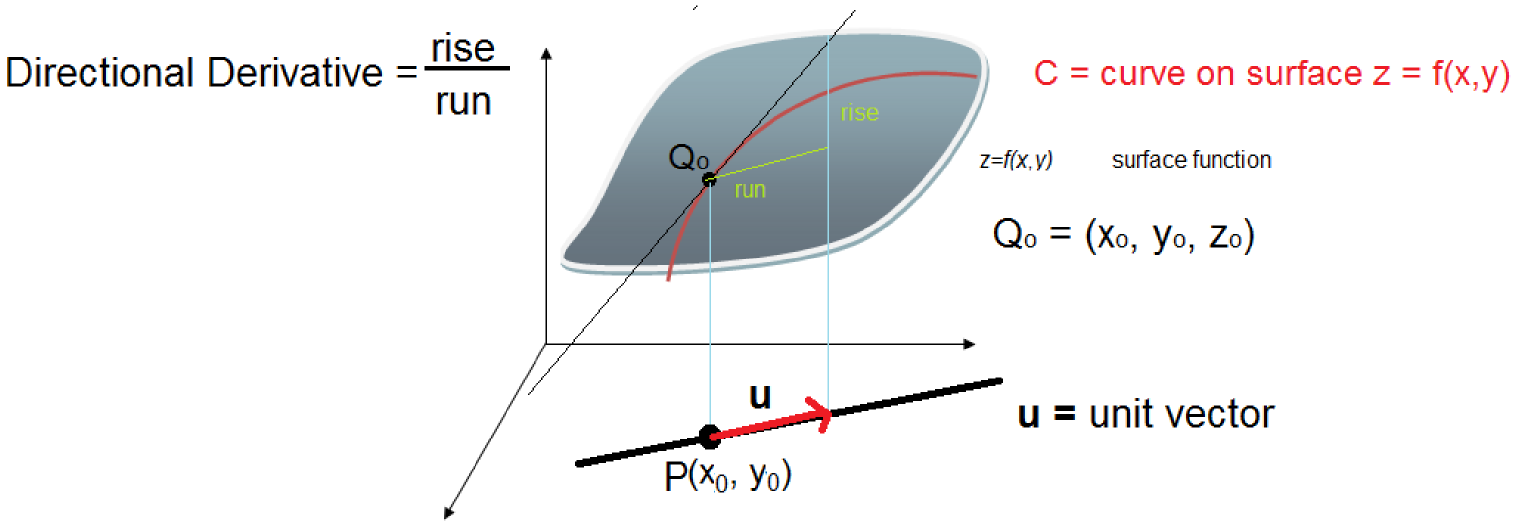
Section 13.6 Directional Derivative

Illustration 1



$D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} =$ directional derivative of $f(x, y)$ at $P(x_0, y_0)$ in the direction of $\mathbf{u}$.

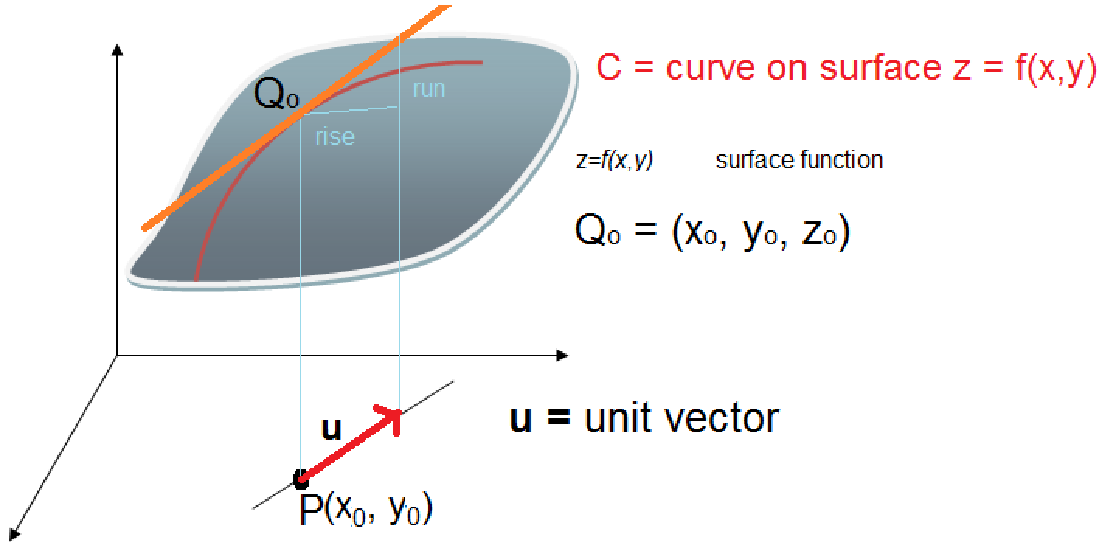
Note: a) When $\mathbf{u} = \langle 1,0 \rangle$, $D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 1,0 \rangle = f_x(x_0, y_0)$

b) When $\mathbf{u} = \langle 0,1 \rangle$, $D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 0,1 \rangle = f_y(x_0, y_0)$

Illustration 2

Directional Derivative

$= \dfrac{\text{rise}}{\text{run}}$



$D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} =$ directional derivative of $f(x, y)$ at $P(x_0, y_0)$ in the direction of $\mathbf{u}$.

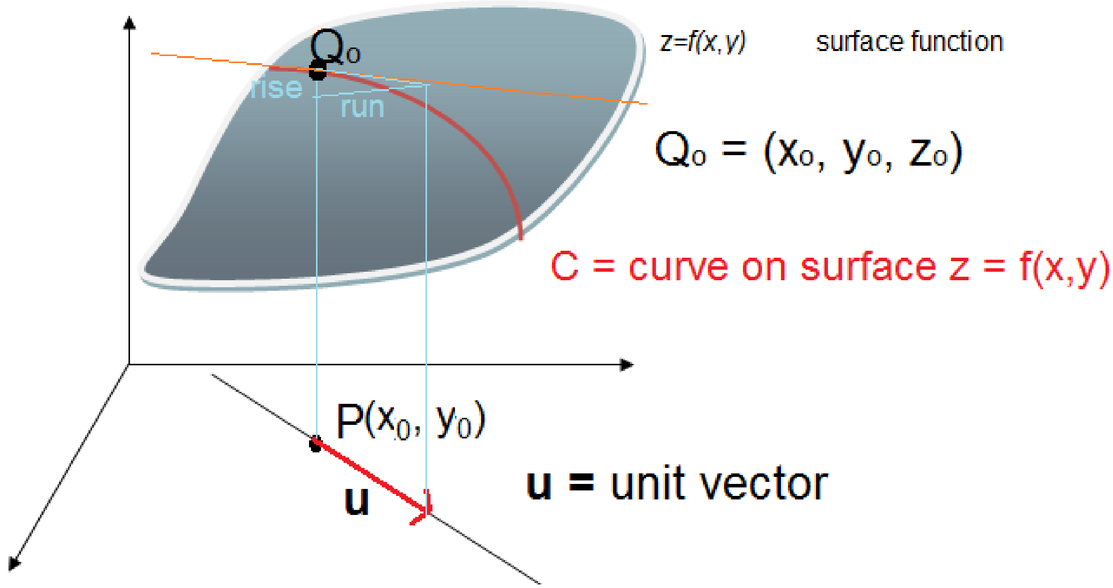
Note: a) When $\mathbf{u} = \langle 1, 0 \rangle$, $D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 1, 0 \rangle = f_x(x_0, y_0)$

b) When $\mathbf{u} = \langle 0, 1 \rangle$, $D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 0, 1 \rangle = f_y(x_0, y_0)$

Illustration 3

Directional Derivative

$$= \frac{\text{rise}}{\text{run}}$$



$D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} =$ directional derivative of $f(x, y)$ at $P(x_0, y_0)$ in the direction of $\mathbf{u}$.

Note: a) When $\mathbf{u} = \langle 1,0 \rangle$, $D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 1,0 \rangle = f_x(x_0, y_0)$

b) When $\mathbf{u} = \langle 0,1 \rangle$, $D_{\mathbf{u}} f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \mathbf{u} = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle \cdot \langle 0,1 \rangle = f_y(x_0, y_0)$

Example 1: Let $f(x, y) = 4 - x^2 - \frac{1}{4}y^2$

Find the directional derivative for $f(x, y)$ at the point $P(1, 2, 2)$ in the direction of the unit vector $\mathbf{u} = \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$.

directional derivative = $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \left\langle -2x, -\frac{1}{2}y \right\rangle$$

$$\nabla f(1, 2) = \langle f_x, f_y \rangle = \left\langle -2(1), -\frac{1}{2}(2) \right\rangle = \langle -2, -1 \rangle$$

directional derivative = $D_{\mathbf{u}} f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = \langle -2, -1 \rangle \cdot \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = -1 - \frac{\sqrt{3}}{2}$

Example 2: Let $f(x, y) = x^2 \sin 2y$

Find the directional derivative for $f(x, y)$ at the point $P\left(1, \frac{\pi}{2}, 0\right)$ in the direction of $\mathbf{v} = \langle 3, -4 \rangle$.

Let $\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 3, -4 \rangle}{\sqrt{(3)^2 + (-4)^2}} = \frac{\langle 3, -4 \rangle}{5} = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$; Note: $\mathbf{u}$ is a unit vector.

directional derivative = $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 2x \sin 2y, 2x^2 \cos 2y \rangle$$

$$\nabla f\left(1, \frac{\pi}{2}\right) = \langle f_x, f_y \rangle = \langle 2(1) \sin 2(\pi/2), 2(1)^2 \cos 2(\pi/2) \rangle = \langle 0, -2 \rangle$$

directional derivative = $D_{\mathbf{u}} f(1, 2) = \nabla f\left(1, \frac{\pi}{2}\right) \cdot \mathbf{u} = \langle 0, -2 \rangle \cdot \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle = \frac{8}{5}$

Example 3: Let $f(x,y) = \sin 2x \cdot \cos y$

Find the directional derivative for $f\left(x,y\right)$ at the point $P(\pi,0,0)$ in the direction from $P\left(\pi,0\right)$ to $Q\left(\pi/2,\pi\right)$.

Let $\mathbf{v} = \overrightarrow{PQ} = \left\langle \pi/2-\pi, \pi-0 \right\rangle = \left\langle -\pi/2, \pi \right\rangle$.

unit vector $\mathbf{u} = \dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \dfrac{\left\langle -\pi/2,\pi \right\rangle}{\left\|\left\langle -\pi/2,\pi \right\rangle\right\|} = \dfrac{\left\langle -\pi/2,\pi \right\rangle}{\sqrt{\left(-\pi/2\right)^2+\left(\pi\right)^2}} = \dfrac{\left\langle -\pi/2,\pi \right\rangle}{\left(\pi/2\right)\sqrt{5}} = \left\langle \dfrac{-\pi/2}{\left(\pi/2\right)\sqrt{5}}, \dfrac{\pi}{\left(\pi/2\right)\sqrt{5}} \right\rangle = \left\langle \dfrac{-1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$

directional derivative = $D_{\mathbf{u}} f\left(x,y\right) = \nabla f\left(x,y\right) \cdot \mathbf{u}$

$\nabla f\left(x,y\right) = \left\langle f_x, f_y \right\rangle = \left\langle 2\cos y \cdot \cos 2x, -\sin 2x \cdot \sin y \right\rangle$

$\nabla f\left(\pi,0\right) = \left\langle f_x, f_y \right\rangle = \left\langle 2\cos y \cdot \cos 2x, -\sin 2x \cdot \sin y \right\rangle = \left\langle 2,0 \right\rangle$

directional derivative = $D_{\mathbf{u}} f\left(\pi,0\right) = \nabla f\left(\pi,0\right) \cdot \mathbf{u} = \left\langle 2,0 \right\rangle \cdot \left\langle \dfrac{-1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle = -\dfrac{2}{\sqrt{5}}$

Maximum Directional Derivative at Given Point

Recall: The angle θ between two vectors $\mathbf{a}$ and $\mathbf{b}$ is as follows: $\mathbf{a}\cdot\mathbf{b}=\|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$

For directional derivative, $D_{\mathbf{u}}f(x,y)=\nabla f(x,y)\cdot\mathbf{u}=\|\nabla f(x,y)\|\cdot\|\mathbf{u}\|\cos\theta$

Since $\mathbf{u}$ is a unit vector, $\|\mathbf{u}\|=1$.

Hence, $D_{\mathbf{u}}f(x,y)=\nabla f(x,y)\cdot\mathbf{u}=\|\nabla f(x,y)\|\cdot 1\cdot\cos\theta=\|\nabla f(x,y)\|\cdot\cos\theta$

Since $-1\le\cos\theta\le 1$, the quantity $\|\nabla f(x,y)\|\cdot\cos\theta$ is a maximum when $\cos\theta=1$.

Therefore, the maximum directional derivative $D_{\mathbf{u}}f(x,y)$ for $f(x,y)$ at a given point P is $\|\nabla f(x,y)\|$.

Example 4: Let $f(x,y)=20-4x^2-y^2$

Find the maximum directional derivative for $f(x,y)$ at the point $P(2,-3,-5)$.

$$\nabla f(x,y)=\langle f_x, f_y\rangle=\langle -8x,-2y\rangle$$

$$\nabla f(2,-3)=\langle f_x, f_y\rangle=\langle -16,6\rangle$$

$$\|\nabla f(2,-3)\|=\sqrt{(-16)^2+(6)^2}=\sqrt{292}=17.09$$

Hence, maximum directional derivative for $f(x,y)$ at the point $P(2,-3,-5)=17.09$.

Example 5: Let $f(x, y) = y \cdot \cos\left(x - y\right)$

Find the maximum directional derivative for $f\left(x, y\right)$ at the point $P(0, \pi/3, \pi/6)$.

$$\nabla f\left(x, y\right) = \left\langle f_x, f_y \right\rangle = \left\langle -y\sin\left(x-y\right), y\sin\left(x-y\right) + \cos\left(x-y\right) \right\rangle$$

$$\nabla f\left(0, \pi/3\right) = \left\langle f_x, f_y \right\rangle = \left\langle \frac{\pi\sqrt{3}}{6}, \frac{-\pi\sqrt{3}}{6} + \frac{1}{2} \right\rangle = \left\langle 0.91, -0.41 \right\rangle$$

$$\left\|\nabla f\left(0, \pi/3\right)\right\| = \sqrt{\left(\frac{\pi\sqrt{3}}{6}\right)^2 + \left(\frac{-\pi\sqrt{3}}{6}\right)^2} = 0.998$$

Hence, maximum directional derivative for $f\left(x, y\right)$ at the point $P(0, \pi/3, \pi/6) = 0.998$.