Example 1: Evaluate $\int_0^x (14x + 2y)\,dy$

Treating $14x$ as constant and integrating with respect to y:

$$\int_0^x (14x + 2y)\,dy = (14x)\,y + 2\frac{y^2}{2} = \left[14xy + y^2\right]_0^x$$

$$= \left[14x(x) + (x)^2\right] - \left[14x(0) + (0)^2\right]$$

$$= 14x^2 + x^2 = 15x^2$$

Example 2: Evaluate $\int_2^{2y} \frac{3y}{x}\,dx = \int_2^{2y} 3y \cdot \frac{1}{x}\,dx$

Treating $3y$ as constant and integrating with respect to x:

$$\int_2^{2y} 3y \cdot \frac{1}{x}\,dx = 3y \cdot \int_2^{2y} \frac{1}{x}\,dx = 3y \cdot \left[\ln x\right]_2^{2y}$$

$$= 3y \cdot \left[\ln(2y)\right] - 3y \cdot \left[\ln(2)\right]$$

$$= 3y\ln(2y) - 3y\ln(2)$$

Example 3: Evaluate $\int_{e^y}^{4y} \frac{5y\ln x}{x}\,dx = \int_{e^y}^{4y} 5y\cdot\frac{\ln x}{x}\,dx.$

Treating $5y$ as constant and integrating with respect to x:

$$\int_{e^y}^{4y} 5y\cdot\frac{\ln x}{x}\,dx = 5y\cdot\int_{e^y}^{4y}\frac{\ln x}{x}\,dx$$

$$Note: \int\frac{\ln x}{x}\,dx = \int \ln x\cdot\frac{1}{x}\,dx$$

Let $u=\ln x \quad \Rightarrow \quad \frac{du}{dx}=\frac{1}{x} \quad \Rightarrow \; du=\frac{1}{x}dx$

$$\int \ln x\cdot\frac{1}{x}\,dx = \int u\cdot du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

Example 3: con't

Hence, $5y \cdot \int_{e^y}^{4y} \frac{\ln x}{x} dx = 5y \cdot \left[\frac{(\ln x)^2}{2} \right]_{e^y}^{4y}$

$$= 5y \cdot \left[\frac{(\ln(4y))^2}{2} \right] - 5y \cdot \left[\frac{(y)^2}{2} \right]$$

Example 4: Evaluate $\int_0^1\int_0^3 (3x+y)\,dy\,dx$.

Integrate with respect to y:

$$\int_0^3 (3x+y)\,dy = \left[3xy+\frac{y^2}{2}\right]_0^3 = \left[3x(3)+\frac{(3)^2}{2}\right]-\left[3x(0)+\frac{(0)^2}{2}\right]$$

$$= 9x+9/2$$

$$\int_0^1\int_0^3 (3x+y)\,dy\,dx = \int_0^1 (9x+9/2)\,dx = \left[\frac{9x^2}{2}+\frac{9}{2}x\right]_0^1$$

$$= \left[\frac{9(1)^2}{2}+\frac{9}{2}(1)\right]-\left[\frac{9(0)^2}{2}+\frac{9}{2}(0)\right]=9$$

Example 5: Evaluate $\int_0^{\pi/3}\int_0^{\cos x}(4+\sin x)\,dy\,dx.$

Integrate with respect to y:

$$\int_0^{\cos x}(4+\sin x)\,dy = \Big[(4+\sin x)\,y\Big]_0^{\cos x}$$

$$=\Big[(4+\sin x)(\cos x)\Big]-\Big[(4+\sin x)(0)\Big]$$

$$= 4\cos x+\sin x\cos x$$

$$\int_0^{\pi/3}\int_0^{\cos x}(4+\sin x)\,dy\,dx = \int_0^{\pi/3}(4\cos x+\sin x\cos x)\,dx$$

Example 5: *con't*

$$Note: \int 4\cos x dx = 4(\sin x) = 4\sin x$$

For $\int (\sin x \cos x) dx$,

let $u = \sin x \Rightarrow \dfrac{du}{dx} = \cos x \Rightarrow du = \cos x dx$

$$\int (\sin x \cos x) dx = \int (u) du = \frac{u^2}{2} = \frac{(\sin x)^2}{2}$$

Example 5: *con't*

$$\int_0^{\pi/3}\int_0^{\cos x}(4+\sin x)\,dy\,dx=\int_0^{\pi/3}(4\cos x+\sin x\cos x)\,dx$$

$$=\left[4\sin x+\frac{(\sin x)^2}{2}\right]_0^{\pi/3}$$

$$=\left[4\sin(\pi/3)+\frac{(\sin(\pi/3))^2}{2}\right]-\left[4\sin(0)+\frac{(\sin(0))^2}{2}\right]$$

$$=\left[4\sin(\pi/3)+\frac{(\sin(\pi/3))^2}{2}\right]$$

Example 6: Evaluate $\int_0^2 \int_0^{\sqrt{1-y^2}} \left(\frac{7}{\sqrt{1-y^2}} \right) dxdy$

Integrate with respect to x:

$$\int_0^{\sqrt{1-y^2}} \left(\frac{7}{\sqrt{1-y^2}} \right) dx = \left[\left(\frac{7}{\sqrt{1-y^2}} \right) x \right]_0^{\sqrt{1-y^2}}$$

$$= \left[\left(\frac{7}{\sqrt{1-y^2}} \right) \left(\sqrt{1-y^2} \right) \right] - \left[\left(\frac{7}{\sqrt{1-y^2}} \right) (0) \right] = 7$$

Example 6: $con't$

$$\int_0^2 \int_0^{\sqrt{1-y^2}} \left(\frac{7}{\sqrt{1-y^2}} \right) dxdy = \int_0^2 7dy = 7y\Big|_0^2$$

$$= 7(2) - 7(0) = 14$$

Example 7: Evaluate $\int_0^{\pi/4}\int_0^{\cos\theta}(5\theta r)\,dr\,d\theta.$

Integrate with respect to r :

$$\int_0^{\cos\theta}(5\theta r)\,dr=\left[(5\theta)\frac{r^2}{2}\right]_0^{\cos\theta}$$

$$=\left[(5\theta)\frac{(\cos\theta)^2}{2}\right]-\left[(5\theta)\frac{(0)^2}{2}\right]$$

$$=\frac{5}{2}\theta(\cos\theta)^2$$

Example 7: *con't*

$$\int_0^{\pi/4}\int_0^{\cos\theta}(5\theta r)\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{5}{2}\theta(\cos\theta)^2\right]d\theta$$

$$= 0.563904609758 \quad \text{Using Trapezoidal Rule}$$

Example 8: Evaluate $\int_1^{\infty}\int_0^{4/x}(6y)\,dydx.$

Integrate with respect to y:

$$\int_0^{4/x}(6y)\,dy = 6\frac{y^2}{2} = \left[3y^2\right]_0^{4/x}$$

$$= \left[3\left(\frac{4}{x}\right)^2\right] - \left[3(0)^2\right] = \frac{48}{x^2} = 48x^{-2}$$

Example 8: *con't*

$$\int_1^\infty \int_0^{4/x} (6y)\,dy\,dx = \int_1^\infty 48x^{-2}dx = 48\frac{x^{-1}}{-1} = -48x^{-1} = \left.\frac{-48}{x}\right|_1^\infty$$

$$= \left[\frac{-48}{\infty}\right] - \left[\frac{-48}{1}\right] = 0 + 48 = 48$$

Example 9: Evaluate $\int_1^\infty \int_1^\infty \left(\frac{15}{xy}\right) dxdy.$

Integrate with respect to x:

$$\int_1^\infty \left(\frac{15}{xy}\right) dx = \frac{15}{y}\int_1^\infty \left(\frac{1}{x}\right) dx = \frac{15}{y}\left[\ln x\right]\Big|_1^\infty$$

$$= \frac{15}{y}\left[\ln(\infty)\right] - \frac{15}{y}\left[\ln(1)\right]$$

$$= \infty - 0 = \infty$$

Therefore, $\int_1^\infty \int_1^\infty \left(\frac{15}{xy}\right) dxdy$ diverges.