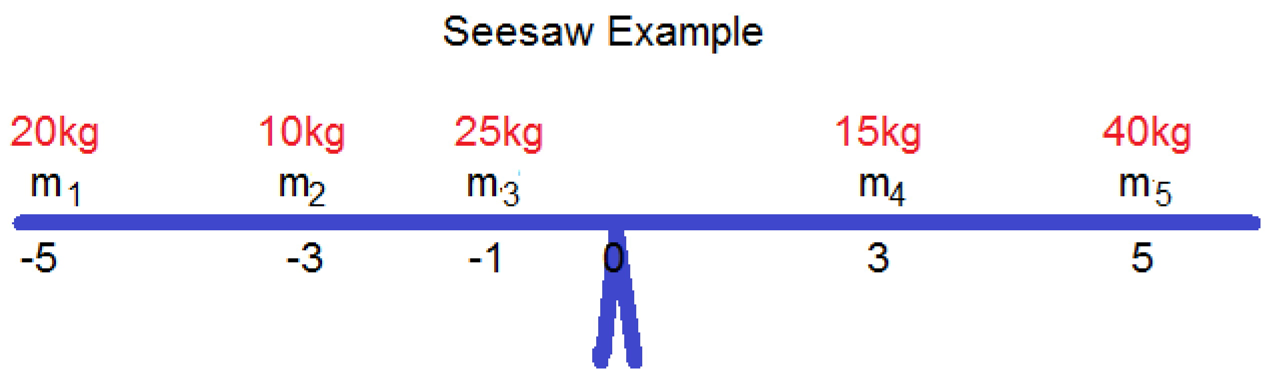
Example 1: Center of Mass in a One-Dimensional System



Moment of mass about the origin = $(\text{mass})(\text{distance from origin})$

moment of mass $m_1 = (20\text{kg})(-5) = -100$

moment of mass $m_2 = (10\text{kg})(-3) = -30$

moment of mass $m_3 = (25\text{kg})(-1) = -25$

moment of mass $m_4 = (15\text{kg})(3) = 45$

moment of mass $m_5 = (40\text{kg})(5) = 200$

M_0 = Sum of Moments = -100 + -30 + -25 + 45 + 200 = 90

Definition of Equilibrium: If $M_0 = 0$, then the seesaw is said to be in equilibrium.

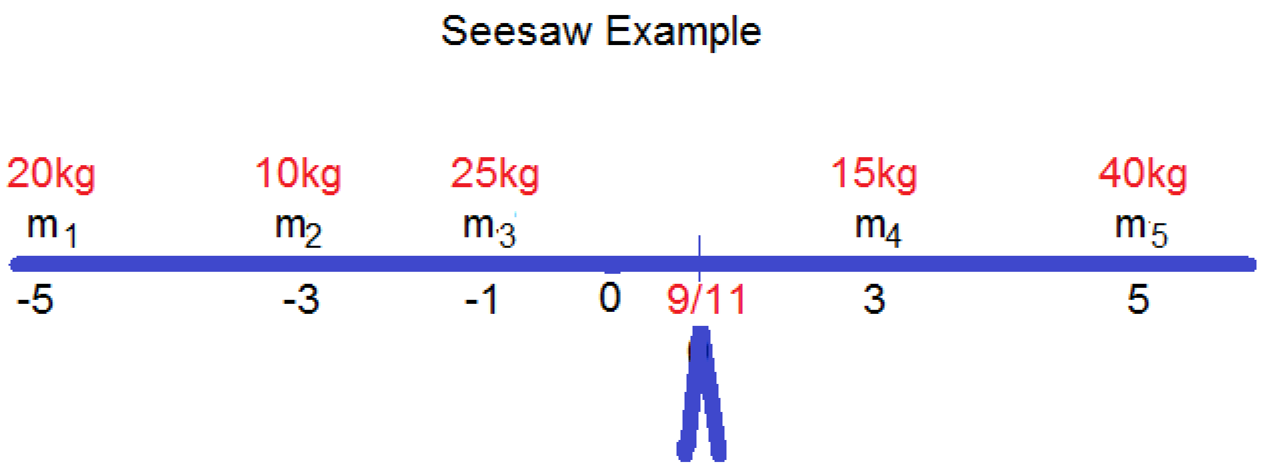
Where should fulcrum be placed so that seesaw system is in equilibrium?

Definition of Center of Mass: location of fulcrum such that $M_0 = 0$.

sum of mass = 20 + 10 + 25 + 15 + 40 = 110kg

$$\text{Center of Mass} = \frac{\text{sum of moments}}{\text{sum of mass}} = \frac{90}{110} = \frac{9}{11}$$

Hence, if fulcrum is placed at a distance of 9/11 to the right of 0, then seesaw system is in equilibrium.



Moment of mass about the origin = $(\text{mass})(\text{distance from origin})$

moment of mass $m_1 = (20\text{kg})(-5-9/11) = -1280/11$

moment of mass $m_2 = (10\text{kg})(-3-9/11) = -420/11$

moment of mass $m_3 = (25\text{kg})(-1-9/11) = -500/11$

moment of mass $m_4 = (15\text{kg})(3-9/11) = 360/11$

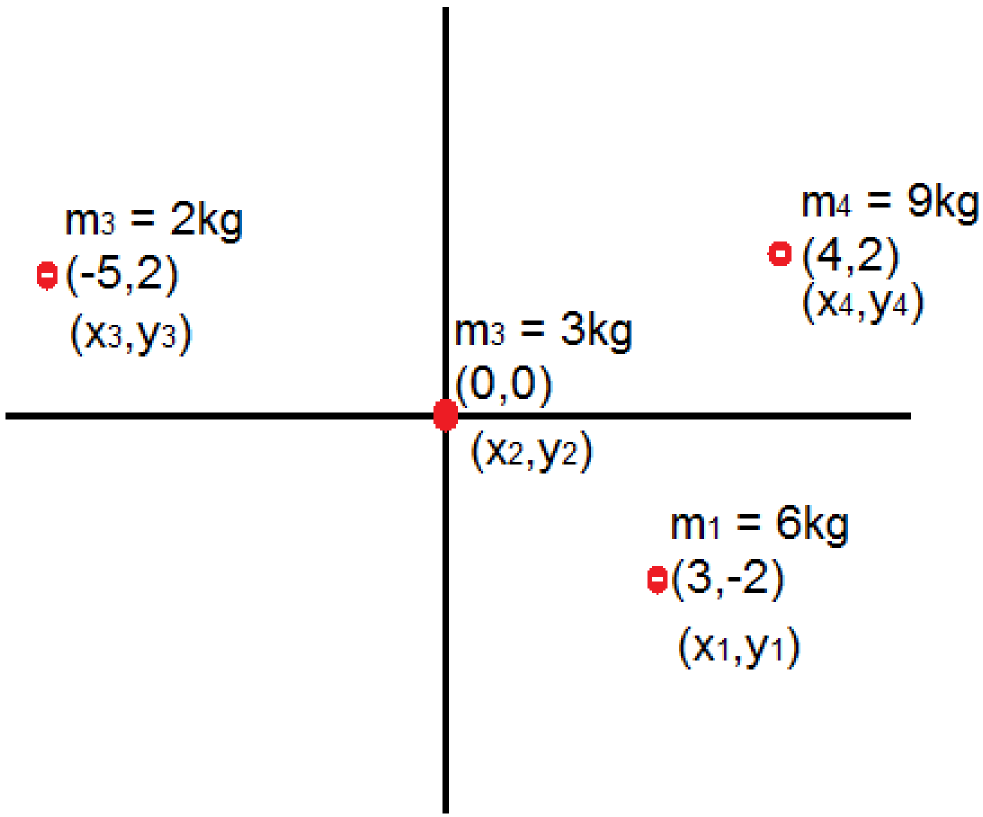
moment of mass $m_5 = (40\text{kg})(5-9/11) = 1840/11$

$$M_0 = \text{Sum of Moments} = \frac{-1280}{11} + \frac{-420}{11} + \frac{-500}{11} + \frac{360}{11} + \frac{1840}{11} = 0$$

Hence, seesaw is in equilibrium when fulcrum is placed at 9/11 unit to the right of 0.

Example 2: Center of Mass in a Two-Dimensional System

Find center of mass for the following system with four point masses.



m = total mass = 6 + 3 + 2 + 9 = 20kg

moment of mass m_1 about the y-axis = $\left(\text{mass of } m_1\right)$(distance of m_1 from y-axis) = $m_1x_1 = \left(6\text{kg}\right)\left(3\right) = 18$

moment of mass m_2 about the y-axis = $\left(\text{mass of } m_2\right)$(distance of m_2 from y-axis) = $m_2x_2 = \left(3\text{kg}\right)\left(0\right) = 0$

moment of mass m_3 about the y-axis = $\left(\text{mass of } m_3\right)$(distance of m_3 from y-axis) = $m_3x_3 = \left(2\text{kg}\right)\left(-5\right) = -10$

moment of mass m_4 about the y-axis = $\left(\text{mass of } m_4\right)$(distance of m_4 from y-axis) = $m_4x_4 = \left(9\text{kg}\right)\left(4\right) = 36$

M_y = Moment about the y-axis = $m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 = 6(3) + 3(0) + 2(-5) + 9(4) = 44$

moment of mass m_1 about the x-axis = $\left(\text{mass of } m_1\right)$(distance of m_1 from x-axis) = $m_1y_1 = \left(6\text{kg}\right)\left(-2\right) = -12$

moment of mass m_2 about the x-axis = $\left(\text{mass of } m_2\right)$(distance of m_2 from x-axis) = $m_2y_2 = \left(3\text{kg}\right)\left(0\right) = 0$

moment of mass m_3 about the x-axis = $\left(\text{mass of } m_3\right)$(distance of m_3 from x-axis) = $m_3y_3 = \left(2\text{kg}\right)\left(2\right) = 4$

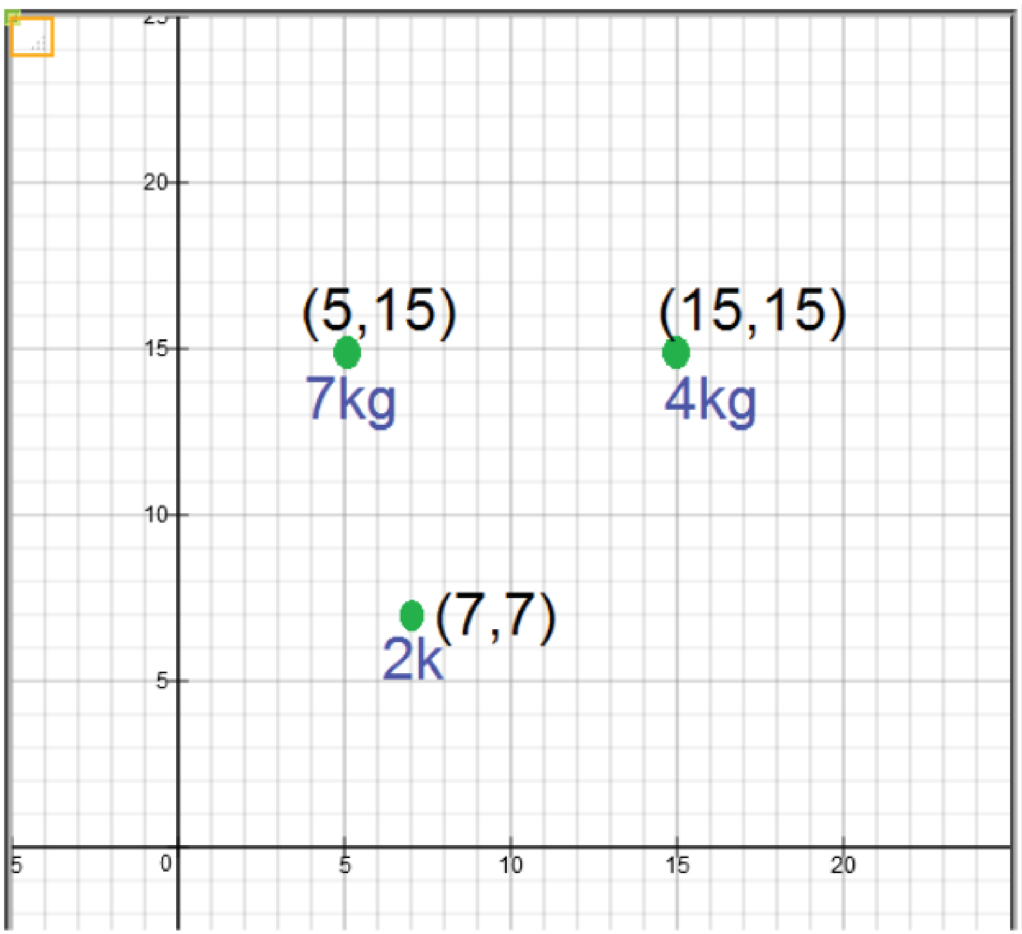
moment of mass m_4 about the x-axis = $\left(\text{mass of } m_4\right)$(distance of m_4 from x-axis) = $m_4y_4 = \left(9\text{kg}\right)\left(2\right) = 18$

M_x = Moment about the x-axis = $m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4 = 6(-2) + 3(0) + 2(2) + 9(2) = 12$

Center of Mass = location where mass is concentrated = $\left(\dfrac{M_y}{m},\ \dfrac{M_x}{m}\right) = \left(\dfrac{44}{20},\ \dfrac{12}{20}\right) = \left(2.2,\ 0.6\right)$

Example 3: Center of Mass in a Two-Dimensional System

Example of Discrete Masses: There are three point masses arranged as follows. What is the center of mass of this system?



m = total mass = 7 + 4 + 2 = 13kg

moment of mass m_1 about the y-axis = $\left(\text{mass of } m_1\right)$(distance of m_1 from y-axis) = $m_1 x_1 = \left(7\text{kg}\right)\left(5\right) = 35$

moment of mass m_2 about the y-axis = $\left(\text{mass of } m_2\right)$(distance of m_2 from y-axis) = $m_2 x_2 = \left(4\text{kg}\right)\left(15\right) = 60$

moment of mass m_3 about the y-axis = $\left(\text{mass of } m_3\right)$(distance of m_3 from y-axis) = $m_3 x_3 = \left(2\text{kg}\right)\left(7\right) = 14$

M_y = Moment about the y-axis = $m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4 = 35 + 60 + 14 = 109$

moment of mass m_1 about the x-axis = $\left(\text{mass of } m_1\right)$(distance of m_1 from x-axis) = $m_1 y_1 = \left(7\text{kg}\right)\left(15\right) = 105$

moment of mass m_2 about the x-axis = $\left(\text{mass of } m_2\right)$(distance of m_2 from x-axis) = $m_2 y_2 = \left(4\text{kg}\right)\left(15\right) = 60$

moment of mass m_3 about the x-axis = $\left(\text{mass of } m_3\right)$(distance of m_3 from x-axis) = $m_3 y_3 = \left(2\text{kg}\right)\left(7\right) = 14$

M_x = Moment about the x-axis = $m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4 = 105 + 60 + 14 = 179$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } y\text{-axis}}{\text{sum of the masses}}$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } x\text{-axis}}{\text{sum of the masses}}$$

Center of mass = $(\bar{x}, \bar{y})$

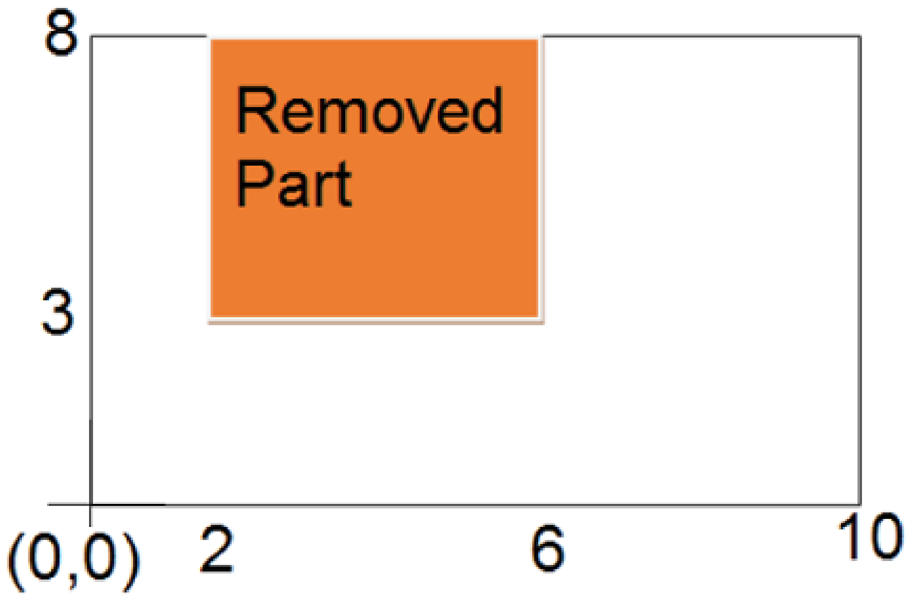
$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{(7)(5)+(4)(15)+(2)(7)}{7+4+2} = 8.4$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{(7)(15)+(4)(15)+(2)(7)}{7+4+2} = 13.8$$

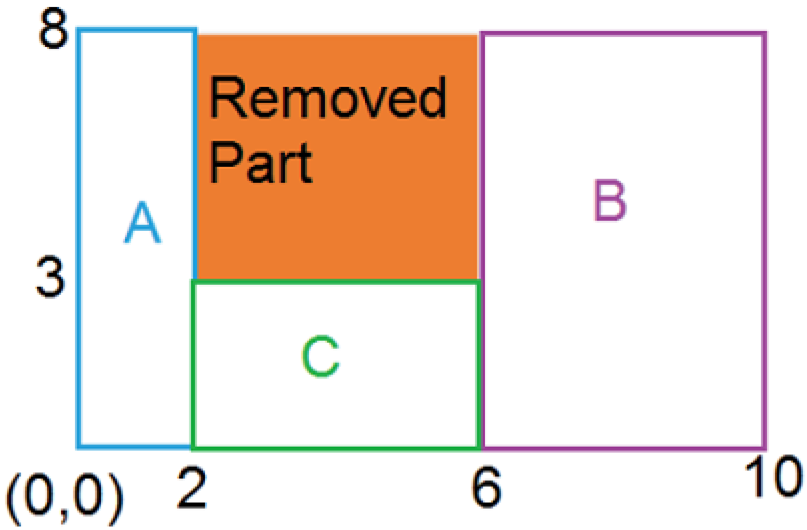
Center of mass = $(\bar{x}, \bar{y}) = (8.4, 13.8)$

Example 4: Center of Mass in a Two-Dimensional System

Suppose we have an 8 in by 10 in piece of paper. We then remove a portion of it (see figure below). Find the center of mass of the remaining piece.



Let ρ be the density of paper.



For piece A:

center of mass = location where mass is concentrated = (1, 4)

m_A = mass of piece A = $\rho\left(\text{Area of piece A}\right) = 16\rho$

For piece B:

center of mass = location where mass is concentrated = (8, 4)

m_B = mass of piece B = $\rho\left(\text{Area of piece B}\right) = 32\rho$

For piece C:

center of mass = location where mass is concentrated = (4, 1.5)

m_C = mass of piece C = $\rho\left(\text{Area of piece C}\right) = 12\rho$

For piece A:

center of mass = location where mass is concentrated = (1, 4) = (x_1, y_1)

m_A = mass of piece A = $\rho\left(\text{Area of piece A}\right) = 16\rho$

moment of m_A about y-axis = $\left(mass\right)\left(\text{distance of mass from } y\text{-axis}\right) = \left(16\rho\right)\left(1\right) = 16\rho$

moment of m_A about x-axis = $\left(mass\right)\left(\text{distance of mass from } x\text{-axis}\right) = \left(16\rho\right)\left(4\right) = 16\rho$

For piece B:

center of mass = location where mass is concentrated = (8, 4) = (x_2, y_2)

m_B = mass of piece B = $\rho\left(\text{Area of piece B}\right) = 32\rho$

moment of m_B about y-axis = $\left(mass\right)\left(\text{distance of mass from } y\text{-axis}\right) = \left(32\rho\right)\left(8\right) = 256\rho$

moment of m_B about x-axis = $\left(mass\right)\left(\text{distance of mass from } x\text{-axis}\right) = \left(32\rho\right)\left(4\right) = 128\rho$

For piece C:

center of mass = location where mass is concentrated = (4, 1.5) = (x_3, y_3)

m_C = mass of piece C = $\rho\left(\text{Area of piece C}\right) = 12\rho$

moment of m_C about y-axis = $\left(mass\right)\left(\text{distance of mass from } y\text{-axis}\right) = \left(16\rho\right)\left(4\right) = 64\rho$

moment of m_C about x-axis = $\left(mass\right)\left(\text{distance of mass from } x\text{-axis}\right) = \left(16\rho\right)\left(1.5\right) = 24\rho$

Center of Mass of remaining piece:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } y\text{-axis}}{\text{sum of the masses}} = \frac{M_y}{\text{Total Mass}}$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } x\text{-axis}}{\text{sum of the masses}} = \frac{M_x}{\text{Total Mass}}$$

$$\text{Center of mass} = (\bar{x}, \bar{y})$$

$m =$ mass of remaining piece $= m_A + m_B + m_C = 16\rho + 32\rho + 12\rho = 60\rho$

$M_y =$ moment about the y-axis $= m_A \cdot x_1 + m_B \cdot x_2 + m_C \cdot x_3 = 16\rho + 256\rho + 48\rho = 320\rho$

$M_x =$ moment about the x-axis $= \; = 64\rho + 128\rho + 18\rho = 210\rho$

$$\text{Center of remaining piece} = (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right) = \left(\frac{m_A \cdot x_1 + m_B \cdot x_2 + m_C \cdot x_3}{m}, \frac{m_A \cdot x_1 + m_B \cdot x_2 + m_C \cdot x_3}{m}\right)$$

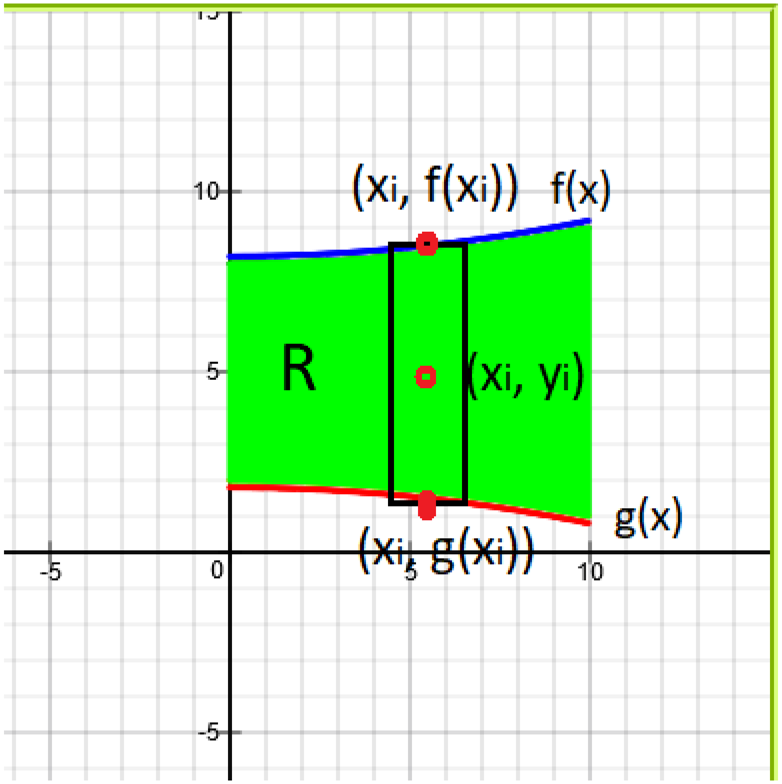
$$\text{Center of remaining piece} = \left(\frac{320\rho}{60\rho}, \frac{210\rho}{60\rho}\right) = \left(\frac{16}{3}, \frac{7}{2}\right)$$

Mass of Planar Lamina with Uniform Density

Find the mass and center of planar lamina that has the shape of region R.

Region R is boundef by $f(x)$ and $f(x)$ with $a \le x \le b$.

Let ρ be density of region R.



To find mass of region R, we will divide region R into n number of rectangles.

For the ith rectangle, center of mass will be at the point $\left(x_i, y_i\right)$; and $y_i = \dfrac{f(x_i)+g(x_i)}{2}$.

In other words, for the ith rectangle, mass is concentrated at the point $\left(x_i, y_i\right) = \left(x_i, \dfrac{f(x_i)+g(x_i)}{2}\right)$.

Area of ith rectangle $=$ (height of ith rectangle)(height of ith rectangle) $= \left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right)$

m_i = mass of ith rectangle $=$ (density)(area of ith rectangle) $= \rho\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right)$

moment of m_i about the x-axis = $\left(m_i\right)$(distance from the point $\left(x_i, y_i\right)$ to the x-axis) $= m_i y_i$

$$= \rho\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right)\left(\frac{f(x_i)+g(x_i)}{2}\right) = \rho\left(\frac{f(x_i)+g(x_i)}{2}\right)\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right)$$

moment of m_i about the y-axis = $\left(m_i\right)$(distance from the point $\left(x_i, y_i\right)$ to the y-axis) $= m_i x_i$

$$= \rho\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right)\left(x_i\right) = \rho\left(x_i\right)\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right)$$

Hence,

$m =$ mass of region $R = \lim_{n\to\infty}\sum_{i=1}^{n}\rho\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right) = \int_a^b \rho\left(f(x)-g(x)\right)dx$

M_x = moment of region R about the x-axis $= \lim_{n\to\infty}\sum_{i=1}^{n}\rho\left(\frac{f(x_i)+g(x_i)}{2}\right)\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right)$

$$= \int_a^b \rho\left(\frac{f(x)+g(x)}{2}\right)\left(f(x)-g(x)\right)dx$$

M_y = moment of region R about the y-axis $= \lim_{n\to\infty}\sum_{i=1}^{n}\rho\left(x_i\right)\left(f(x_i)-g(x_i)\right)\left(\Delta x_i\right) = \int_a^b \rho\left(x\right)\left(f(x)-g(x)\right)dx$

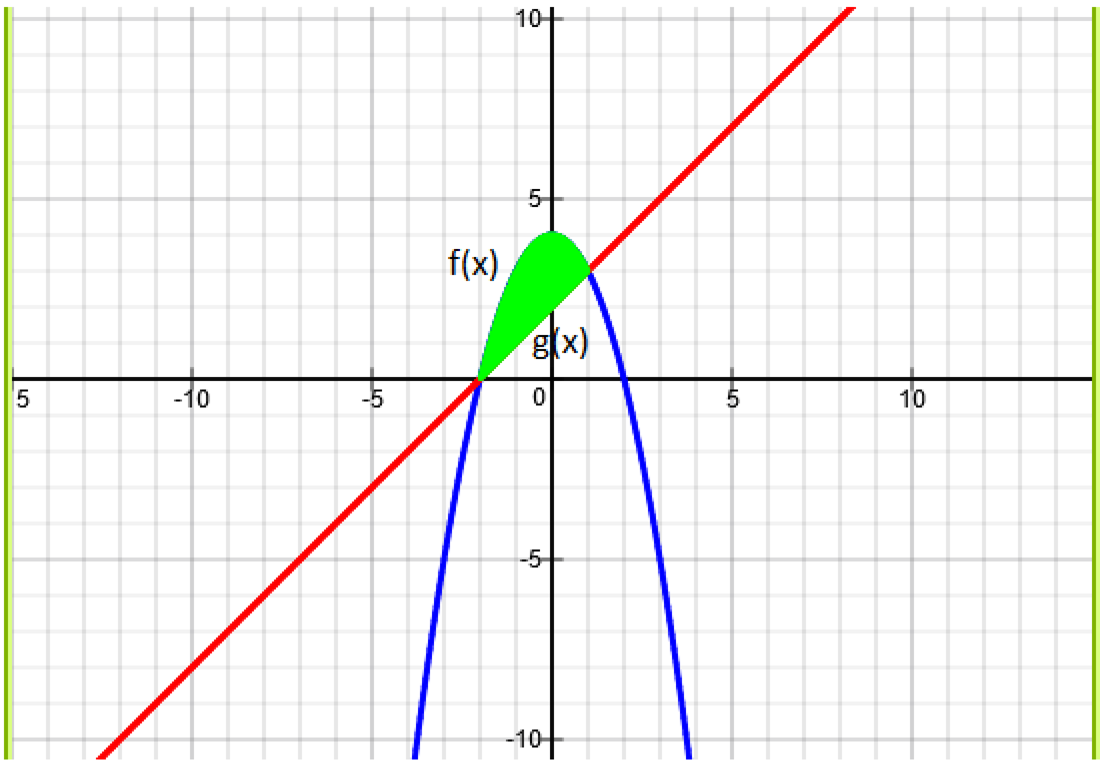
Center of mass of region $R = \left(\frac{M_y}{m},\ \frac{M_x}{m}\right)$

Example 5: Center of Mass of Planar Lamina

Find the mass and center of mass of region R bounded by $f(x)$ and $g(x)$.

$f(x) = 4 - x^2$ and $g(x) = x + 2$.

Let ρ be density of region R.



$$m = \text{ mass of region } R = \int_a^b \rho\left(f(x) - g(x)\right) dx = \int_{-2}^{1} \rho\left[\left(4 - x^2\right) - \left(x + 2\right)\right] dx = 4.5\rho$$

$$M_x = \text{moment of region } R \text{ about the } x\text{-axis} = \int_a^b \rho\left(\frac{f(x) + g(x)}{2}\right)\left(f(x) - g(x)\right) dx$$

$$M_x = \text{moment of region } R \text{ about the } x\text{-axis} = \int_a^b \rho\left(\frac{\left(4 - x^2\right) + \left(x + 2\right)}{2}\right)\left(\left(4 - x^2\right) - \left(x + 2\right)\right) dx = 10.8\rho$$

$$M_y = \text{moment of region } R \text{ about the } y\text{-axis} = \int_a^b \rho\left(x\right)\left(f(x) - g(x)\right) dx$$

$$M_y = \text{moment of region } R \text{ about the } y\text{-axis} = \int_{-2}^{1} \rho\left(x\right)\left(\left(4 - x^2\right) - \left(x + 2\right)\right) dx = -9\rho$$

$$\text{Center of mass of region } R = \left(\frac{M_y}{m},\ \frac{M_x}{m}\right) = \left(\frac{-9\rho}{4.5\rho}, \frac{10.8\rho}{4.5\rho}\right) = \left(-0.5,\ 2.4\right)$$

For lamina with non-homogeneous density:

mass of lamina: $m = \iint\limits_R \rho(x, y)\,dA = \underline{\quad ? \quad}$

moment of mass with respect to x-axis: $M_x = \iint\limits_R y\rho(x, y)\,dA$

moment of mass with respect to y-axis: $M_y = \iint\limits_R x\rho(x, y)\,dA$

Center of Mass: $\bar{x} = \dfrac{M_y}{m} \qquad \bar{y} = \dfrac{M_x}{m}$

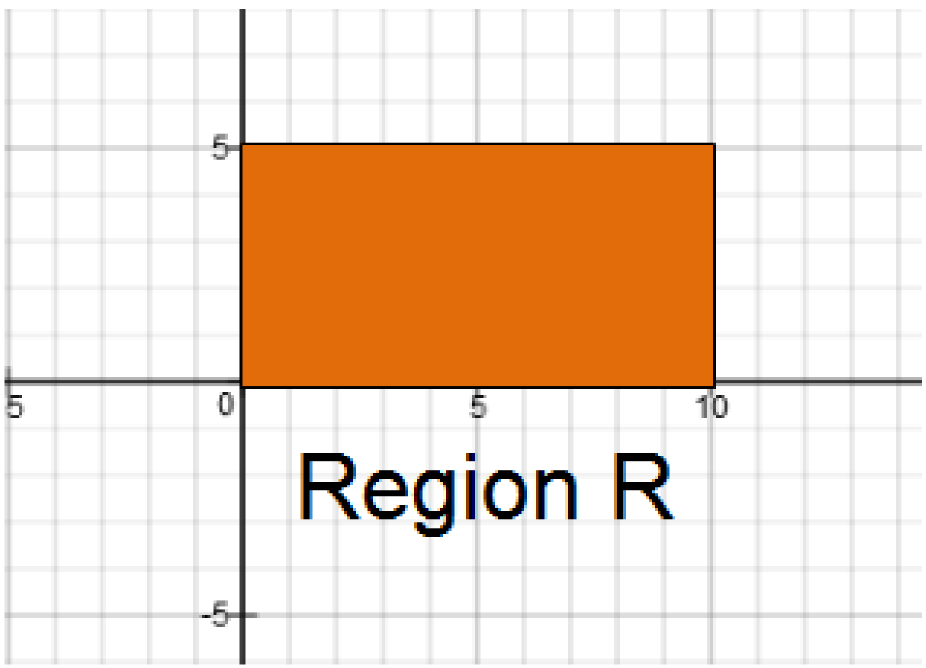
Hence, $\bar{x} = \dfrac{M_y}{m} = \dfrac{\iint\limits_R x\rho(x, y)\,dA}{\iint\limits_R \rho(x, y)\,dA} \qquad \bar{y} = \dfrac{M_x}{m} = \dfrac{\iint\limits_R y\rho(x, y)\,dA}{\iint\limits_R \rho(x, y)\,dA}$

Example 6 : Find the mass (m) of the lamina corresponding to region R

with the given desisty function ρ.

$\rho(x, y) = 8xy.$

$m = \iint_R \rho(x, y)\,dA = \underline{\ ?\qquad\qquad\qquad\qquad\qquad}$



$$m = \iint_R \rho(x, y)\,dA = \int_0^5 \int_0^{10} 8xy\,dx\,dy$$

Evaluate $\int_0^{10} 8xy\,dx = 8y \int_0^{10} x\,dx = 8y \left[\frac{x^2}{2} \right]_0^{10} = 400y$

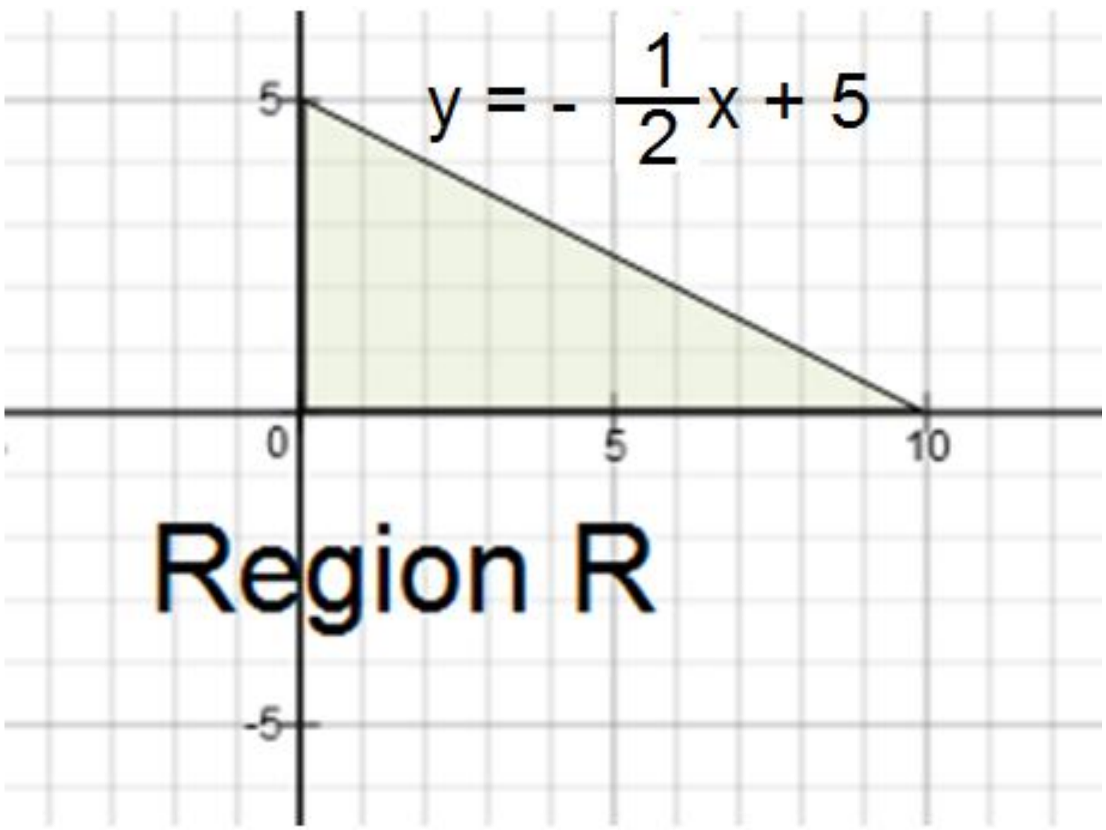
Hence, $\int_0^5 \int_0^{10} 8xy\,dx\,dy = \int_0^5 400y\,dy = 400 \left[\frac{y^2}{2} \right]_0^5 = 500$

Example 7 : Find the mass (m) of the lamina corresponding to region R

with the given desisty function ρ.

$\rho(x, y) = 14xy.$

$m = \iint_R \rho(x, y)\,dA = \underline{\ ?\qquad}$



$$m=\iint_R \rho(x,y)\,dA=\iint_R 14kxydydx=\int_0^{10}\int_{y=0}^{y=-0.5x+5} 14xydydx$$

Evaluate $\displaystyle\int_{y=0}^{y=-0.5x+5} 14xydy = 14x\int_{0}^{-0.5x+5} ydy = 14x\left[\frac{y^2}{2}\right]_0^{-0.5x+5}$

$$=14x\left[\frac{\left(-0.5x+5\right)^2}{2}\right]-\ 4kx\left[\frac{(0)}{2}\right]=14x\left[\frac{\left(-0.5x+5\right)^2}{2}\right]$$

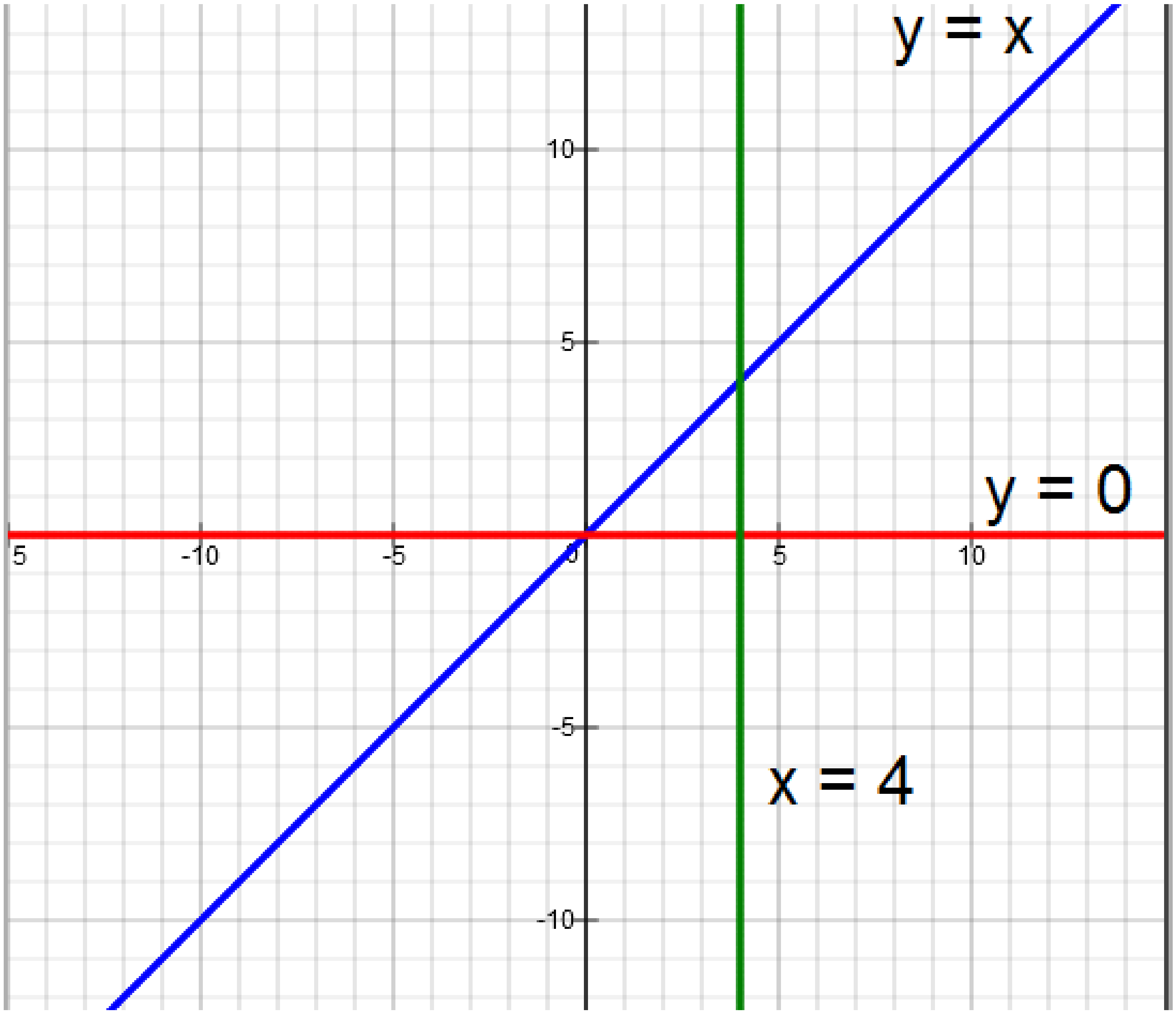
Hence, $\displaystyle\int_0^{10}\int_{y=0}^{y=-0.5x+5} 14xydydx=\int_0^{10}14x\left[\frac{\left(-0.5x+5\right)^2}{2}\right]dx$

$$=\int_0^{10}7x\left(-0.5x+5\right)^2 dx=1458.333333333$$

Example 8: Find the mass and the center of mass of the lamina bounded by the following graphs.

$y = x; \quad y = 0; \quad x = 4;$

Density Function: $\rho(x, y) = 8x.$


y = x
y = 0
x = 4
10
5
-5
-10
5
-10
-5
0
5
10

mass of lamina: $m = \iint\limits_R \rho(x, y)\,dydx = \int_0^4 \int_{y=0}^{y=x} 8x\,dydx$

Evaluate inside integral: $\int_{y=0}^{y=x} 8x\,dy = \left[8xy\right]_0^x = 8x^2$

$$m = \int_0^4 \int_{y=0}^{y=x} 8x\,dydx = \int_0^4 8x^2\,dx = \left[8 \cdot \frac{x^3}{3}\right]_0^4 = \frac{512}{3}$$

moment of mass with respect to x-axis:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_0^4 \int_{y=0}^{y=x} 8xy\,dy\,dx$$

Evaluate inside integral: $\displaystyle\int_{y=0}^{y=x} 8xy\,dy = \left[8x \cdot \frac{y^2}{2}\right]_0^x = 4x^3$

$$M_x = \int_0^4 \int_{y=0}^{y=x} 8xy\,dy\,dx = \int_0^4 4x^3\,dx = \left[4 \cdot \frac{x^4}{4}\right]_0^4 = 256$$

moment of mass with respect to y-axis:

$$M_y = \iint_R x\rho(x,y)\,dA = \int_0^4 \int_{y=0}^{y=x} 8xx\,dy\,dx$$

Evaluate inside integral: $\displaystyle\int_{y=0}^{y=x} 8xx\,dy = \int_{y=0}^{y=x} 8x^2\,dy = \left[8x^2 y\right]_0^x = 8x^3$

$$M_y = \int_0^4 \int_{y=0}^{y=x} 8xx\,dy\,dx = \int_0^4 8x^3\,dx = 8\left[\frac{x^4}{4}\right]_0^4 = 512$$

Center of Mass: $\displaystyle\bar{x} = \frac{M_y}{m} = \frac{512}{\left(\frac{512}{3}\right)} = 3$ $\qquad\displaystyle\bar{y} = \frac{M_x}{m} = \frac{256}{\left(\frac{512}{3}\right)} = 1.5$

Moments of Inertia (rotational inertia):

For a point mass the **moment of inertia** is the mass times the square of perpendicular distance to the rotation axis, $I = mr^2$.

Illustration of Moments of Inertia:

https://www.youtube.com/watch?v=fHDB7PMUdZE

https://www.youtube.com/watch?v=W9fPGXyvrVM

I_x = moment of inertia with respect to x-axis of a lamina

of variable density

$$I_x = \iint_R y^2 \rho(x, y)\,dA \quad ; \qquad \text{Note: mass} = \iint_R \rho(x, y)\,dA$$

I_y = moment of inertia with respect to y-axis of a lamina

of variable density

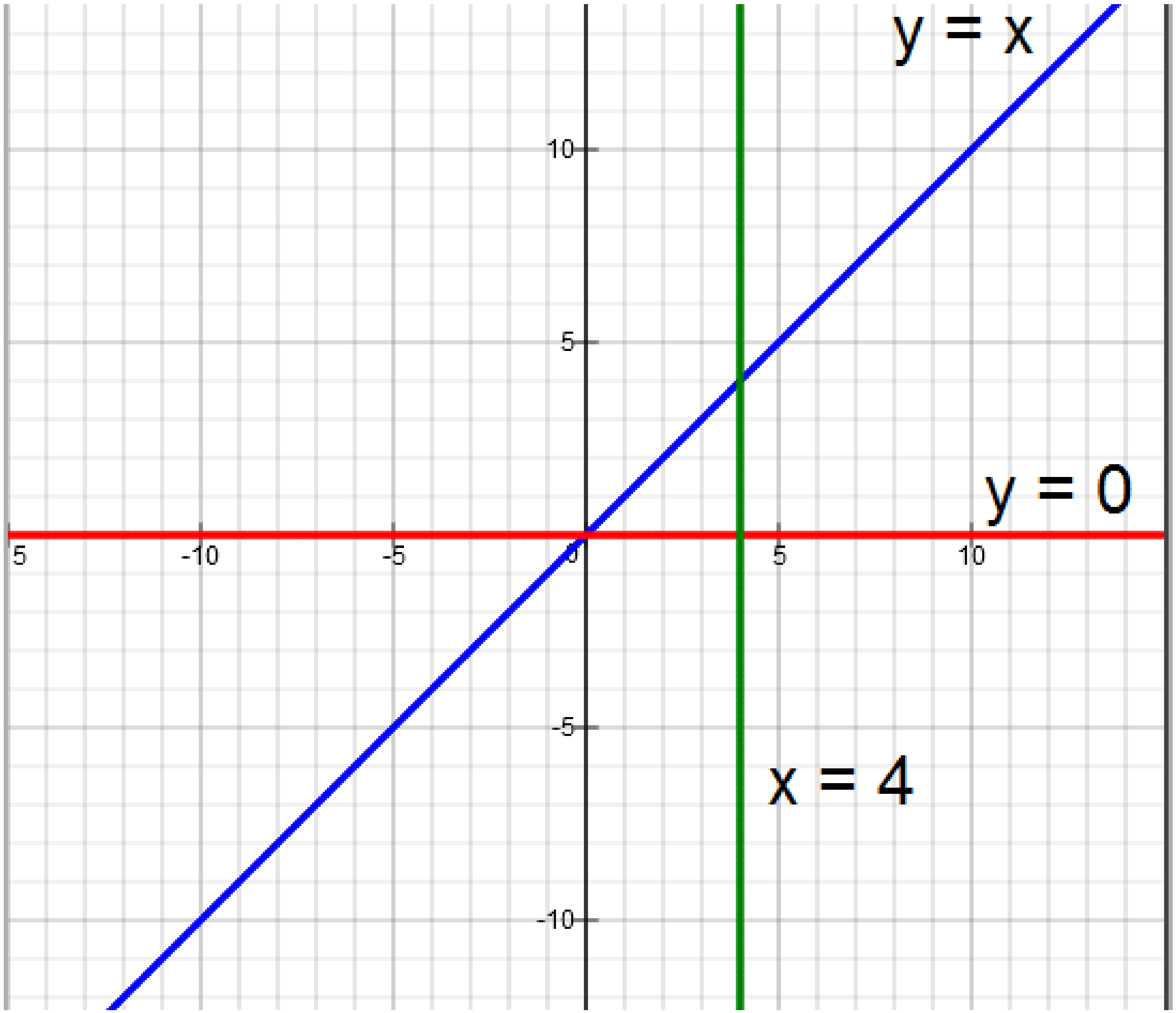
$$I_y = \iint_R x^2 \rho(x, y)\,dA \qquad \text{Note: mass} = \iint_R \rho(x, y)\,dA$$

Example 4: Find the moments of inertia of the lamina

bounded by the following graphs

$$y = x; \quad y = 0; \quad x = 4;$$

Density Function: $\rho(x, y) = 8x.$


y = x
y = 0
x = 4
10
5
-5
-10
5
-10
-5
0
5
10

Example 9 : Find the moments of inertia of the lamina

bounded by the following graphs

$$y = x; \quad y = 0; \quad x = 4;$$

Density Function: $\rho(x, y) = 8x.$

I_x = moment of inertia with respect to x-axis $= \iint_R y^2 \rho(x, y)\,dA$

$$I_x = \int_0^4 \int_{y=0}^{y=x} y^2 8x\,dy\,dx$$

Evaluate inside integral: $\displaystyle\int_{y=0}^{y=x} y^2 8x\,dy = \left[8x \cdot \frac{y^3}{3} \right]_0^x = \frac{8}{3} x^4$

$$I_x = \int_0^4 \int_{y=0}^{y=x} y^2 8x\,dy\,dx = I_x = \int_0^4 \frac{8}{3} x^4\,dx = \left[\frac{8}{3} \frac{x^5}{5} \right]_0^4 = \frac{8}{15}(1024) = 546.133333$$

$$I_x = \text{moment of inertia with respect to } y\text{-axis} = \iint_R x^2 \rho(x, y)\, dA$$

$$I_x = \int_0^4 \int_{y=0}^{y=x} x^2 8x\, dy\, dx = \int_0^4 \int_{y=0}^{y=x} 8x^3\, dy\, dx$$

Evaluate inside integral: $\displaystyle\int_{y=0}^{y=x} 8x^3\, dy = \left[8x^3 y\right]_0^x = 8x^4$

$$I_x = \int_0^4 \int_{y=0}^{y=x} 8x^3\, dy\, dx = I_x = \int_0^4 8x^4\, dx = \left[8\frac{x^5}{5}\right]_0^4 = \frac{8(1024)}{5} = 1638.4$$