-(x+y)(0)}{(z-3y)^2}=\frac{(z-3y)}{(z-3y)^2}=\frac{1}{(z-3y)}$$

$$\frac{\partial w}{\partial y}=\frac{(z-3y)\frac{\partial}{\partial y}(x+y)-(x+y)\frac{\partial}{\partial y}(z-3y)}{(z-3y)^2}=\frac{\partial w}{\partial y}=\frac{(z-3y)(1)-(x+y)(-3)}{(z-3y)^2}=\frac{z+3x}{(z-3y)^2}$$

$$\frac{\partial w}{\partial z}=\frac{(z-3y)\frac{\partial}{\partial z}(x+y)-(x+y)\frac{\partial}{\partial z}(z-3y)}{(z-3y)^2}=\frac{(z-3y)(0)-(x+y)(1)}{(z-3y)^2}=\frac{-(x+y)}{(z-3y)^2}$$

$$dw=\frac{\partial w}{\partial x}\cdot dx+\frac{\partial w}{\partial y}\cdot dy+\frac{\partial w}{\partial z}\cdot dz$$

$$dw=\left(\frac{1}{(z-3y)}\right)dx+\left(\frac{z+3x}{(z-3y)^2}\right)dy+\left(\frac{-(x+y)}{(z-3y)^2}\right)dz$$

Example 3: Let $w = e^{y}\cos x + z^{2}$

$$\frac{\partial w}{\partial x} = e^{y}\left(-\sin x\right) = -e^{y}\sin x$$

$$\frac{\partial w}{\partial y} = \left(\cos x\right)e^{y}$$

$$\frac{\partial w}{\partial z} = 2z$$

$$dw = \frac{\partial w}{\partial x}\cdot dx + \frac{\partial w}{\partial y}\cdot dy + \frac{\partial w}{\partial z}\cdot dz$$

$$dw = \left(-e^{y}\sin x\right)dx + \left(\left(\cos x\right)e^{y}\right)dy + \left(2z\right)dz$$

Example 4: Let $z = f(x, y) = x^2 + y^2$

Find Δz when x changes from $x = 2$ to $x = 2.1$; and y changes from $y = 1$ to $y = 1.05$.

$$f(2,1) = (2)^2 + (1)^2 = 5$$

$$f(2.1, 1.05) = (2.1)^2 + (1.05)^2 = 5.5125$$

$$\Delta z = f(2.1, 1.05) - f(2,1) = 5.5125 - 5 = 0.5125$$

$$\frac{\partial z}{\partial x} = 2x \quad \text{and} \quad \frac{\partial z}{\partial y} = 2y$$

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy = 2x \cdot dx + 2y \cdot dy$$

We will dz by letting $dx = \Delta x = 2.1 - 2 = 0.1$;

$dx = \Delta y = 1.05 - 1 = 0.05$; $x = 2$; and $y = 1$.

$$dz = \frac{\partial z}{\partial x} \cdot dx + \frac{\partial z}{\partial y} \cdot dy = 2x \cdot dx + 2y \cdot dy = 2(2)(0.1) + 2(1)(0.05) = 0.5$$

Note: $\Delta z \approx dz$

Example 5: Let $T(x,y)=\dfrac{10}{1+x^2+y^2}$ be the temperature at the point

(x,y) on a metal plate, where T is measured in $_\circ C$ and x,y in meters.

Find ΔT when x changes from $x=2$ to $x=2.1$; and y changes from $y=1$ to $y=1.05$.

$$T(2,1)=\frac{10}{1+(2)^2+(1)^2}=\frac{10}{6}=\frac{5}{3}=1.66666666666 \text{ and } T(2.1,1.05)=\frac{10}{1+(2.1)^2+(1.05)^2}=1.53550863723$$

$$\Delta T=T(2.1,1.05)-T(2,1)=5.5125-5=1.53550863723-1.6666666666= -0.13115802942$$

$$\frac{\partial T}{\partial x}=\frac{\left(1+x^2+y^2\right)(0)-(10)(2x)}{\left(1+x^2+y^2\right)^2}=\frac{-20x}{\left(1+x^2+y^2\right)^2}; \quad \text{and } \frac{\partial T}{\partial x}=\frac{\left(1+x^2+y^2\right)(0)-(10)(2y)}{\left(1+x^2+y^2\right)^2}=\frac{-20y}{\left(1+x^2+y^2\right)^2}$$

$$dz=\frac{\partial z}{\partial x}\cdot dx+\frac{\partial z}{\partial y}\cdot dy=\frac{-20x}{\left(1+x^2+y^2\right)^2}\cdot dx+\frac{-20y}{\left(1+x^2+y^2\right)^2}\cdot dy$$

We will dz by letting $dx=\Delta x=2.1-2=0.1$; $dx=\Delta y=1.05-1=0.05$; $x=2$; and $y=1$.

$$dz=\frac{-20(2)}{\left(1+2^2+1^2\right)^2}\cdot(0.1)+\frac{-20(1)}{\left(1+2^2+1^2\right)^2}\cdot(0.05)=-0.16666666666666$$

$Note$: $\Delta z\approx dz$