13.8 Extremum

Example 1

$f(x, y) = 2x^2 + y^2 + 8x - 6y + 20$

$f_x = 4x + 8$

$f_y = 2y - 6$

set $f_x = 0$; $f_y = 0$

$4x + 8 = 0$; $2y - 6 = 0$

$x = -2$; $y = 3$

$z = f(-2, 3) = 2(-2)^2 + (3)^2 + 8(-2) - 6(3) + 20 = 3$

~~Ext~~ $f(x, y)$ has extremum at $(-2, 3, 3)$

Example 2

$$f(x,y) = 1 - (x^2+y^2)^{1/3}$$

$$f_x = 0 - \frac{1}{3}(x^2+y^2)^{-2/3}(2x)$$

$$f_x = -\frac{2x}{3} \cdot \frac{1}{(x^2+y^2)^{2/3}} = \frac{-2x}{3(x^2+y^2)^{2/3}}$$

$$f_y = 0 - \frac{1}{3}(x^2+y^2)^{-2/3}(2y)$$

$$= \frac{-2y}{3(x^2+y^2)^{2/3}}$$

Set $f_x = 0$; $f_y = 0$

$$\frac{-2x}{3(x^2+y^2)^{2/3}} = 0 \qquad \Big| \qquad \frac{-2y}{3(x^2+y^2)^{2/3}} = 0$$

$$\Longrightarrow -2x = 0 \quad ; \quad -2y = 0$$
$$x = 0 \qquad\qquad y = 0$$

$$3(x^2+y^2)^{2/3} = 0$$

$\Longrightarrow$ Critical Point $= (x=0,\ y=0)$

$$z = f(x,y) = 1 - (x^2+y^2)^{1/3}$$
$$= 1 - (0^2+0^2)^{1/3} = 1$$

Extremum at $(0,0,1)$

Example 3

$$f(x,y) = -x^3 + 4xy - 2y^2 + 1$$

$$f_x = -3x^2 + 4y$$

$$f_y = 4x - 4y$$

Set $f_x = 0$

$f_y = 0$

$$\begin{aligned} -3x^2 + 4y &= 0 \quad (1) \\ + \quad 4x - 4y &= 0 \quad (2) \\ \hline -3x^2 + 4x &= 0 \end{aligned}$$

$$x \cdot (-3x + 4) = 0$$

$x = 0$	$-3x + 4 = 0$
	$x = 4/3$
(2) $4x - 4y = 0$	(2) $4x - 4y = 0$
$4(0) - 4y = 0$	$4(4/3) - 4y = 0$
$y = 0$	$y = 4/3$
$(0, 0)$	

Critical Points: $(0,0)$; $\left(\frac{4}{3}, \frac{4}{3}\right)$

$\Rightarrow (0,0,1)$ $\Rightarrow \left(\frac{4}{3}, \frac{4}{3}, \frac{59}{27}\right)$

Determinine if $(0,0,1)$ is relative max, min, or saddle point.

$f_{xx} = -6x$ $f_{xx}(0,0) = -6(0) = 0$

$f_{yy} = -4$ $f_{yy}(0,0) = -4$

$f_{xy} = 4$ $f_{xy}(0,0) = 4$

$d = f_{xx}(0,0) \cdot f_{yy}(0,0) - [f_{xy}(0,0)]^2$

$d = (0)(-4) - [4]^2 = -16 < 0$

$\therefore (0,0,1)$ is a ~~relative max~~. saddle point

$f_{xx}(4/3, 4/3) = -6(4/3) = -8$

$f_{yy}(4/3, 4/3) = -4$

$f_{xy}(4/3, 4/3) = 4$

$d = (-8)(-4) - [4]^2 = 16 > 0$

Note: $d > 0$ and $f_{xx} < 0$

$\Rightarrow (4/3, 4/3, 59/27)$ is a relative max.

Example 4

Section 13.8 : Find Extremum

$z = f(x, y) = x^2 y^2 \geq 0$

$f_x = 2x y^2$

$f_y = 2y x^2$

$f_{xx} = 2y^2$

$f_{yy} = 2x^2$

$f_{xy} = 4xy$

To find critical point:

set $f_x = 0$

$f_y = 0$

$(2x)(y^2) = 0 \Rightarrow 2x = 0 \; ; \; y^2 = 0$

$x = 0 \quad y = 0$

$(2y)(x^2) = 0$

$2y = 0 \; ; \; x^2 = 0$

$y = 0 \; ; \; x = 0$

Critical point $(0, 0)$

$x = 0, y = 0, \; z = x^2 y^2 = 0$

$(0, 0, 0)$

Determine what is happening at $(0,0,0)$:

$$f_{xx}(0,0) = 0$$

$$f_{yy}(0,0) = 0$$

$$f_{xy}(0,0) = 0$$

$$d = f_{xx} \cdot f_{yy} - [f_{xy}]^2 = (0)(0) - (0)^2 = 0$$

$(0,0,0)$ is a relative minimum because

$$z = f(x,y) = x^2 y^2 \geq 0$$

Example 5

$f(x,y) = \sin(x \cdot y)$ $\qquad 0 \le x \le \pi$

$\qquad\qquad\qquad\qquad 0 \le y \le 1$

$f_x = \cos(x \cdot y) \cdot y = y\cos(xy)$

$f_y = \cos(xy) \cdot x = x \cdot \cos(xy)$

To find critical point: set $f_x = 0$ and $f_y = 0$

$\cancel{y = co}\,[\,y\,][\cos(xy)] = 0$

$y = 0 \;;\; \cos(xy) = 0$

$y = 0 \;;\; xy = \pi/2$

$x \cdot \cos(xy) = 0$

$x = 0 \;|\; \cos(xy) = 0$

$\qquad\quad xy = \pi/2$

Critical Points: $(x=0, y=0)$ $\quad \{(x,y) : xy = \pi/2\}$

For $x=0$, $y=0$, $z = f(x,y) = \sin(xy) = \sin(0) = 0$

$(x=0,\ y=0,\ z=0)$

$$\boxed{z = f(x,y) = \sin(xy) \geq 0 \qquad \begin{array}{l} 0 \leq x \leq \pi \\ 0 \leq y \leq 1 \end{array}}$$

Therefore $(0,0,0)$ is a relative min.

For $\{(x,y) : xy = \pi/2\}$

$z = f(x,y) = \sin(xy) = \sin(\pi/2) = 1$

$\{(x, y, 1) : xy = \pi/2\}$ is absolute max.