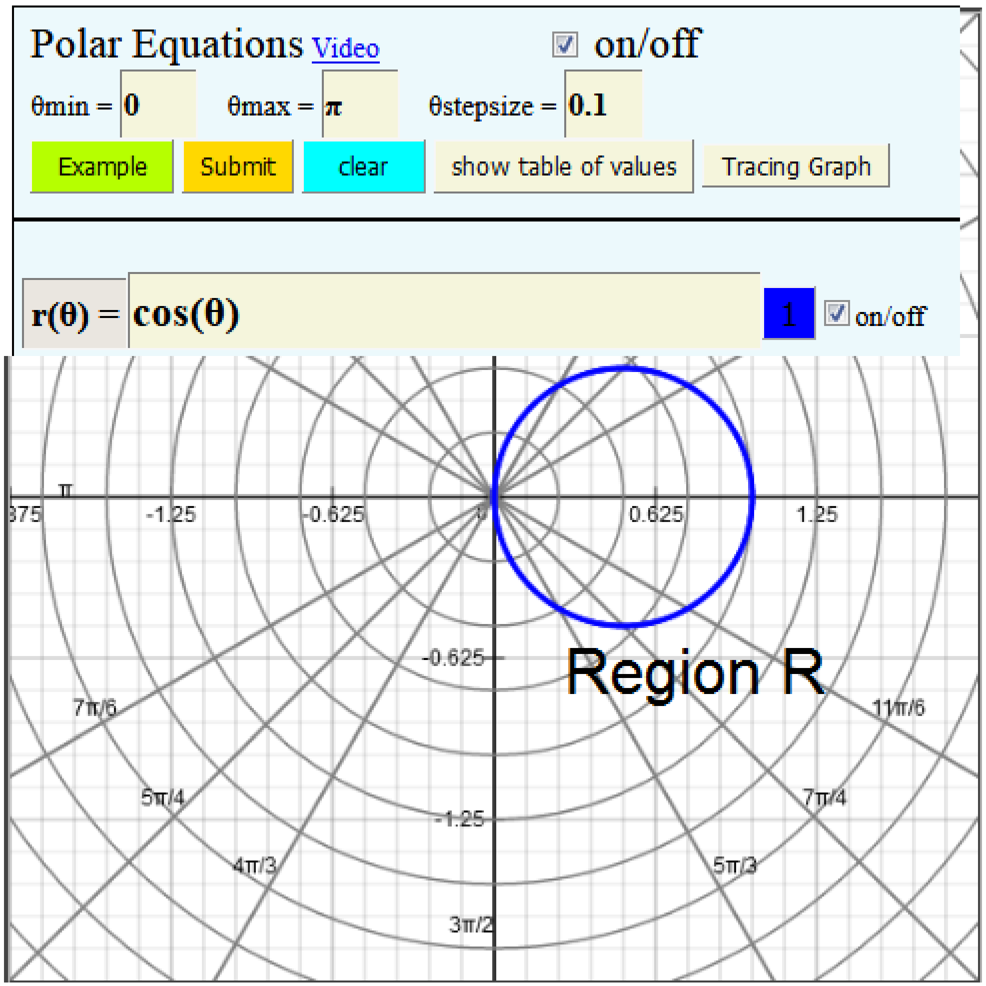
Example 1:

Find the volume of the solid between $f(r,\theta)=1$ and region $R=\{0\le r\le\cos\theta;\quad 0\le\theta\le\pi\}$

Draw the region R:



volume of the solid $= \int_0^{\pi} \int_0^{\cos\theta} f(r,\theta)\, r\,dr\,d\theta = \int_0^{\pi} \int_0^{\cos\theta} r\,dr\,d\theta$

Note the solid is a cylinder with radius 0.5 and height of 1.

Evaluating the double integral $\int_0^{\pi} \int_0^{\cos\theta} r\,dr\,d\theta$:

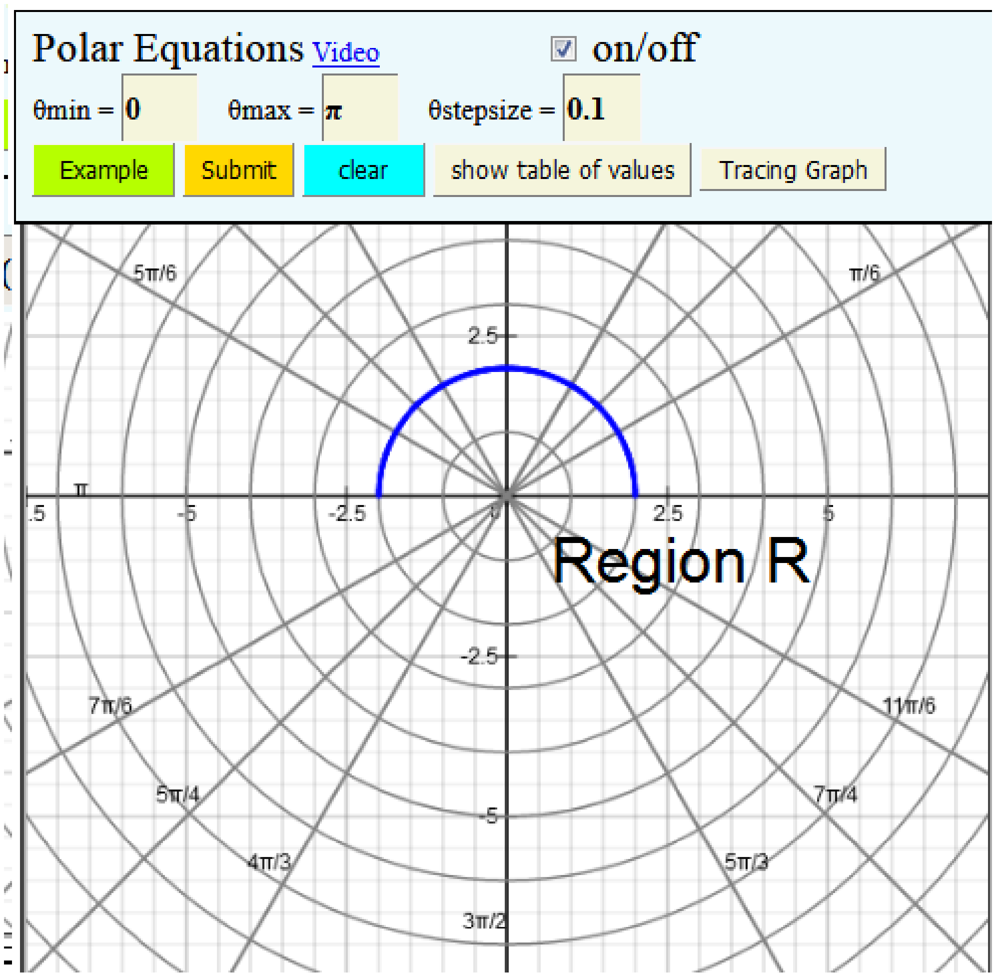
$$\int_0^{\cos\theta} r\,dr = \left[\frac{r^2}{2}\right]_0^{\cos\theta} = \frac{1}{2}(\cos\theta)^2$$

$$\int_0^{\pi} \int_0^{\cos\theta} r\,dr\,d\theta = \int_0^{\pi} \left[\frac{1}{2}(\cos\theta)^2\right] d\theta = 0.785398$$

Example 2:

Find the volume of the solid between $f(r,\theta) = 2r\sin$ and region $R = \{0 \le r \le 2;\ \ 0 \le \theta \le \pi\}$

Draw the region R:



volume of the solid $=\int_0^{\pi}\int_0^2 f\left(r,\theta\right) r dr d\theta = \int_0^{\pi}\int_0^2 2r\sin\theta r dr d\theta = \int_0^{\pi}\int_0^2 2r^2\sin\theta dr d\theta$

Evaluating I $= \int_0^{\pi}\int_0^2 2r^2\sin\theta dr d\theta$:

Let $I_1 = \int_0^2 2r^2\sin\theta dr = \sin\theta\int_0^2 2r^2 dr = \sin\theta\left[2\cdot\frac{r^3}{3}\right]_0^2 = \frac{16}{3}\sin\theta$

So I $= \int_0^{\pi}\frac{16}{3}\sin\theta d\theta = 10\frac{2}{3} = 10.666666667$

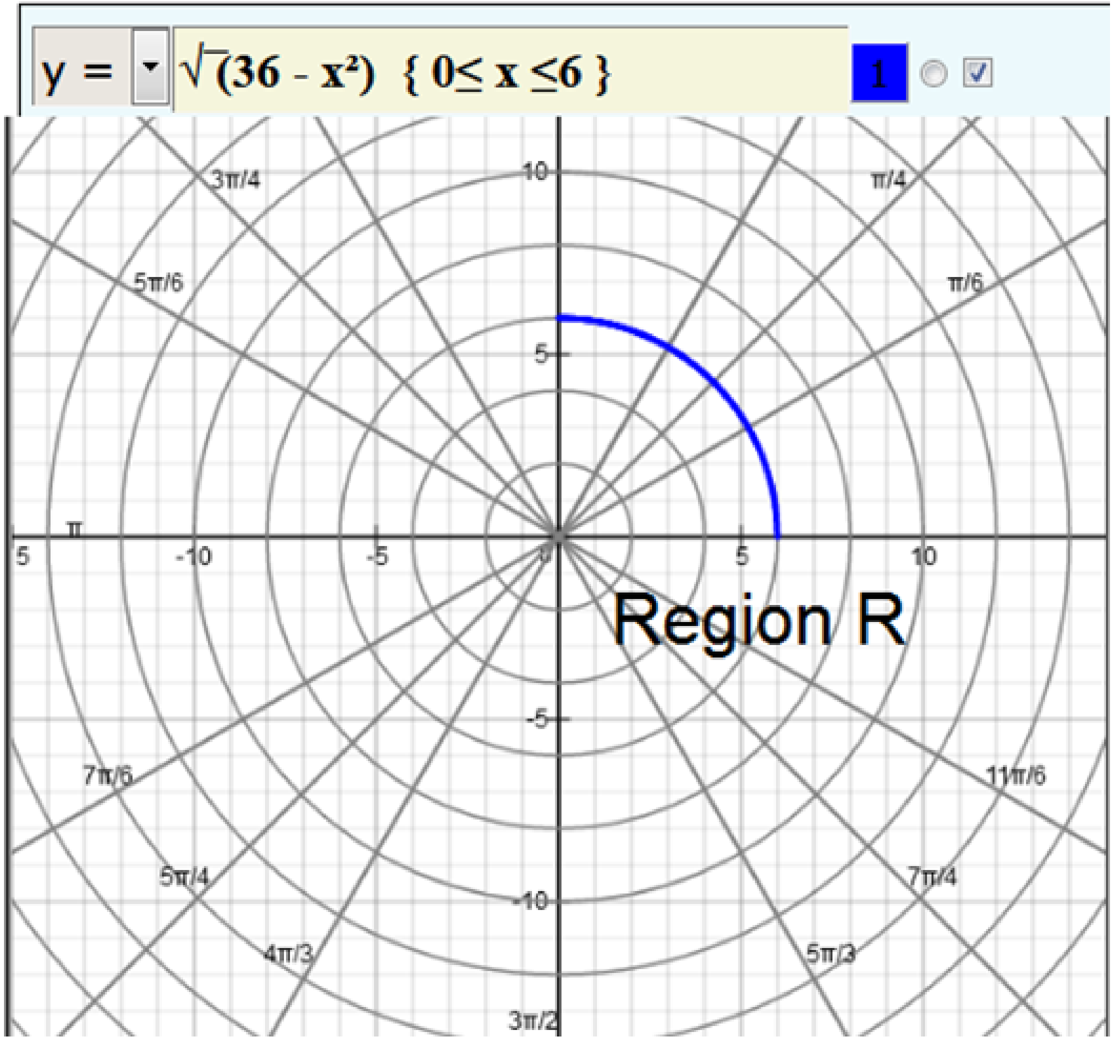
Therefore, $\int_0^{\pi}\int_0^2 2r^2\sin\theta dr d\theta = 10.666666667$

Example 3:

Evaluate the iterated integral $\int_0^6 \int_0^{\sqrt{36-x^2}} \left(x^2+y^2\right)^2 dydx$ by converting to polar coordinates.

Region R in rectangular coordinates: $0 \le x \le 6; \quad 0 \le y \le \sqrt{36-x^2}$

Region R in polar coordinates: $\underline{\ 0\ } \le \theta \le \underline{\ \pi/2\ }; \quad \underline{\ 0\ } \le r \le \underline{\ 6\ }$



Graph of $y = \sqrt{36 - x^2}$

Region R in rectangular coordinates:

$0 \le x \le 6; \quad 0 \le y \le \sqrt{36 - x^2}$

Region R in polar coordinates:

$\underline{\ 0\ } \le \theta \le \underline{\ \pi/2\ }; \quad \underline{\ 0\ } \le r \le \underline{\ 6\ }$

Change of Variables to Polar Form:

$$\int_R \int f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Hence, $\displaystyle \int_0^6 \int_0^{\sqrt{36-x^2}} \left(x^2+y^2\right)^2 dy\,dx = \int_0^{\pi/2} \int_0^6 \left(r^2\right)^2 r\,dr\,d\theta$

Evaluating $\displaystyle \mathrm{I} = \int_0^{\pi/2} \int_0^6 \left(r^2\right)^2 r\,dr\,d\theta$:

Let $\displaystyle \mathrm{I}_1 = \int_0^6 \left(r^2\right)^2 r\,dr = \int_0^6 r^5 dr = \frac{r^6}{6} = \frac{6^6}{6} = 6^5 = 7776$

So $\displaystyle \mathrm{I} = \int_0^{\pi/2} 7776\,d\theta = \frac{7776}{2}\pi$

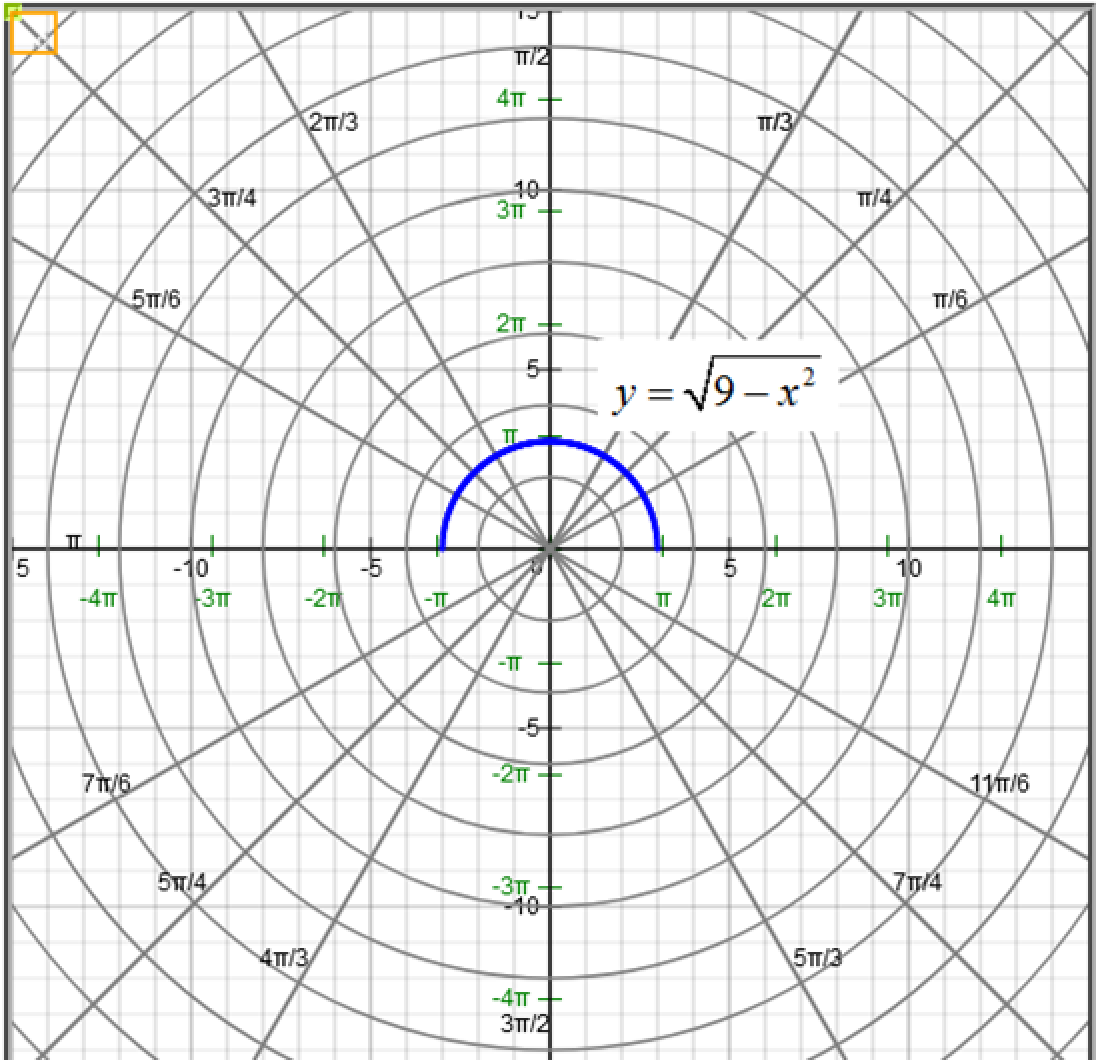
Therefore, $\displaystyle \int_0^{\pi/2} \int_0^6 \left(r^2\right)^2 r\,dr\,d\theta = \frac{7776}{2}\pi$

Example 4:

Evaluate the iterated integral $\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}} \cos\left(x^2+y^2\right) dydx$ by converting to polar coordinates.

Region R in rectangular coordinates: $-3 \le x \le 3; \quad 0 \le y \le \sqrt{9-x^2}$

Region R in polar coordinates: $\underline{\;0\;} \le \theta \le \underline{\;\pi\;}; \quad \underline{\;0\;} \le r \le \underline{\;3\;}$



Graph of $y = \sqrt{9 - x^2}$

Region R in rectangular coordinates:

$-3 \le x \le 3; \quad 0 \le y \le \sqrt{9 - x^2}$

Region R in polar coordinates:

$\underline{\ 0\ } \le \theta \le \underline{\ \pi\ }; \quad \underline{\ 0\ } \le r \le \underline{\ 3\ }$

Change of Variables to Polar Form:

$$\int_R \int f(x,y)\,dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Hence, $\displaystyle\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \cos(x^2+y^2)\,dy\,dx = \int_0^{\pi}\int_0^3 \cos(r^2)\,r\,dr\,d\theta$

Evaluating $\displaystyle I = \int_0^{\pi}\int_0^3 \cos(r^2)\,r\,dr\,d\theta$:

Let $\displaystyle I_1 = \int_0^3 \cos(r^2)\,r\,dr = \frac{1}{2}\sin(r^2)\Big|_0^3 = 0.2060592426208783$

So $\displaystyle I = \int_0^{\pi} 0.2060592426208783\,d\theta = 0.647354202822028$

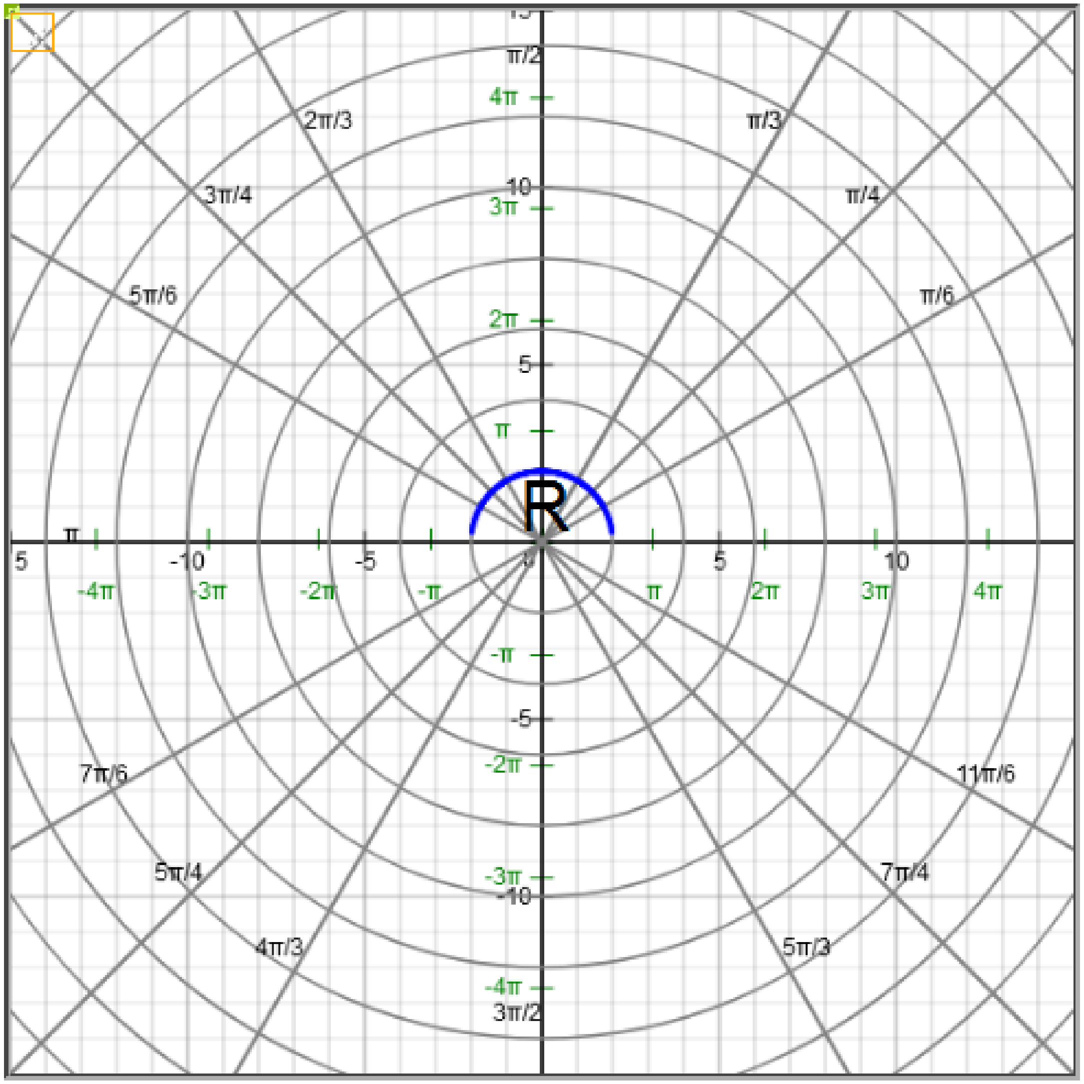
Therefore, $\displaystyle\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \cos(x^2+y^2)\,dy\,dx = \int_0^{\pi}\int_0^3 \cos(r^2)\,r\,dr\,d\theta = 0.647354202822028$

Example 5:

Evaluate the iterated integral $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)dydx$ by converting to polar coordinates.

Region R in rectangular coordinates: $-2 \le x \le 2; \quad 0 \le y \le \sqrt{4-x^2}$

Region R in polar coordinates: $\underline{\quad 0 \quad} \le \theta \le \underline{\quad \pi \quad}; \quad \underline{\quad 0 \quad} \le r \le \underline{\quad 2 \quad}$



Graph of $y = \sqrt{4 - x^2}$

Region R in rectangular coordinates:

$$-2 \leq x \leq 2; \quad 0 \leq y \leq \sqrt{4 - x^2}$$

Region R in polar coordinates:

$$0 \quad \leq \theta \leq \quad \pi \quad ; \quad 0 \quad \leq r \leq \quad 2$$

Change of Variables to Polar Form:

$$\int\limits_R \int f\left(x,y\right)dA = \int_{\alpha}^{\beta} \int_{g_1(\theta)}^{g_2(\theta)} f\left(r\cos\theta, r\sin\theta\right) r\,dr\,d\theta$$

Hence, $\displaystyle\int_{-2}^{2} \int_{0}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)dy\,dx = \int_{0}^{\pi} \int_{0}^{3} \cos\left(r^2\right) r\,dr\,d\theta$

Evaluating $\displaystyle\mathrm{I} = \int_{0}^{\pi} \int_{0}^{3} \sin\left(r^2\right) r\,dr\,d\theta$:

Let $\displaystyle\mathrm{I}_1 = \int_{0}^{3} \sin\left(r^2\right) r\,dr = -\frac{1}{2}\cos\left(r^2\right)\Big|_{0}^{2} = 0.826821810431806$

So $\displaystyle\mathrm{I} = \int_{0}^{\pi} 0.826821810431806\,d\theta = 2.597537325480374$

Therefore, $\displaystyle\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \cos\left(x^2+y^2\right)dy\,dx = \int_{0}^{\pi} \int_{0}^{3} \cos\left(r^2\right) r\,dr\,d\theta = 2.597537325480374$