13.5 : Chain Rule

Example 1

$$z = f(x, y)$$

$z(t) = \dfrac{x^2}{y}$ z is a function of x and y

$x = t^2$ x is a function of t

$y = t+1$ y is a function of t

Find $\dfrac{dz}{dt} = ?$

$$z(t) = \frac{x^2}{y} = \frac{(t^2)^2}{t+1} = \frac{t^4}{t+1}$$

$$\frac{dz}{dt} = \frac{(t+1)(4t^3) - t^4\cdot(1)}{(t+1)^2} = \frac{3t^4 + 4t^3}{(t+1)^2}$$

Chain Rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$z = \frac{x^2}{y} = \frac{1}{y} \cdot x^2$$

$$\frac{\partial z}{\partial x} = \left(\frac{1}{y}\right)(2x) = \frac{2x}{y}$$

$$= \frac{2t^2}{t+1}$$

$x = t^2$

$$\frac{dx}{dt} = 2t$$

$y = t+1$

$$\frac{dy}{dt} = 1$$

$$\frac{\partial z}{\partial y} = (x^2)\left(-\frac{1}{y^2}\right) = -\frac{x^2}{y^2}$$

$$= -\frac{(t^4)}{(t+1)^2}$$

$$\frac{dz}{dt} = \left(\frac{2t^2}{t+1}\right)(2t) + \frac{-(t^4)}{(t+1)^2} \cdot (1)$$

$$= \frac{4t^3}{(t+1)} \cdot \frac{(t+1)}{(t+1)} + \frac{-t^4}{(t+1)^2}$$

$$= \frac{4t^4 + 4t^3 - t^4}{(t+1)^2}$$

$$= \frac{3t^4 + 4t^3}{(t+1)^2}$$

Example 2

$w = \cos(x - y)$; $x = t^2$; $y = 1$

Find $\frac{dw}{dt} = ?$; $\frac{dx}{dt} = 2t$; $\frac{dy}{dt} = 0$

$$\frac{\partial w}{\partial x} = -\sin(x - y)(1) = -\sin(x-y) = -\sin(t^2-1)$$

$$\frac{\partial w}{\partial y} = -\sin(x-y)(-1) = \sin(x-y) = \sin(t^2-1)$$

Chain Rule: $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$

$$= \cancel{(-\sin(x-y))(2t)}$$

$$= -\sin(t^2-1)(2t) + \sin(t^2-1)(0)$$

$$= -2t\cdot\sin(t^2-1)$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

$$= (8t^2)(2t) + (4t^3+4)(2) + (t^4+8t)(0)$$

Example 3

$$w = x \cdot y^2 + x^2 \cdot z + y \cdot z^2$$

$x = t^2$	$y = 2t$	$z = 2$
$\frac{dx}{dt} = 2t$	$\frac{dy}{dt} = 2$	$\frac{dz}{dt} = 0$

Find $\frac{dw}{dt}$?

$$\frac{\partial w}{\partial x} = (y^2)(1) + (z)(2x) + 0 = y^2 + 2xz = (2t)^2 + 2(t^2)(2)$$

$$= 4t^2 + 4t^2 = 8t^2$$

$$\frac{\partial w}{\partial y} = (x)(2y) + 0 + (z^2)(1)$$

$$= 2xy + z^2 = 2(t^2)(2t) + 4 = 4t^3 + 4$$

$$\frac{\partial w}{\partial z} = (x^2)(1) + (y)(2z) = x^2 + 2yz$$

$$= t^4 + 2(2t)(2) = t^4 + 8t$$

$$x^2 + y^2 = 0$$

Find y'

$$2x + 2y \cdot \frac{dy}{dx} = 0$$

$$2x + 2y \cdot y' = 0$$

$$2y \cdot y' = -2x$$

$$y' = \frac{-2x}{2y} = \frac{-x}{y}$$

Example 4

$$x \cdot y^2 + y^3 = \cancel{\;}\,3x \quad \text{Find } y' = \frac{dy}{dx}$$

$$\underline{(x)(2y \cdot y') + y^2 \cdot (1)} + \underline{3y^2 \cdot y'} = \cancel{\;}\,3.$$

$$y' \cdot (2xy + 3y^2) = 3 - y^2$$

$$y' = \frac{3 - y^2}{2xy + 3y2}$$

Form $F(x,y) = x \cdot y^2 + y^3 - 3x = 0$

$$F_x = y^2 + \cancel{3y^2} - 3$$

$$F_y = \cancel{\;}\,x \cdot (2y) + 3y^2$$

$$-F_x = 3 - y^2$$

$$y' = \frac{-F_x}{F_y} = \frac{3 - y^2}{2xy + 3y2}$$

Example 5

$$x^2 \cdot y^2 + 4y^2 = 6x \qquad \text{Find } y' = \frac{dy}{dx}$$

Form $F(x, y) = x^2 \cdot y^2 + 4y^2 - 6x = 0$

$$F_x = (y^2)(2x) + 0 - 6 = 2xy^2 - 6$$

$$-F_x = -2xy^2 + 6$$

$$F_y = (x^2)(2y) + 8y = 2x^2y + 8y$$

$$y' = \frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-2xy^2 + 6}{2x^2y + 8y}$$

Example 6

$$xz + yz + xy = 0$$

$$F(x,y,z) = xz + yz + xy = 0$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$$

$$= \frac{-z-y}{x+y}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$$

$$= \frac{-z-x}{x+y}$$