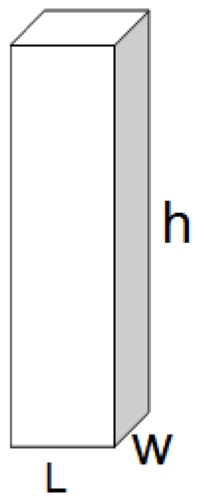
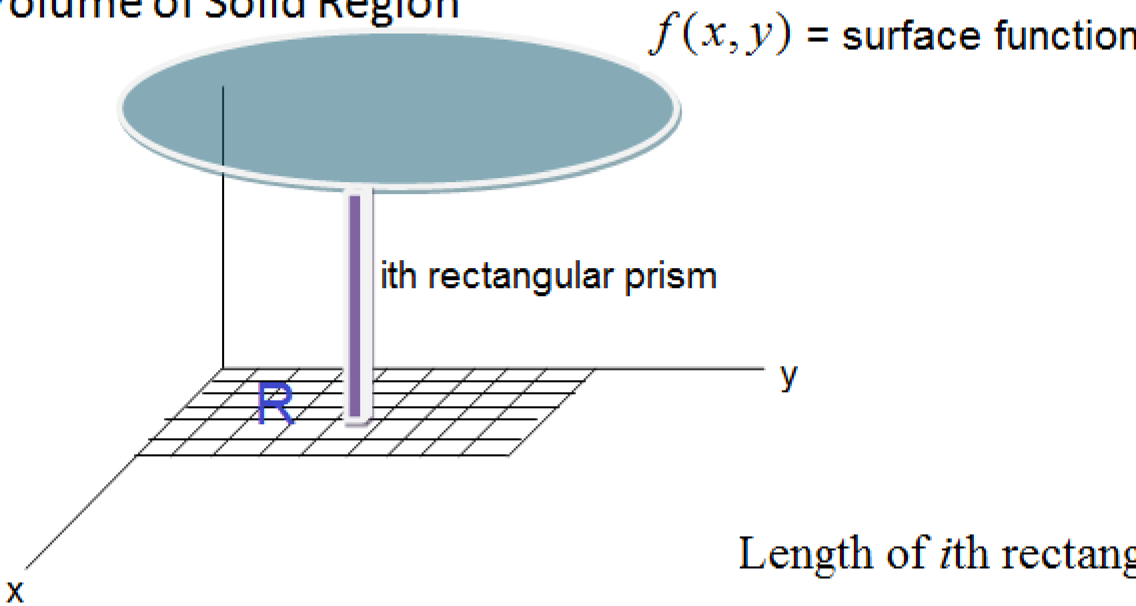
Double Integrals and Volume

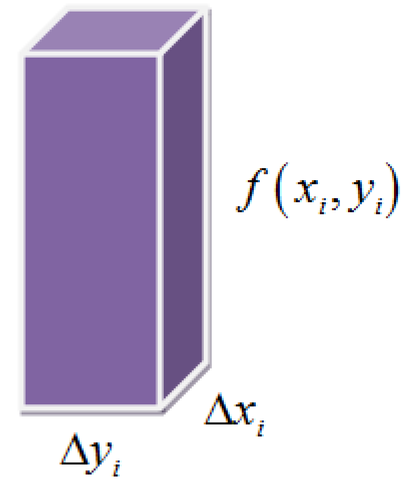
Volume of rectangular prism = $L \cdot w \cdot h$



Volume of Solid Region



ith rectangular prism



Length of ith rectangular prism $= \Delta x_i$

Width of ith rectangular prism $= \Delta y_i$

Height of ith rectangular prism $= f(x_i, y_i)$

Volume of ith rectangular prism $= f(x_i, y_i)\,\Delta x_i \Delta y_i$

Process of finding volume of solid region:

a) Partition region R into many n small rectangles.

b) Volume of ith rectangular prism = $f\left(x_i, y_i\right)\Delta x_i \Delta y_i = f\left(x_i, y_i\right)\Delta A_i$

c) Volume of Solid Region $\approx f\left(x_1, y_1\right)\Delta A_1 + f\left(x_2, y_2\right)\Delta A_2 + \ldots + f\left(x_n, y_n\right)\Delta A_n = \sum_{i=1}^{n} f\left(x_i, y_i\right)\Delta A_i$

d) Volume of Solid Region = $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f\left(x_i, y_i\right)\Delta A_i$

where $\|\Delta\| =$ length of longest diagonal of the n rectangles inside of region R.

Note: As $\|\Delta\| \to 0$, $n \to \infty$ (n is the number of rectangles)

If $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} f\left(x_i, y_i\right)\Delta A_i$ exists, $\lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} f\left(x_i, y_i\right)\Delta A_i$ is defined as $\int_R\int f(x,y)dA$

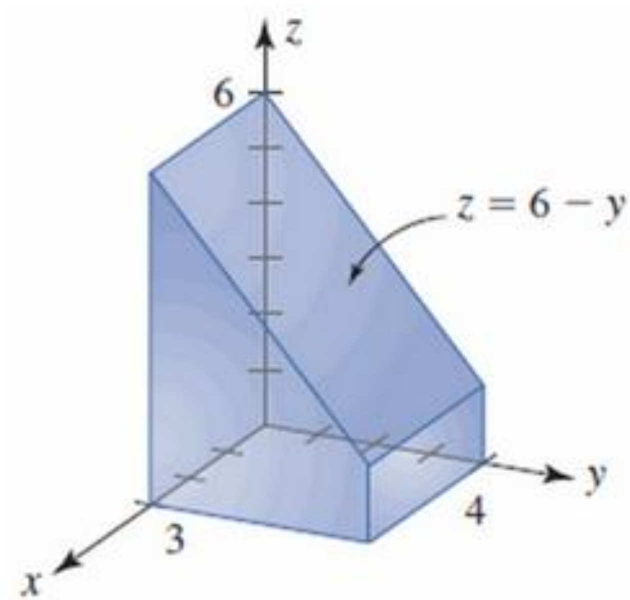
In other words, $\int_R\int f(x,y)dA = \lim\limits_{\|\Delta\|\to 0} \sum\limits_{i=1}^{n} f\left(x_i, y_i\right)\Delta A_i$

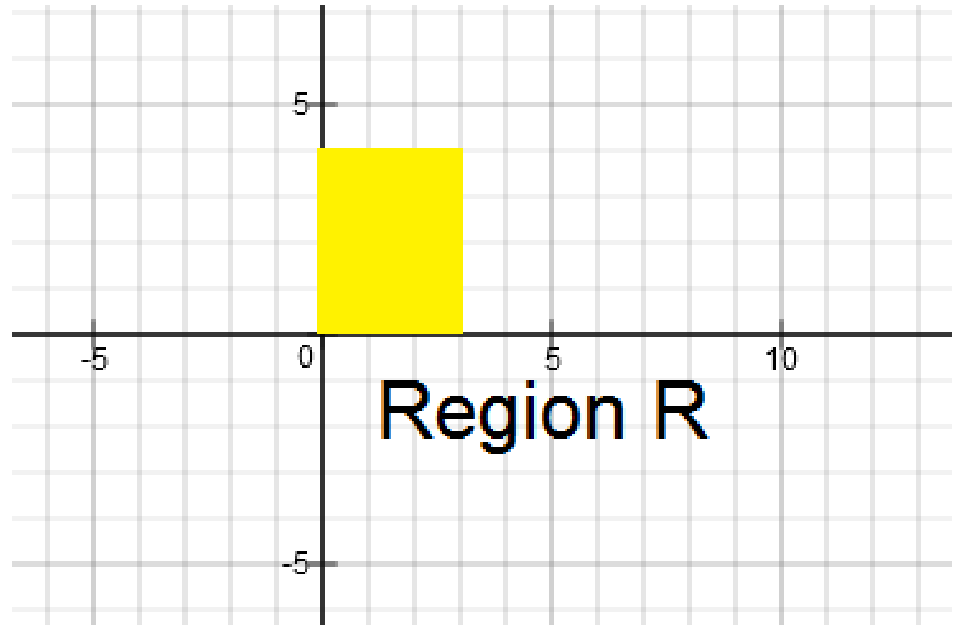
Fubini's Theorem:

$$\int_R\int f(x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)dxdy$$

Example 1: Use a double integral to find the volume of the following solid

Surface Function: $f(x, y) = z = 6 - y$





Order of Integration: *dxdy*

$$\text{Volume} = \int_0^4 \int_0^3 f\left(x+y\right) dxdy = \int_0^4 \int_0^3 \left(6-y\right) dxdy$$

Order of Integration: *dydx*

$$\text{Volume} = \int_0^3 \int_0^4 f\left(x+y\right) dydx = \int_0^3 \int_0^4 \left(6-y\right) dydx$$

Evaluating the double integral:

$$\text{Volume} = \int_0^4 \int_0^3 f\left(x+y\right) dxdy = \int_0^4 \int_0^3 \left(6-y\right) dxdy$$

$$\int_0^3 \left(6-y\right) dx = \left[\left(6-y\right)x\right]_0^3 = \left(6-y\right)\left(3\right) = 18-3y$$

$$\int_0^4 \int_0^3 \left(6-y\right) dxdy = \int_0^4 \left(18-3y\right) dy = \left[18y - 3\cdot\frac{y^2}{2}\right]_0^4 = 48$$

Evaluating the double integral:

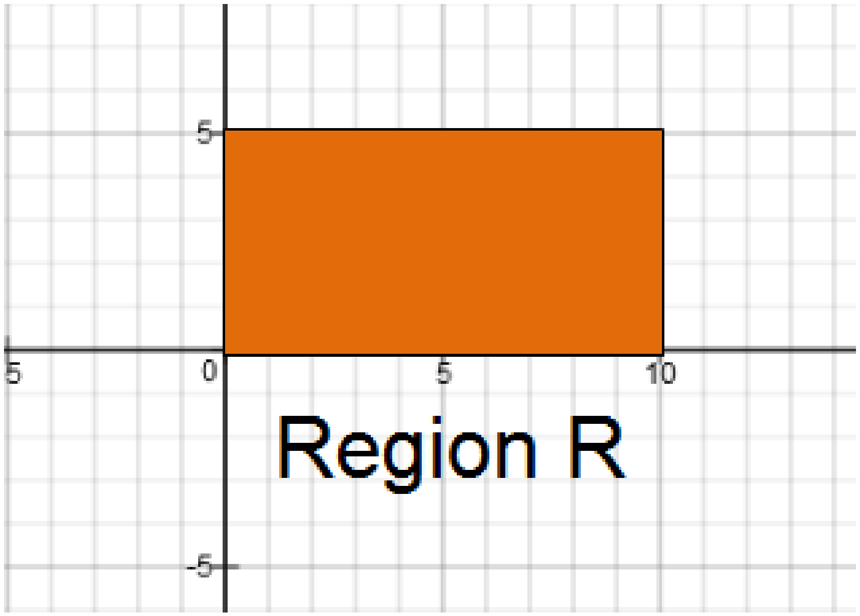
$$\text{Volume} = \int_0^3 \int_0^4 f\left(x+y\right) dy dx = \int_0^3 \int_0^4 \left(6-y\right) dy dx$$

$$\int_0^4 \left(6-y\right) dy = \left[\left(6y - \frac{y^2}{2}\right)\right]_0^4 = 24 - 8 = 16$$

$$\int_0^3 \int_0^4 \left(6-y\right) dy dx = \int_0^3 16 dx = 48$$

Example 2: Use a double integral to find the volume of the solid that is formed by the following surface function and region *R*.

Surface Function: $f(x,y)=1+4y$



Order of Integration: *dxdy*

$$\text{Volume} = \int_0^5 \int_0^{10} f\left(x+y\right) dxdy = \int_0^5 \int_0^{10} \left(1+4y\right) dxdy$$

Order of Integration: *dydx*

$$\text{Volume} = \int_0^{10} \int_0^{5} f\left(x+y\right) dydx = \int_0^{10} \int_0^{5} \left(1+4y\right) dydx$$

Evaluating the double integral:

$$\text{Volume} = \int_0^5 \int_0^{10} f\left(x+y\right) dxdy = \int_0^5 \int_0^{10} \left(1+4y\right) dxdy$$

$$\int_0^{10} \left(1+4y\right) dx = \left[\left(1+4y\right)x\right]_0^{10} = \left(1+4y\right)\left(10\right) = 10+40y$$

$$\int_0^5 \int_0^{10} \left(1+4y\right) dxdy = \int_0^5 \left(10+40y\right) dy = \left[10y+20y^2\right]_0^5 = 550$$

Evaluating the double integral:

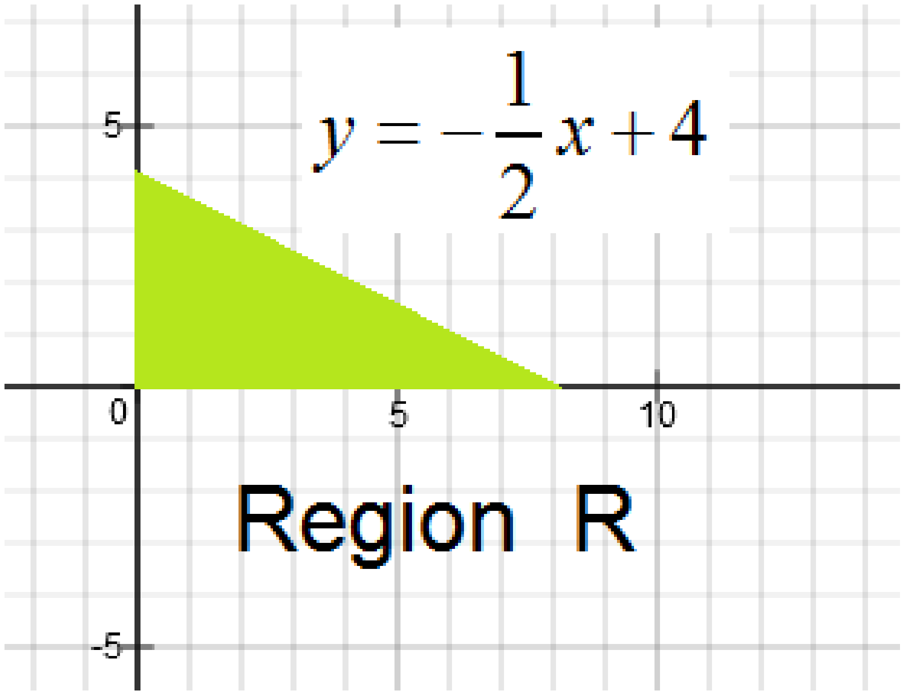
$$\text{Volume} = \int_0^{10}\int_0^5 f\left(x+y\right)dydx = \int_0^{10}\int_0^5 \left(1+4y\right)dydx$$

$$\int_0^5 \left(1+4y\right)dy = \left[y+2y^2\right]_0^5 = 5+50 = 55$$

$$\int_0^{10}\int_0^5 \left(1+4y\right)dydx = \int_0^{10} 55dx = \left[55x\right]_0^{10} = 550$$

Example 3: Use a double integral to find the volume of the solid formed by the following surface function and region *R*.

Surface Function: $f(x, y) = 5 - 4xy$



Order of Integration: $dy \cdot dx$

$$\text{Volume} = \int_{0}^{8} \int_{y=0}^{y=-\frac{1}{2}x+4} f\left(x+y\right) dy dx = \int_{0}^{8} \int_{y=0}^{y=-\frac{1}{2}x+4} \left(5-4xy\right) dy dx$$

Order of Integration: $dy \cdot dx$

$$\text{Volume} = \int_0^8 \int_{y=0}^{y=-\frac{1}{2}x+4} f(x+y)\,dydx = \int_0^8 \int_{y=0}^{y=-\frac{1}{2}x+4} (5-4xy)\,dydx$$

$$\int_{y=0}^{y=-\frac{1}{2}x+4} (5-4xy)\,dy = \left[5y - 4x\frac{y^2}{2}\right]_0^{-\frac{1}{2}x+4} = \left[5y-2xy^2\right]_0^{-\frac{1}{2}x+4}$$

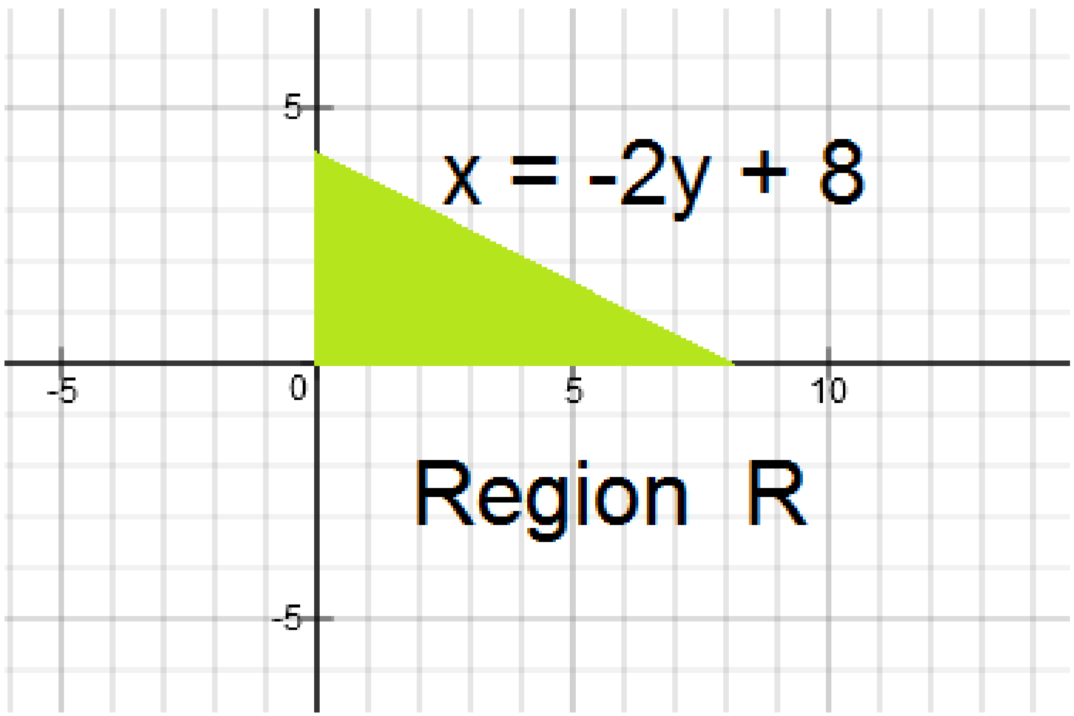
$$= 5\left(-\frac{1}{2}x+4\right) - 2x\left(-\frac{1}{2}x+4\right)^2 = -\frac{5}{2}x + 20 - 2x\left(-\frac{1}{2}x+4\right)^2$$

$$\int_0^8 \int_{y=0}^{y=-\frac{1}{2}x+4} (5-4xy)\,dydx = \int_0^8 \left[-\frac{5}{2}x + 20 - 2x\left(-\frac{1}{2}x+4\right)^2\right]dx = -90\frac{2}{3}$$

Note: $y = \frac{-1}{2}x + 4$

$-2y = x - 8$ Multiply each term by -2

$x = -2y + 8$



Order of Integration: $dx \cdot dy$

$$\text{Volume} = \int_0^4 \int_{x=0}^{x=-2y+8} f(x+y)\,dxdy = \int_0^4 \int_{x=0}^{x=-2y+8} (5-4xy)\,dxdy$$

$$\int_{x=0}^{x=-2y+8} (5-4xy)\,dx = \left[5x - 4y\frac{x^2}{2}\right]_0^{-2y+8} = \left[5x - 2yx^2\right]_0^{-2y+8}$$

$$= 5(-2y+8) - 2y(-2y+8)^2$$

$$\int_0^4 \int_{x=0}^{x=-2y+8} (5-4xy)\,dxdy = \int_0^4 \left[5(-2y+8) - 2y(-2y+8)^2\right]dy = -90\frac{2}{3}$$

Example 4: Use a double integral to find the volume of the solid formed by the following surface function and region *R*.

Surface Function: $f(x, y) = e^{-2x-3y}$

Region $R: \left\{ \left(x, y \right) : 0 \le x \le \infty;\ 0 \le y \le \infty \right\}$

$$\text{Volume of solid} = \int_0^\infty \int_0^\infty f\left(x + y \right) dydx = \int_0^\infty \int_0^\infty e^{-2x-3y} dydx$$

$$\text{Volume of solid} = \int_0^\infty \int_0^\infty f\left(x + y \right) dxdy = \int_0^\infty \int_0^\infty e^{-2x-3y} dxdy$$

Evaluating the double integral with integration order *dydx*:

Volume of solid = $\int_0^{\infty}\int_0^{\infty} f\left(x+y\right)dydx = \int_0^{\infty}\int_0^{\infty} e^{-2x-3y}dydx$

Evaluating $\int_0^{\infty} e^{-2x-3y}dy$:

Recall: $\int e^{ax}dx = \frac{1}{a}e^{ax}; \qquad \int e^{ay}dy = \frac{1}{a}e^{ay}; \quad e^{-\infty} = 0$

$$\int_0^{\infty} e^{-2x-3y}dy = \int_0^{\infty} e^{-2x}e^{-3y}dy = e^{-2x}\int_0^{\infty} e^{-3y}dy = e^{-2x}\left[\frac{-1}{3}e^{-3y}\right]_0^{\infty}$$

$$= \left(e^{-2x}\right)\left(\frac{-1}{3}e^{-\infty}\right) - \left(e^{-2x}\right)\left(\frac{-1}{3}e^{0}\right) = \frac{1}{3}e^{-2x}$$

$$\int_0^{\infty}\int_0^{\infty} e^{-2x-3y}dydx = \int_0^{\infty}\frac{1}{3}e^{-2x}dx = \frac{1}{3}\left[\frac{-1}{2}e^{-2x}\right]_0^{\infty} = \frac{1}{3}\left[\frac{-1}{2}e^{-\infty}\right] - \frac{1}{3}\left[\frac{-1}{2}e^{0}\right] = \frac{1}{6}$$

Evaluating the double integral with integration order *dxdy*:

$$\text{Volume of solid} = \int_0^\infty \int_0^\infty f\left(x+y\right) dxdy = \int_0^\infty \int_0^\infty e^{-2x-3y} dxdy$$

$$\text{Recall: } \int e^{ax} dx = \frac{1}{a} e^{ax}; \qquad \int e^{ay} dy = \frac{1}{a} e^{ay}; \quad e^{-\infty} = 0$$

$$\int_0^\infty e^{-2x-3y} dx = \int_0^\infty e^{-2x} e^{-3y} dx = e^{-3y} \int_0^\infty e^{-2x} dx = e^{-3y} \left[\frac{-1}{2} e^{-2x} \right]_0^\infty$$

$$= \left(e^{-3y}\right)\left(\frac{-1}{2} e^{-\infty}\right) - \left(e^{-3y}\right)\left(\frac{-1}{2} e^{0}\right) = \frac{1}{2} e^{-3y}$$

$$\int_0^\infty \int_0^\infty e^{-2x-3y} dxdy = \int_0^\infty \frac{1}{2} e^{-3y} dy = \frac{1}{2}\left[\frac{-1}{3} e^{-3y} \right]_0^\infty = \frac{1}{2}\left[\frac{-1}{3} e^{-\infty} \right] - \frac{1}{2}\left[\frac{-1}{3} e^{0} \right] = \frac{1}{6}$$