Section 13.7 Tangent Plane and Normal Line

Space Curve and Chain Rule Review

Space Curve Example: $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle=\langle t,t^2,t^3\rangle;$

Tangent Vector: $\mathbf{r}'(t)=\langle x'(t),y'(t),z'(t)\rangle=\langle 1,2t,3t^2\rangle$

Chain Rule:

Let $F(x,y,z)=x^2+y^2+z^2;$ and $x(t)=t;$ $y(t)=t^2;$ $z(t)=t^3;$

By Chain Rule: $\dfrac{dF}{dt}=\dfrac{\partial F}{\partial x}\cdot\dfrac{dx}{dt}+\dfrac{\partial F}{\partial y}\cdot\dfrac{dy}{dt}+\dfrac{\partial F}{\partial z}\cdot\dfrac{dx}{dt}$

$$=\frac{\partial F}{\partial x}\cdot x'(t)+\frac{\partial F}{\partial y}\cdot y'(t)+\frac{\partial F}{\partial z}\cdot z'(t)$$

$$=\left\langle\frac{\partial F}{\partial x},\frac{\partial F}{\partial y},\frac{\partial F}{\partial z}\right\rangle\cdot\langle x'(t),y'(t),z'(t)\rangle$$

$$=\nabla F(x,y,z)\cdot\mathbf{r}'(t)$$

$$=\langle 2x,2y,2z\rangle\cdot\langle 1,2t,3t^2\rangle$$

Let $z = 9 - x^2 - y^2$ Surface Function in the shape of paraboloid.

Let $F(x, y, z) = 9 - x^2 - y^2 - z = 0$ Surface Function in the shape of paraboloid.

At the point $P(1,1,7)$, there are many space curves $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ on the paraboloid that pass through $P(1,1,7)$.

For each $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ passing through $P(1,1,7)$, there is a

corresponding tangent vector $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ passing through $P(1,1,7)$.

Note that all tangent vectors $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ passing through $P(1,1,7)$ lie on the tangent plane passing through $P(1,1,7)$.

Note that x, y, z are functions of t; hence, $F(x, y, z)$ is also a function of t.

By Chain Rule: $$\frac{dF(x,y,z)}{dt} = \frac{\partial F(x,y,z)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F(x,y,z)}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F(x,y,z)}{\partial z} \cdot \frac{dx}{dt} = 0$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot x'(t) + \frac{\partial F}{\partial y} \cdot y'(t) + \frac{\partial F}{\partial z} \cdot z'(t) = 0$$

$$\frac{dF}{dt} = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \langle x'(t), y'(t), z'(t) \rangle = 0$$

Hence, $\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$

Meaning, $\nabla F(x, y, z) = \langle -2x, -2y, -1 \rangle$ is orthogonal to each tangent vector $\mathbf{r}'(t)$ passing through $P(1,1,7)$.

Therefore, $\nabla F(x, y, z) = \langle -2x, -2y, -1 \rangle$ is orthogonal to the tangent plane at $P(1,1,7)$.

$\nabla F(1,1,7) = \langle -2(1), -2(1), -1 \rangle = \langle -2, -2, -1 \rangle$

Equation of tangent plane is $-2(x-1) - 2(y-1) - 1(z-7) = 0$

Equation of normal line is $x - 1 = -2t$; $y - 1 = -2t$; $z - 7 = -1t$

In General:

Let $z = f(x, y)$ Surface Function

Let $F(x, y, z) = f(x, y) - z$ Surface Function

At the point $P(x_0, y_0, z_0)$, there are many space curves $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ on surface function $F(x, y, z)$ that pass through $P(x_0, y_0, z_0)$.

For each $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ passing through (x_0, y_0, z_0), there is a

corresponding tangent vector $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ passing through (x_0, y_0, z_0).

Note that all tangent vectors $\mathbf{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$ passing through (x_0, y_0, z_0) lie on the tangent plane passing through $P(x_0, y_0, z_0)$.

Note that x, y, z are functions of t; hence, $F(x, y, z)$ is also a function of t.

By Chain Rule: $$\frac{dF(x,y,z)}{dt} = \frac{\partial F(x,y,z)}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial F(x,y,z)}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial F(x,y,z)}{\partial z} \cdot \frac{dx}{dt} = 0$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial x} \cdot x'(t) + \frac{\partial F}{\partial y} \cdot y'(t) + \frac{\partial F}{\partial z} \cdot z'(t) = 0$$

$$\frac{dF}{dt} = \left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle \cdot \langle x'(t), y'(t), z'(t) \rangle = 0$$

Hence, $\nabla F(x, y, z) \cdot \mathbf{r}'(t) = 0$

Meaning, $\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle$ is orthogonal to each tangent vector $\mathbf{r}'(t)$ passing through $P(x_0, y_0, z_0)$.

Therefore, $\nabla F(x, y, z)$ is orthogonal to the tangent plane at $P(x_0, y_0, z_0)$.

Equation of tangent plane is $F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$

Equation of normal line is $x - x_0 = F_x \cdot t$; $y - y_0 = F_y \cdot t$; $z - z_0 = F_z \cdot t$

Example 1: $z^2 - 2x^2 - 2y^2 = 12$ Surface Function

Hence, $F(x, y, z) = z^2 - 2x^2 - 2y^2 - 12 = 0$ Surface Function

Find the tangent plane and normal line passing through $P(1, -1, 4)$

$$\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle = \langle -4x, -4y, 2z \rangle$$

$$\nabla F(1, -1, 4) = \langle -4(1), -4(-1), 2(4) \rangle = \langle -4, 4, 8 \rangle$$

Equation of tangent plane is $-4(x-1) + 4(y+1) + 8(z-4) = 0$

Equation of normal line is $x - 1 = -4t$; $y + 1 = 4t$; $z - 4 = 8t$

Example 2: $z = 1 - \frac{1}{10}\left(x^2 + 4y^2\right)$ Surface Function

Hence, $F(x, y, z) = 1 - \frac{1}{10}\left(x^2 + 4y^2\right) - z = 0$ Surface Function

Find the tangent plane and normal line passing through $P(1,1,1/2)$

$$\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle = \left\langle -\frac{2}{10}x, -\frac{8}{10}y, -1 \right\rangle$$

$$\nabla F(1,1,1/2) = \left\langle -\frac{2}{10}(1), -\frac{8}{10}(1), -1 \right\rangle = \left\langle -\frac{1}{5}, -\frac{4}{5}, -1 \right\rangle$$

Equation of tangent plane is $-\frac{1}{5}(x-1) - \frac{4}{5}(y-1) - 1\left(z - \frac{1}{2}\right) = 0$

Equation of normal line is $x - 1 = -\frac{1}{5}t;\quad y - 1 = -\frac{4}{5}t;\quad z - \frac{1}{2} = -1t$

Example 3: $xyz = 12$ Surface Function

Hence, $F(x, y, z) = xyz - 12 = 0$ Surface Function

Find the tangent plane and normal line passing through $P(2, -2, -3)$

$$\nabla F(x, y, z) = \langle F_x, F_y, F_z \rangle = \langle yz, xz, xy \rangle$$

$$\nabla F(2, -2, -3) = \langle (-2)(-3), (2)(-3), (2)(-2) \rangle = \langle 6, -6, -4 \rangle$$

Equation of tangent plane is $6(x-2) - 6(y+2) - 4(z+3) = 0$

Equation of normal line is $x - 2 = 6t$; $y + 2 = -6t$; $z + 3 = -4t$