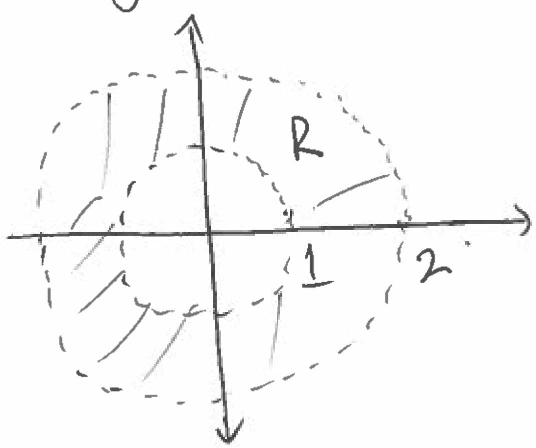
Double Integrals in Polar Coordinates

Let $f(x, y)$ be height function

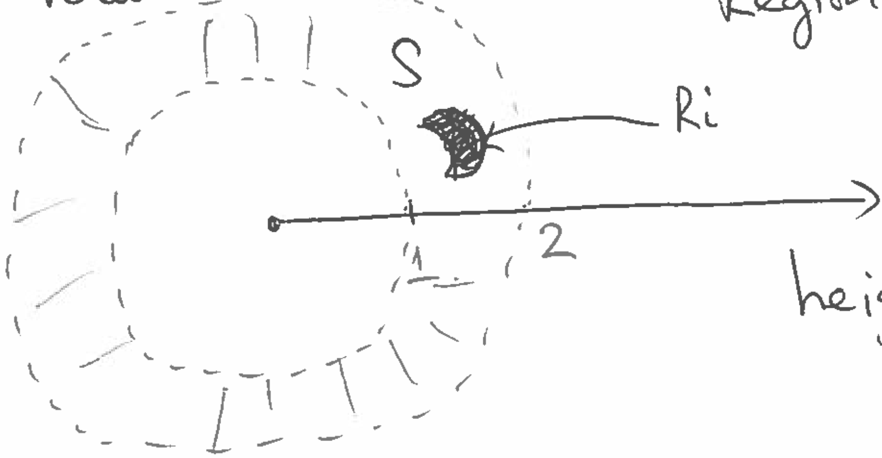
Let region R be area between $x^2+y^2=1$ and $x^2+y^2=4$



Volume of Solid Region

$$= \iint_R f(x,y) \cdot dA$$

Polar Coordinates:



Region S: $1 \le r \le 2$

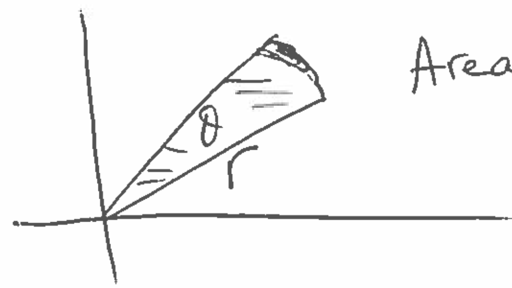
$0 \le \theta \le 2\pi$

height function $= f(r, \theta)$.

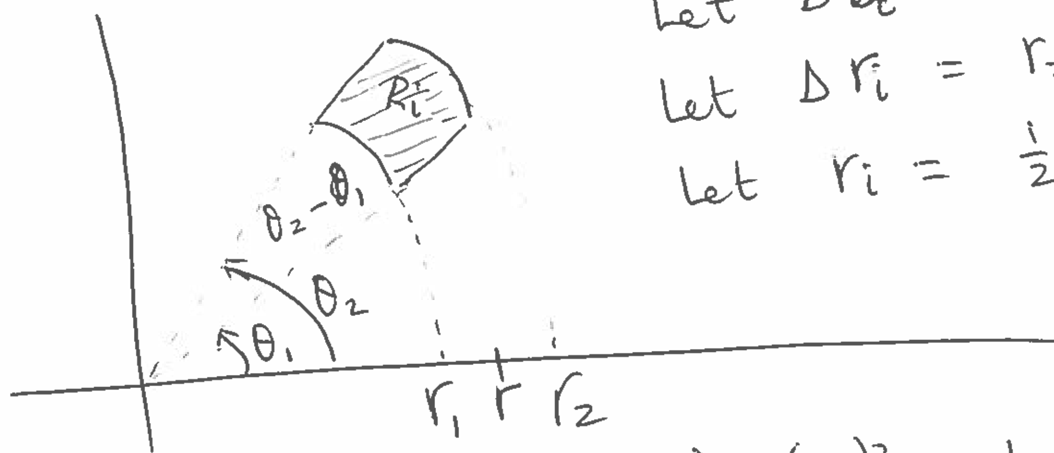
R_i is a sector inside of Region S.

Find area of R_i:

Recall:



Area of Sector $= \frac{1}{2}\cdot\theta\cdot r^2$



Let $\Delta\theta_i = \theta_2 - \theta_1$

Let $\Delta r_i = r_2 - r_1$

Let $r_i = \frac{1}{2}(r_2 + r_1)$

Area of Sector $R_i = \frac{1}{2}\cdot(\Delta\theta)\cdot(r_2)^2 - \frac{1}{2}(\Delta\theta_i)(r_1)^2$

$= \frac{1}{2}(\Delta\theta)(r_2^2 - r_1^2) = \frac{1}{2}\Delta\theta\cdot(r_2 - r_1)(r_2 + r_1)$

$= (\Delta\theta_i)(\Delta r_i)\left(\frac{1}{2}(r_2 + r_1)\right) = \Delta\theta_i\cdot\Delta r_i\cdot r_i$

For R_i, area $= A_i = r_i \cdot \Delta r_i \cdot \Delta\theta_i$

Volume of Solid Region $\approx f(r_1, \theta_1) A_1 + f(r_2, \theta_2) \cdot A_2 + \dots + f(r_n, \theta_n) A_n$

$$= \sum_{i=1}^{n} f(r_i, \theta_i) \cdot A_i$$

$$= \sum_{i=1}^{n} f(r_i, \theta_i) \cdot r_i \cdot \Delta r_i \cdot \Delta\theta_i$$

Volume of Solid Region $= \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(r_i, \theta_i) \cdot r_i \cdot \Delta r_i \cdot \Delta\theta_i$

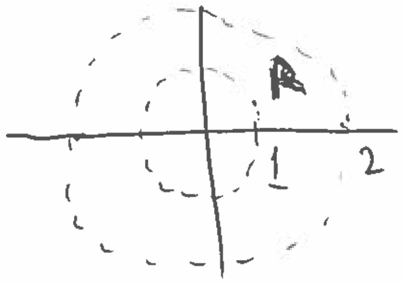
If this limit exists, then $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(r_i, \theta_i) \cdot r_i \cdot \Delta r_i \cdot \Delta\theta_i$ is defined $\iint_S f(r, \theta) \cdot r \cdot dr \cdot d\theta$

In other words, $\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(r_i, \theta_i) \cdot r_i \cdot \Delta r_i \cdot \Delta\theta_i = \iint_S f(r, \theta) r \cdot dr \cdot d\theta$

Example 1: Let $f(x,y) = x^2 + y^2$

Region R = area between $x^2+y^2=1$ and $x^2+y^2=4$

Find Volume of Solid Region using Polar Coordinates

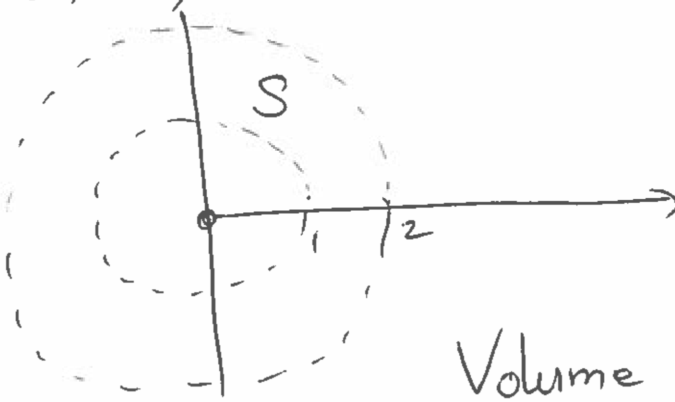


In Polar Coordinates:

$f(x,y) = x^2 + y^2$

$f(r,\theta) = r^2$

Note: $x^2 + y^2 = r^2$



Region S in Polar Coordinates:

$1 \le r \le 2$; $0 \le \theta \le 2\pi$

Volume of Solid Region $= \int_S \int f(r,\theta) \cdot r \cdot dr \cdot d\theta$

$$V = \int_0^{2\pi} \underbrace{\int_1^2 r^2 \cdot r \cdot dr}_{I_2} \, d\theta$$

Example 1 (con't):

$$I_2 = \int_1^2 r^3\,dr = \frac{1}{4}r^4\Big|_1^2 = \frac{1}{4}(16) - \frac{1}{4}(1)$$

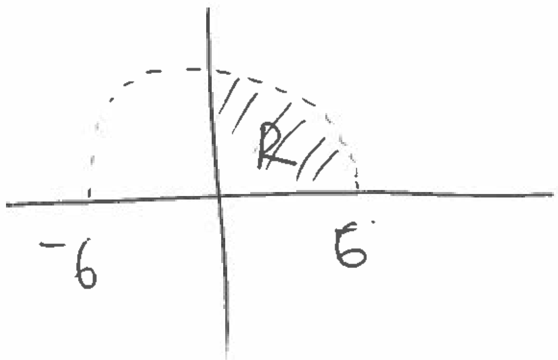
$$= 15/4$$

$$V = \int_0^{2\pi} \frac{15}{4}\,d\theta = \frac{15}{4}(2\pi) - \frac{15}{4}(0) = \frac{15}{2}\pi$$

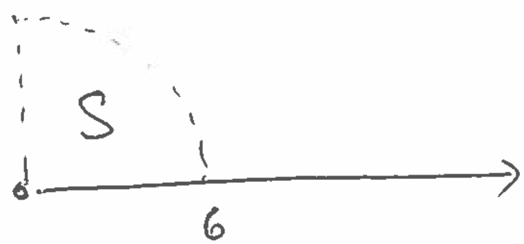
Example 2: Let $f(x,y) = (x^2+y^2)^2$

Region R: $0 \le x \le 6$; $0 \le y \le \sqrt{36-x^2}$

Find volume of Solid Region using Polar Coordinates.



Region R in Rectangular Coordinates



Region S in Polar Coordinates

$0 \le r \le 6$; $0 \le \theta \le \pi/2$

$$\text{Volume of Solid Region} = \iint_S f(r,\theta) \cdot r \cdot dr \cdot d\theta$$

$$V = \int_0^{\pi/2} \underbrace{\int_0^6 (r^2)^2 \, r \cdot dr}_{I_2} \cdot d\theta$$

Example 2 (con't) : Note: $f(x,y) = (x^2+y^2)^2$

$$f(r,\theta) = (r^2)^2$$

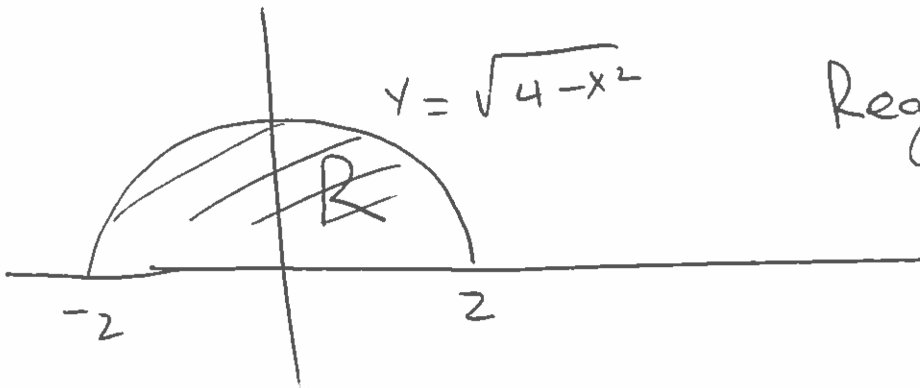
$$I_2 = \int_0^6 r^4 \cdot r\,dr = \int_0^6 r^5\,dr = \frac{1}{6} r^6 \Big|_0^6 = 7776$$

$$V = \int_0^{\pi/2} 7776\,d\theta = 7776\,\theta \Big|_0^{\pi/2} = \frac{7776\pi}{2}$$

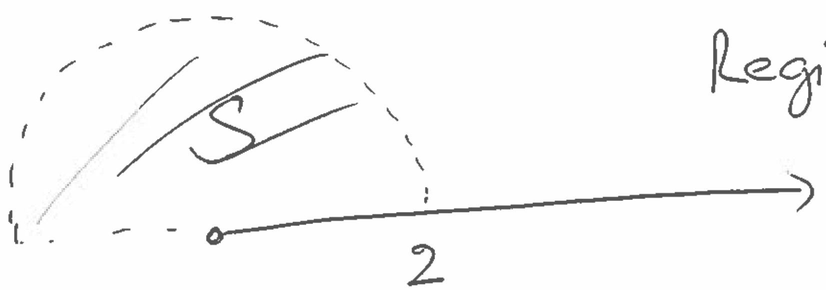
Example 3: Convert $\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (x^2+y^2)\,dy\,dx$ to Polar Coordinates

Note: R: $-2 \le x \le 2$; $0 \le y \le \sqrt{4-x^2}$

Graph $y = \sqrt{4-x^2}$



Region R in Rectangular Coordinates



Region S in Polar Coordinates

$0 \le r \le 2$

$0 \le \theta \le \pi$

$f(x,y) = x^2 + y^2 \Rightarrow f(r,\theta) = r^2$

Example 3 (con't):

Volume of Solid Region $= \int_S \int f(r,\theta) \cdot r \cdot dr \cdot d\theta$

$$V = \int_0^{\pi} \int_0^{2} r^2 \cdot r \cdot dr \cdot d\theta = \int_0^{\pi} \underbrace{\int_0^{2} r^3 \cdot dr}_{I_2} \, d\theta$$

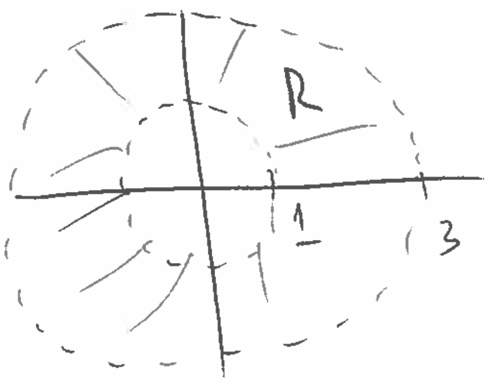
$$I_2 = \int_0^2 r^3 dr = \frac{1}{4} r^4 \Big|_0^2 = 4$$

$$V = \int_0^{\pi} 4 \, d\theta = 4\theta \Big|_0^{\pi} = 4\pi$$

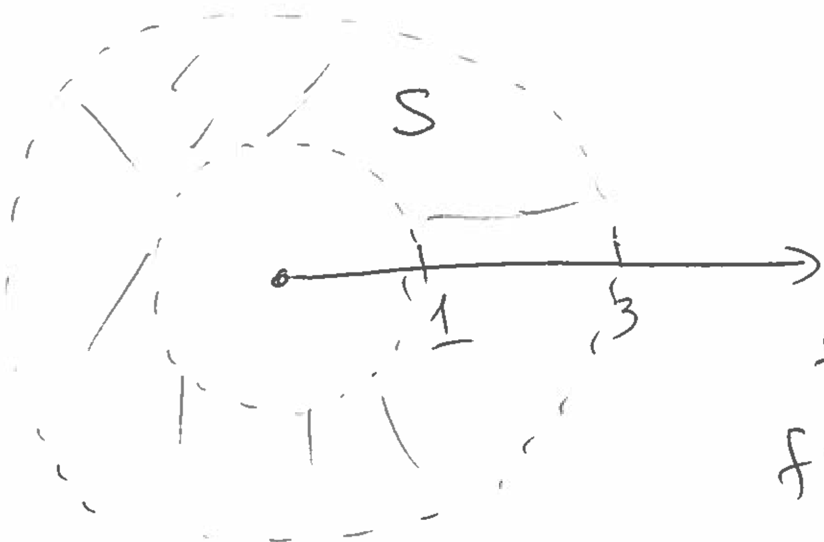
Example 4: Let $f(x,y) = x^2 + y$

Region R = Area between $x^2+y^2=1$ and $x^2+y^2=9$

Find volume of solid Region using Polar Coordinates



Region R in Rectangular Coordinates



Region S in Polar Coordinates:

$$1 \le r \le 3$$

$$0 \le \theta \le 2\pi$$

$$f(x,y) = x^2 + y$$

$$f(r,\theta) = (r\cos\theta)^2 + r\sin\theta$$

Example 4 (con't):

Volume of Solid Region $= \int\int_S f(r,\theta)\cdot r\cdot dr\cdot d\theta$

$$V = \int_0^{2\pi} \underbrace{\int_1^3 \left[r^2\cdot\cos^2\theta + r\cdot\sin\theta\right] r\cdot dr}_{I_2}\, d\theta$$

$$I_2 = \int_1^3 \left(r^3\cos^2\theta + r^2\cdot\sin\theta\right) dr$$

$$= \left(\tfrac{1}{4}r^4\cos^2\theta + \tfrac{1}{3}r^3\sin\theta\right)\Big|_1^3$$

$$= \left[\tfrac{1}{4}(81)\cos^2\theta + \tfrac{1}{3}(27)\sin\theta\right] - \left[\tfrac{1}{4}(1)\cos^2\theta + \tfrac{1}{3}\sin\theta\right]$$

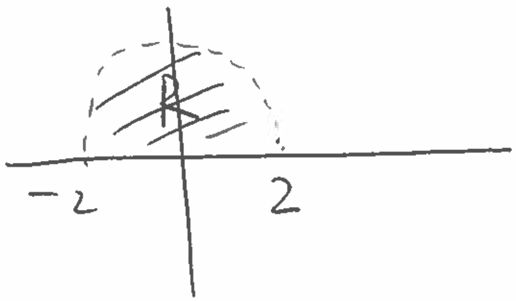
$$I_2 = 20\cos^2\theta + \frac{26}{3}\sin\theta$$

$$V = \int_0^{2\pi}\left(20\cos^2\theta + \frac{26}{3}\sin\theta\right) d\theta = 62.831853$$

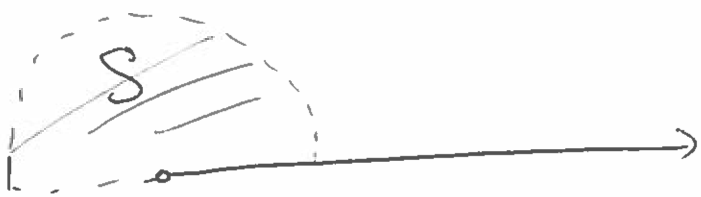
Example 5: Evaluate $V = \int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} \cos(x^2+y^2)\,dy\,dx$ by using Polar Coordinates

Region R: $-2 \le x \le 2$; $0 \le y \le \sqrt{4-x^2}$

Graph $y = \sqrt{4-x^2}$



Region R in Rectangular Coordinates



Region S in Polar Coordinates

$0 \le r \le 2$

$0 \le \theta \le \pi$

$f(x,y) = \cos(x^2+y^2)$

$f(r,\theta) = \cos(r^2)$

Example 5 (con't):

$$V = \int_S \int f(r, \theta) \cdot r \cdot dr \cdot d\theta$$

$$V = \int_0^{\pi} \underbrace{\int_0^{2} \cos(r^2) \cdot dr}_{I_2} \, d\theta$$

$$I_2 = \int_0^{2} \cos(r^2) \cdot r \, dr$$

Use u-substitution method

Let $u = r^2$

$$\frac{du}{dr} = 2r$$

$$du = 2r\,dr$$

$$\frac{1}{2} du = r\,dr$$

$$I_2 = \int \cos u \cdot \frac{1}{2} du$$

$$I_2 = \frac{1}{2}(-\sin u)$$

$$I_2 = -\frac{1}{2} \sin r^2 \Big|_0^2$$

$$I_2 = -\frac{1}{2} \sin 4$$

Example 5 (con't):

$$V = \int_0^{\pi} \frac{1}{2} \sin 4 \; d\theta$$

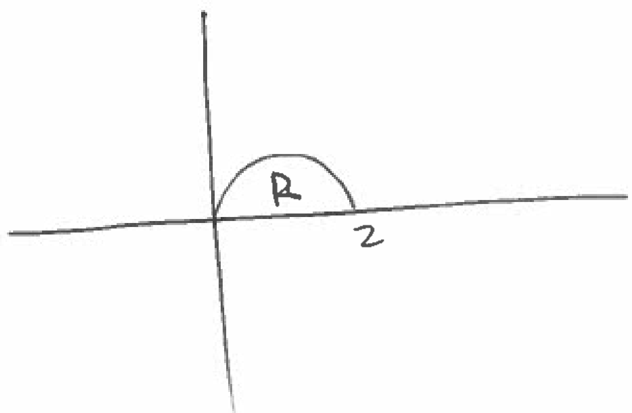
$$V = \left(\frac{1}{2} \sin 4\right)\theta \Big|_0^{\pi}$$

$$V = \left(\frac{1}{2} \sin 4\right)\pi \approx -1.188\,7825$$

Example 6: Evaluate $V = \int_0^2 \int_0^{\sqrt{2x - x^2}} xy \, dy\, dx$ using Polar Coordinates

Region R: $0 \le x \le 2$; $0 \le y \le \sqrt{2x - x^2}$

Graph $y = \sqrt{2x - x^2}$



Region R in Rectangular Coordinates

Region S in Polar Coordinates

$$y = \sqrt{2x - x^2}$$

$$r\sin\theta = \sqrt{2r\cos\theta - r^2\cos^2\theta}$$

$$r^2\sin^2\theta = 2r\cos\theta - r^2\cos^2\theta$$

$$r^2\sin^2\theta + r^2\cos^2\theta = 2r\cos\theta$$

$$r^2(\sin^2\theta + \cos^2\theta) = 2r\cos\theta$$

Example 6 (con't): $r^2(1) = 2r\cos\theta$

$$r = 2\cos\theta$$

Graph $r = 2\cos\theta$, $0 \le \theta \le \pi/2$

Volume of Solid Region $= \iint_S f(r,\theta)\, r\, dr\, d\theta$

$$V = \int_0^{\pi/2} \underbrace{\int_0^{2\cos\theta} r\cos\theta \cdot r\sin\theta \cdot r \cdot dr}_{I_2}\, d\theta$$

Note: $f(x,y) = x \cdot y$

$f(r,\theta) = r\cos\theta \cdot r\sin\theta$

$$I_2 = \int_0^{2\cos\theta} r^3 \cos\theta \cdot \sin\theta \, dr$$

Example 6 (con't)

$$I_2 = \cos\theta \cdot \sin\theta \cdot \left(\tfrac{1}{4} r^4\right)\Big|_0^{2\cos\theta}$$

$$I_2 = \tfrac{1}{4} \cos\theta \cdot \sin\theta \cdot (2\cos\theta)^4 - 0$$

$$I_2 = 4 (\cos\theta)^5 \cdot \sin\theta$$

$$V = \int_0^{\pi/2} 4 (\cos\theta)^5 \sin\theta \, d\theta = \tfrac{2}{3}$$