Section 13.5 Chain Rule

Example 1: Let $z = f\left(x, y\right) = \dfrac{x^2}{y}$; and $x(t) = t^2$ and $y(t) = t+1$

Find $\dfrac{dz}{dt}$.

Method 1 (Direct Method):

$$z = \frac{x^2}{y} = \frac{\left(t^2\right)^2}{t+1} = \frac{t^4}{t+1}$$

$$\frac{dz}{dt} = \frac{(t+1)\cdot D_t\left(t^4\right) - \left(t^4\right)\cdot D_t\left(t+1\right)}{(t+1)^2} = \frac{(t+1)\cdot\left(4t^3\right) - \left(t^4\right)\cdot(1)}{(t+1)^2} = \frac{3t^4+4t^3}{(t+1)^2}$$

Example 1 (con't): Let $z = f\left(x, y\right) = \dfrac{x^2}{y}$; and $x(t) = t^2$ and $y(t) = t + 1$

Method 2 (Chain Rule Method):

$$z = \frac{x^2}{y}$$

$$\frac{\partial z}{\partial x} = \frac{y \cdot \frac{\partial}{\partial x}\left(x^2\right) - x^2 \cdot \frac{\partial}{\partial x}\left(y\right)}{y^2} = \frac{y \cdot \left(2x\right) - x^2 \cdot \left(0\right)}{y^2} = \frac{2xy}{y^2} = \frac{2x}{y}$$

$$\frac{\partial z}{\partial y} = \frac{y \cdot \frac{\partial}{\partial y}\left(x^2\right) - x^2 \cdot \frac{\partial}{\partial y}\left(y\right)}{y^2} = \frac{y \cdot \left(0\right) - x^2 \cdot \left(1\right)}{y^2} = \frac{-x^2}{y^2}$$

$\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 1$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \left(\frac{2x}{y}\right)\left(2t\right) + \left(\frac{-x^2}{y^2}\right)\left(1\right)$$

$$\frac{dz}{dt} = \left(\frac{2x}{y}\right)\left(2t\right) + \left(\frac{-x^2}{y^2}\right) = \left(\frac{2t^2}{t+1}\right)\left(2t\right) + \left(\frac{-(t^2)^2}{(t+1)^2}\right) = \frac{3t^4 + 4t^3}{(t+1)^2}$$

Example 2: Let $w = f\left(x, y\right) = \cos\left(x - y\right)$; and $x(t) = t^2$ and $y(t) = 1$

Find $\dfrac{dw}{dt}$.

Method 1 (Direct Method):

$$w = \cos\left(x - y\right) = \cos\left(t^2 - 1\right)$$

$$\frac{dw}{dt} = -\sin\left(t^2 - 1\right)\left(2t\right) = -2t\sin\left(t^2 - 1\right)$$

Method 2 (Chain Rule):

$$w = \cos(x - y)$$

$$\frac{\partial w}{\partial x} = -\sin(x - y)\cdot\frac{\partial}{\partial x}(x - y) = -\sin(x - y)\cdot\left(1\right) = -\sin(x - y)$$

$$\frac{\partial w}{\partial y} = -\sin(x - y)\cdot\frac{\partial}{\partial y}(x - y) = -\sin(x - y)\cdot\left(-1\right) = \sin(x - y)$$

$\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 0$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} = \left(-\sin(x - y)\right)\left(2t\right) + \left(\sin(x - y)\right)\left(0\right)$$

$$\frac{dw}{dt} = -2t\sin(x - y) = -2t\sin(t^2 - 1)$$

Example 3: Let $w = xy^2 + x^2 z + yz^2$; and $x(t) = t^2$; $y(t) = 2t$ and $z(t) = 2$

Find $\dfrac{dw}{dt}$.

Method 1 (Direct Method):

$$w = xy^2 + x^2 z + yz^2 = w = \left(t^2\right)\left(2t\right)^2 + \left(t^2\right)^2\left(2\right) + \left(2t\right)\left(2\right)^2 = 4t^4 + 2t^4 + 8t = 6t^4 + 8t$$

$$\frac{dw}{dt} = 24t^3 + 8$$

Method 2 (Chain Rule):

$$w = xy^2 + x^2 z + yz^2$$

$$\frac{\partial w}{\partial x} = y^2 + 2xz; \qquad \frac{\partial w}{\partial y} = 2xy + z^2; \qquad \frac{\partial w}{\partial z} = x^2 + 2yz$$

$$\frac{dx}{dt} = 2t \qquad \text{and} \qquad \frac{dy}{dt} = 2 \qquad \text{and} \qquad \frac{dz}{dt} = 0$$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt} = \left(y^2 + 2xz\right)\left(2t\right) + \left(2xy + z^2\right)\left(2\right) + \left(x^2 + 2yz\right)\left(0\right)$$

$$\frac{dw}{dt} = \left(y^2 + 2xz\right)\left(2t\right) + \left(2xy + z^2\right)\left(2\right) = \left(\left(2t\right)^2 + 2\left(t^2\right)\left(2\right)\right)\left(2t\right) + \left(2\left(t^2\right)\left(2t\right) + \left(2\right)^2\right)\left(2\right) = 24t^3 + 8$$

Example 4: $x^2 + y^2 = 0$ Find $y' = \dfrac{dy}{dx}$.

Method 1 (Implicit Differentiation):

$$2x \cdot \frac{dx}{dx} + 2y \cdot \frac{dy}{dx} = 0$$

$$2x + 2y \cdot y' = 0 \qquad \Rightarrow \qquad 2yy' = -2x \qquad \Rightarrow \qquad y' = \frac{-2x}{2y} = \frac{-x}{y}$$

Method 2 (Partial Derivative):

Let $F(x, y) = x^2 + y^2$

$F_x = 2x$ and $F_y = 2y$

$$y' = \frac{-F_x}{F_y} = \frac{-2x}{2y} = -\frac{x}{y}$$

Example 5: $xy^2 + y^3 = 3x$ Find $y' = \dfrac{dy}{dx}$.

Method 1 (Implicit Differentiation):

$$x \cdot D_x\left(y^2\right) + \left(y^2\right)D_x\left(x\right) + 3y^2 \cdot \frac{dy}{dx} = 3$$

$$x \cdot \left(2y \cdot \frac{dy}{dx}\right) + \left(y^2\right)\left(1 \cdot \frac{dx}{dx}\right) + 3y^2 \cdot \frac{dy}{dx} = 3$$

$$2xyy' + y^2 + 3y^2 y' = 3$$

$$2xyy' + 3y^2 y' = 3 - y^2$$

$$y'\left(2xy + 3y^2\right) = 3 - y^2$$

$$y' = \frac{3 - y^2}{\left(2xy + 3y^2\right)}$$

Method 2 (Partial Derivative):

Let $F(x, y) = xy^2 + y^3 - 3x = 0$

$F_x = y^2 - 3$ and $F_y = 2xy + 3y^2$

$$y' = \frac{-F_x}{F_y} = \frac{-\left(y^2 - 3\right)}{2xy + 3y^2} = \frac{3 - y^2}{2xy + 3y^2}$$

Example 6: $x^2y^2+4y^2=6x$ Find $y'=\dfrac{dy}{dx}$.

Method 1 (Implicit Differentiation):

$$x^2\cdot D_x\left(y^2\right)+\left(y^2\right)D_x\left(x^2\right)+8y\cdot\frac{dy}{dx}=6$$

$$x^2\cdot\left(2y\cdot\frac{dy}{dx}\right)+\left(y^2\right)\left(2x\cdot\frac{dx}{dx}\right)+8y\cdot\frac{dy}{dx}=6$$

$$2x^2yy'+2xy^2+8yy'=6$$

$$y'\left(2x^2y+8y\right)=6-2xy^2$$

$$y'=\frac{6-2xy^2}{\left(2x^2y+8y\right)}$$

Method 2 (Partial Derivative):

Let $F(x,y)=x^2y^2+4y^2-6x=0$

$F_x=2xy^2-6$ and $F_y=2x^2y+8y$

$$y'=\frac{-F_x}{F_y}=\frac{-\left(2xy^2-6\right)}{2x^2y+8y}=\frac{6-2xy^2}{2x^2y+8y}$$

Example 7: $xz + yz + xy = 0$ Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Let $F(x, y, z) = xz + yz + xy = 0$

$F_x = z + y$ and $F_y = z + x$ and $F_z = x + y$

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(z+y)}{x+y} = \frac{-z-y}{x+y}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(z+x)}{x+y} = \frac{-z-x}{x+y}$$