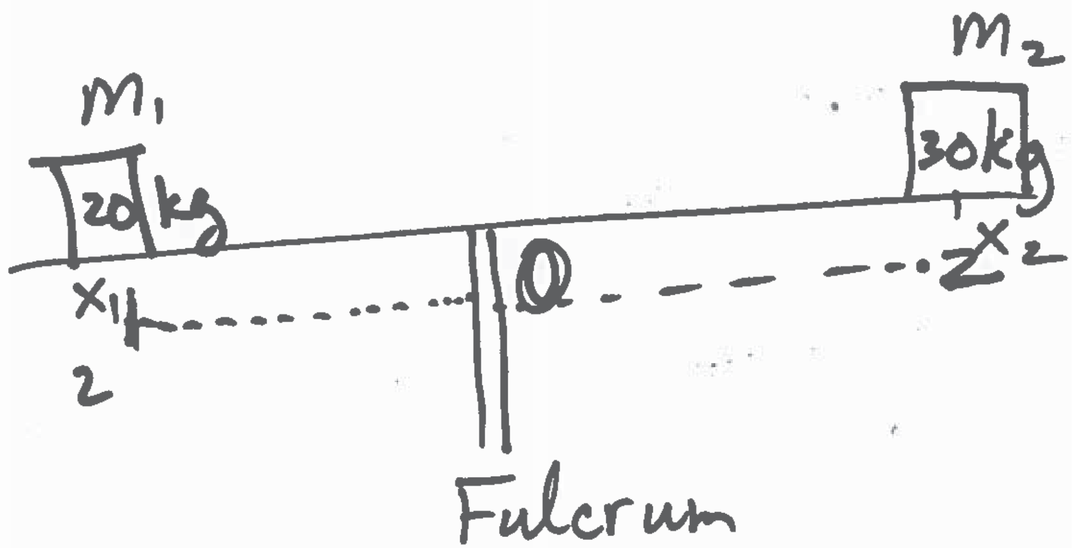
14.4 Center of Mass & Moments

Seesaw Example:

Ex. 1



Moment = mass · distance

Left Moment = $m_1 x_1 = 20(-2) = -40\,kg \cdot m$

Right Moment = $m_2 \cdot x_2 = 30(2) = 60\,kg \cdot m$

M_0 = Moment about point O = $m_1 x_1 + m_2 x_2$

$= -40 + 60 = 20$

Question : If $m_1 = 20\,kg$ $\quad X_1 = -2$

$m_2 = 30\,kg$ $\quad X_2 = ?$

What ~~what~~ would X_2 be so that the seesaw is in equilibrium?

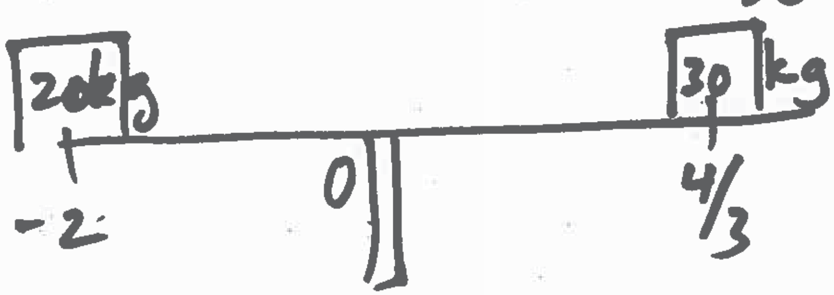
For equilibrium: $\quad M_0 = 0$

$$m_1 X_1 + m_2 X_2 = 0$$

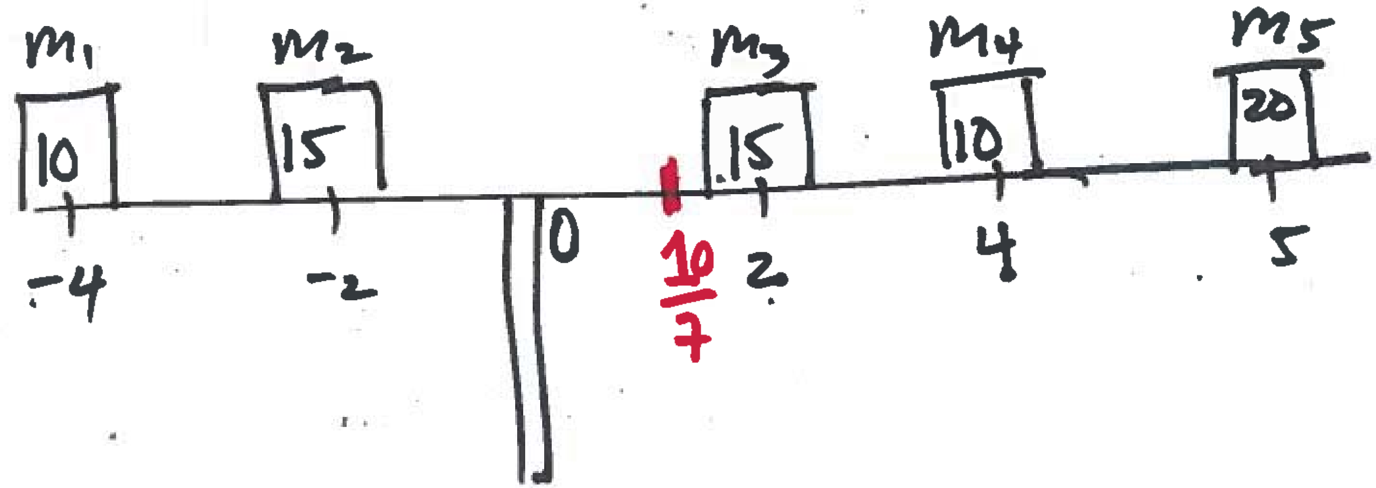
$$20(-2) + 30(X_2) = 0$$

$$-40 + 30X_2 = 0$$

$$X_2 = \frac{40}{30} = 4/3\ m$$



Ex. 2



M_0 = sum of moments = $m_1x_1 + m_2x_2 + \dots + m_5x_5$

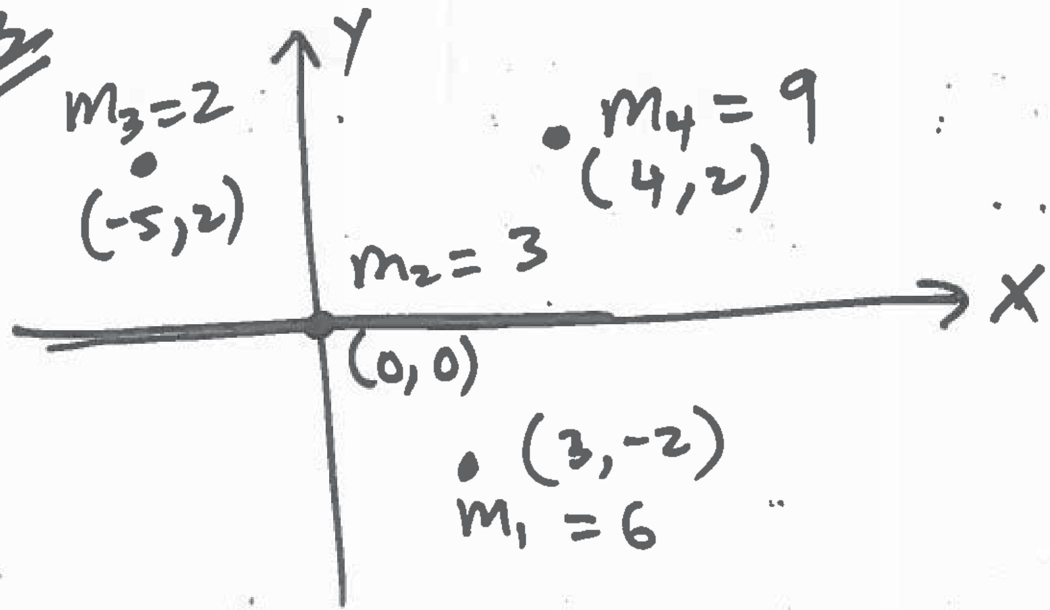
$= 10(-4) + 15(-2) + (15)(2) + 10(4) + 20(5) = 100$

m = total mass = $m_1 + m_2 + m_3 + \dots + m_5$

$= 10 + 15 + 15 + 10 + 20 = 70$

Center of Mass $= \dfrac{M_0}{m} = \dfrac{100}{70} = 10/7$

Ex. 3



(x, y)

$m =$ total mass $= 6 + 3 + 2 + 9 = 20$ kg

$x_1 = 3$, $x_2 = 0$, $x_3 = -5$, $x_4 = 4$

$y_1 = -2$, $y_2 = 0$, $y_3 = 2$; $y_4 = 2$

$M_x =$ Moment about the x-axis

$= m_1 y_1 + m_2 y_2 + m_3 y_3 + m_4 y_4$

$= 6(-2) + 3(0) + 2(2) + 9(2) = 12$ kg.m

$M_y =$ Moment about the Y-axis

$m_1 x_1 + m_2 x_2 + m_3 x_3 + m_4 x_4$

$$M_y = 6(3) + 3(0) + 2(-5) + 9(4) = 44$$

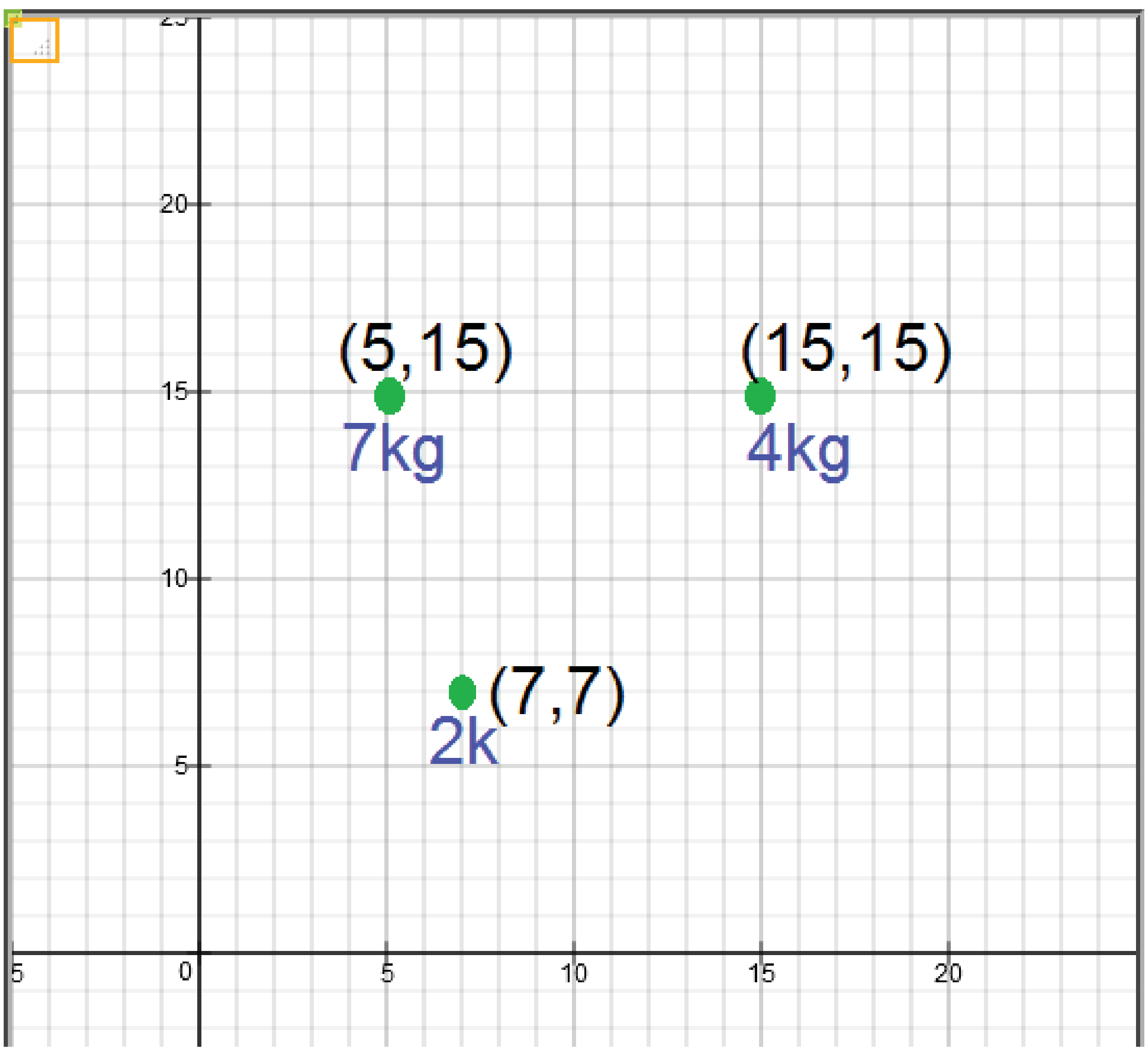
Center of Mass :

$$\overline{\mathrm{x}} = \frac{M_y}{m} = \frac{44}{20} = \frac{11}{5}$$

$$\overline{\mathrm{y}} = \frac{M_x}{m} = \frac{12}{20} = \frac{3}{5}$$

Center of Mass Illustration 1

Example of Discrete Masses: There are three point masses arranged as follows. What is the center of mass of this system?



$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } y\text{-axis}}{\text{sum of the masses}}$$

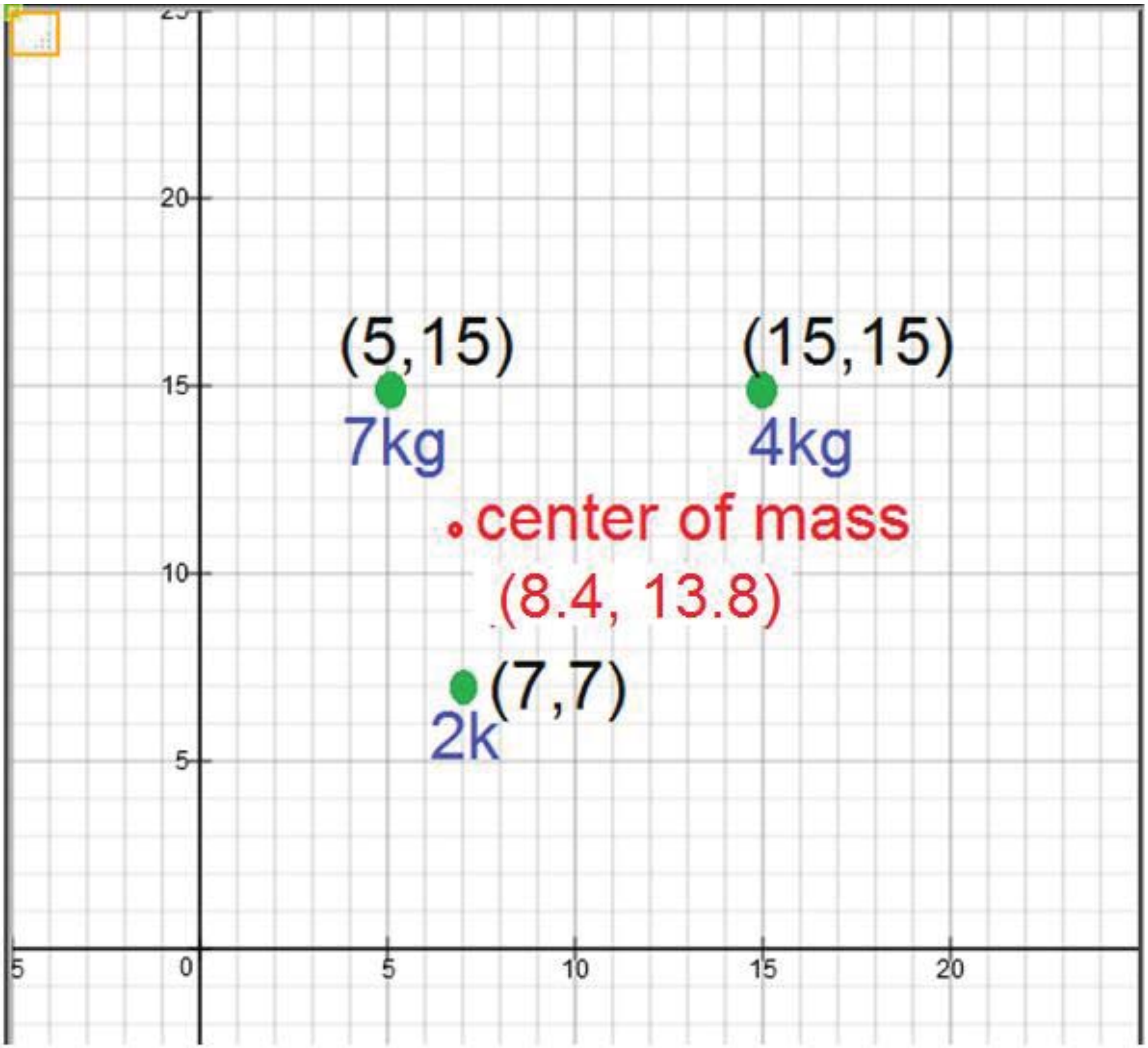
$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } x\text{-axis}}{\text{sum of the masses}}$$

$$\text{Center of mass} = \left(\bar{x}, \bar{y}\right)$$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{(7)(5)+(4)(15)+(2)(7)}{7+4+2} = 8.4$$

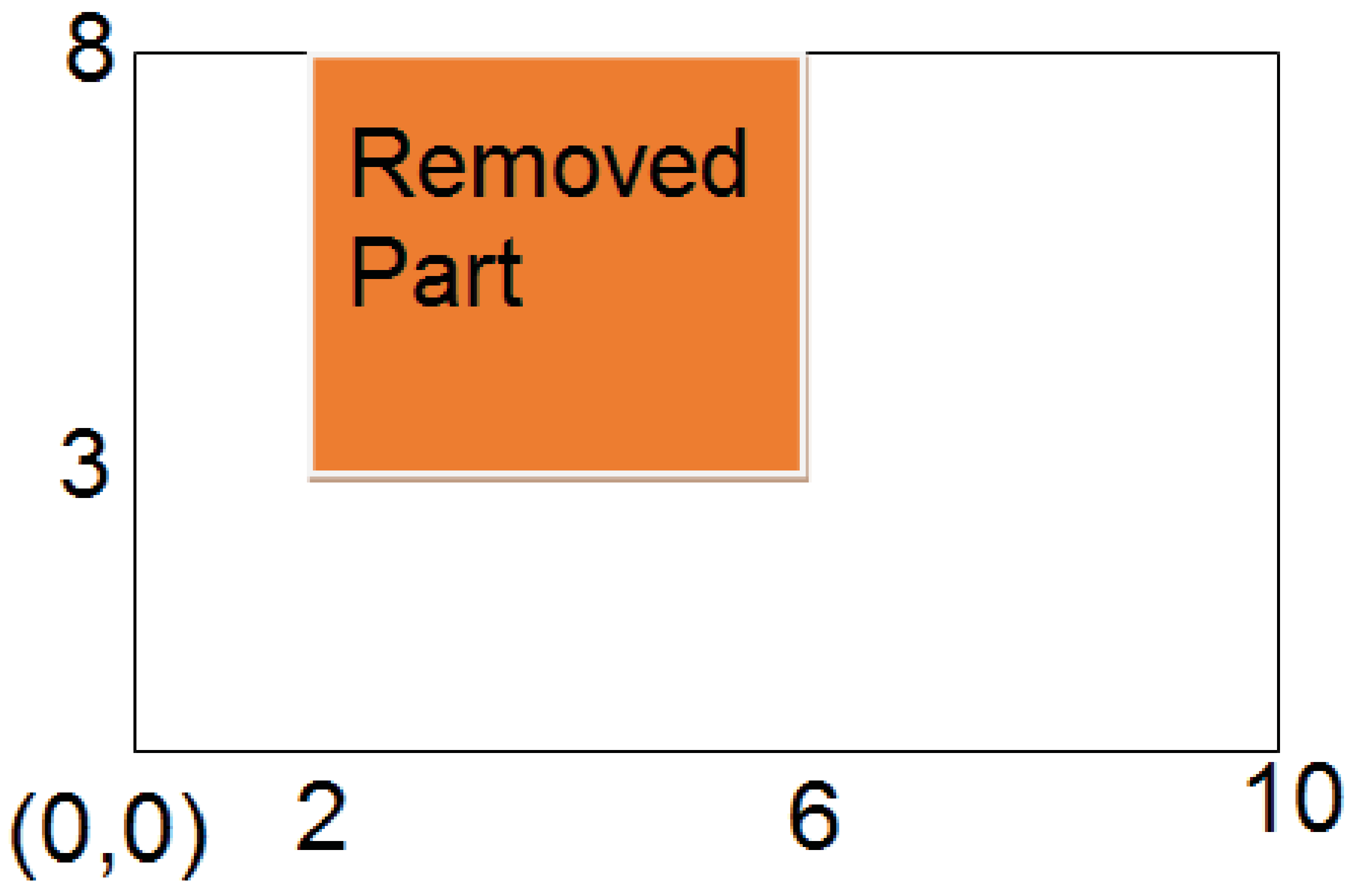
$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{(7)(15)+(4)(15)+(2)(7)}{7+4+2} = 13.8$$

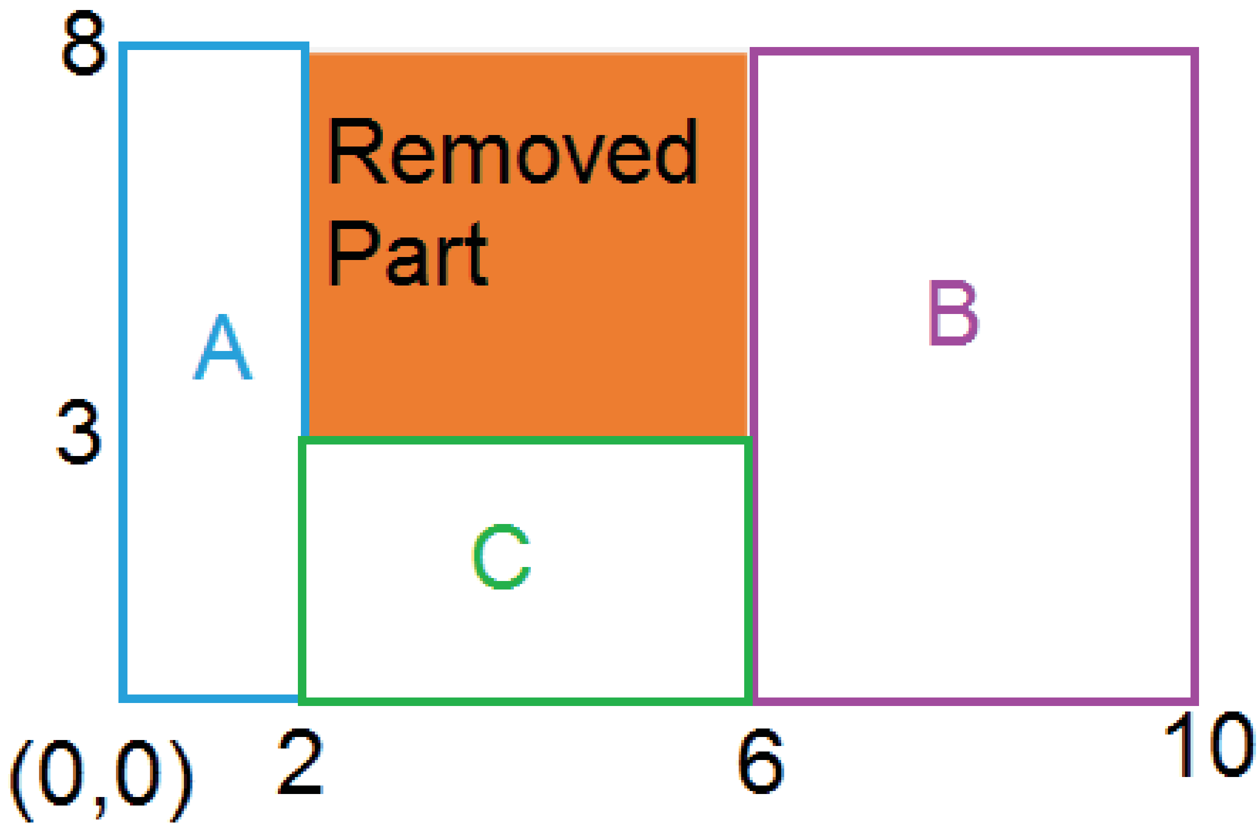
$$\text{Center of mass} = \left(\bar{x}, \bar{y}\right) = (8.4, 13.8)$$


(5,15)
7kg
(15,15)
4kg
center of mass
(8.4, 13.8)
(7,7)
2k
20
15
10
5
0
5
10
15
20

Center of Mass Illustration 2

Suppose we have an 8 in by 10 in piece of paper. We then remove a portion of it (see figure below). Find the center of mass of the remaining piece.





For piece A, center of mass = (1, 4); mass = (area)(density) = 16ρ

For piece B, center of mass = (4, 1.5). mass = (area)(density) = 12ρ

For piece C, center of mass = (8, 4). mass = (area)(density) = 32ρ

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{(16\rho)(1) + (12\rho)(4) + (32\rho)(8)}{16\rho + 12\rho + 32\rho}$$

$$= \frac{320\rho}{60\rho} = 5.33333333$$

$$\bar{y} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{(16\rho)(4) + (12\rho)(1.5) + (32\rho)(4)}{16\rho + 12\rho + 32\rho}$$

$$= \frac{210\rho}{60\rho} = 3.5$$

Center of mass = $(\bar{x}, \bar{y}) = (5.33333333, 3.5)$

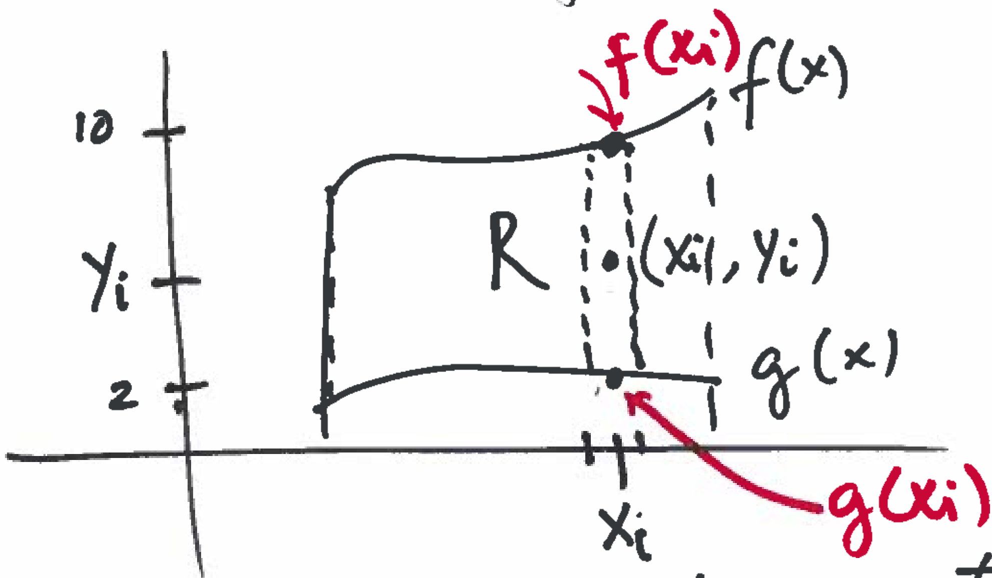
Center of Mass:

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } y\text{-axis}}{\text{sum of the masses}} = \frac{M_y}{\text{Total Mass}}$$

$$\bar{y} = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3} = \frac{\text{moment about the } x\text{-axis}}{\text{sum of the masses}} = \frac{M_x}{\text{Total Mass}}$$

Center of mass $= (\bar{x}, \bar{y})$

14.4 Mass of Planar Lamina with uniform density



Let ρ = density of Region R

$$y_i = \frac{f(x_i) + g(x_i)}{2}$$

For the ith rectangle, center of mass (x_i, y_i)

Or mass is "concentrated" at (x_i, y_i)

mass of ith rectangle:

mass $= m_i =$ (density)(Area)

$$m_i = \rho \cdot \underbrace{[f(x_i) - g(x_i)]}_{\text{height}} \cdot \underbrace{\Delta x_i}_{\text{width}}$$

So the moment of m_i about the x-axis is :

$$\text{Moment} = (\text{mass}) \cdot (\text{distance})$$
$$= m_i \cdot y_i$$
$$= \rho \cdot [f(x_i) - g(x_i)] \cdot \Delta x_i \cdot \left[\frac{f(x_i)+g(x_i)}{2}\right]$$

"Moment of m_i"

Moment of Region R about the x-axis

$$M_x \approx \text{Moment of Region } R$$
$$\approx \sum_{i=1}^{n} \rho \left[\frac{f(x_i)+g(x_i)}{2}\right][f(x_i)-g(x_i)] \cdot \Delta x_i$$

$$M_x = \sum_{i=1}^{\infty} \rho \left[\frac{f(x_i)+g(x_i)}{2}\right][f(x_i)-g(x_i)] \Delta x_i$$

$$M_x = \int_a^b \rho \left[\frac{f(x)+g(x)}{2}\right][f(x)-g(x)]\, dx$$

mass of region $R = \rho \int_a^b [f(x) - g(x)]\, dx$

$$= \rho \int_{-2}^{1} [(4 - x^2) - (x+2)]\, dx$$

$$= \rho \cdot [9/2] = \frac{9}{2}\rho$$

M_y = Moment about Y-axis = (mass)("x" distance)

$$= \rho \int_a^b x\, [f(x) - g(x)]\, dx$$

$$= \rho \int_{-2}^{1} x\, [(4 - x^2) - (x+2)]\, dx = \rho(-9) = -9\rho$$

M_x = moment about x-axis = (mass)("y" distance)

$$= \rho \int_a^b \left[\frac{f(x) + g(x)}{2}\right] [f(x) - g(x)]\, dx$$

M_y = Moment about the y-axis

= (mass)("x" distance)

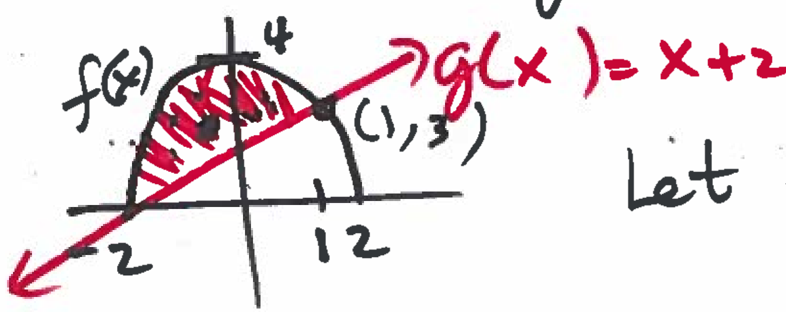
$$M_y = \int_a^b x \cdot \rho [f(x) - g(x)]\, dx$$

$$= \rho \int_a^b x [f(x) - g(x)]\, dx$$

$$\bar{x} = \frac{M_y}{m} \qquad \bar{y} = \frac{M_x}{m}$$

center of mass $= (\bar{x}, \bar{y})$

Let Region R be the region bounded by $f(x) = 4 - x^2$ and $g(x) = x + 2$



Let ρ = density of region R

$$= \rho \int_{-2}^{1} \left[\frac{(4-x^2)+(x+2)}{2} \right] \left[(4-x^2)-(x+2) \right] dx$$

$$= \rho(10.8) = 10.8\rho$$

$$\bar{x} = \frac{M_y}{m} = \frac{-9\rho}{9/2\rho} = -\frac{1}{2}$$

$$\bar{y} = \frac{M_x}{m} = \frac{10.8\rho}{\frac{9}{2}\rho} = 2.4$$

Center of Mass $= \left(-\frac{1}{2}, 2.4\right)$

Calculus III Section 14.4 Homework

Note: mass of lamina $= density \cdot area$

$$m = \rho \cdot Area$$

$$m = \iint_R \rho(x, y)\,dA \qquad \text{(for non-uniform density)}$$

For lamina with non-homogeneous density,

mass of lamina: $m = \iint\limits_R \rho(x,y)\,dA = \underline{\ \ ?\ \ \ \ \ \ }$

moment of mass with respect to x-axis: $M_x = \iint\limits_R y\rho(x,y)\,dA$

moment of mass with respect to y-axis: $M_y = \iint\limits_R x\rho(x,y)\,dA$

Center of Mass: $\bar{x} = \dfrac{M_y}{m} \qquad \bar{y} = \dfrac{M_x}{m}$

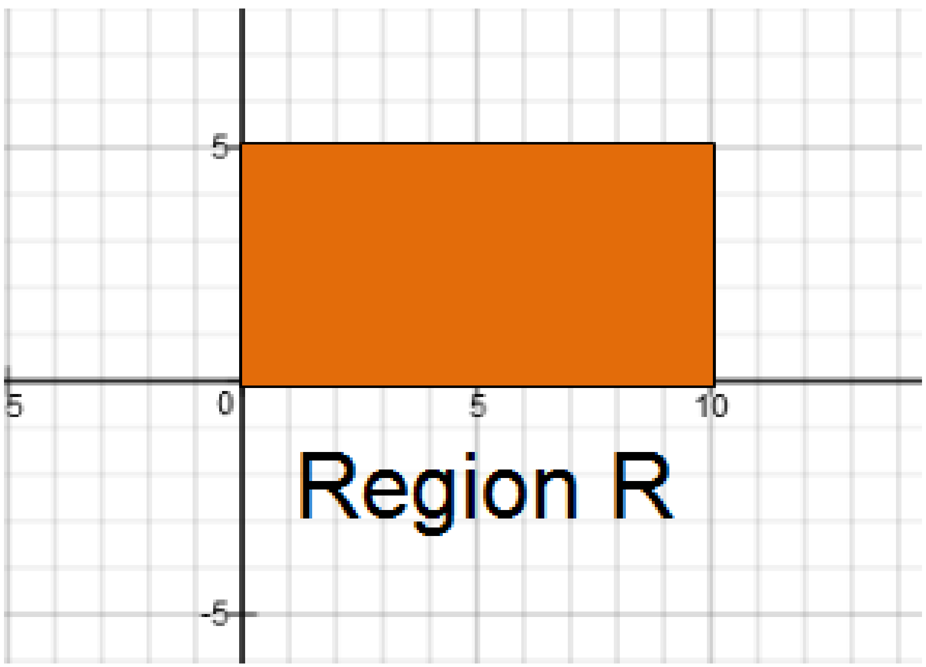
Hence, $\bar{x} = \dfrac{M_y}{m} = \dfrac{\iint\limits_R x\rho(x,y)\,dA}{\iint\limits_R \rho(x,y)\,dA} \qquad \bar{y} = \dfrac{M_x}{m} = \dfrac{\iint\limits_R y\rho(x,y)\,dA}{\iint\limits_R \rho(x,y)\,dA}$

Example 1: Find the mass (m) of the lamina corresponding to region R

with the given desisty function ρ.

$\rho(x, y) = 8xy.$

$m = \iint_R \rho(x, y)\,dA =$? ______________________



$$m = \iint_R \rho(x, y)\, dA = \int_0^5 \int_0^{10} 8xy\, dx\, dy$$

Evaluate $\displaystyle\int_0^{10} 8xy\, dx = 8y \int_0^{10} x\, dx = 8y \left[\frac{x^2}{2} \right]_0^{10} = 400y$

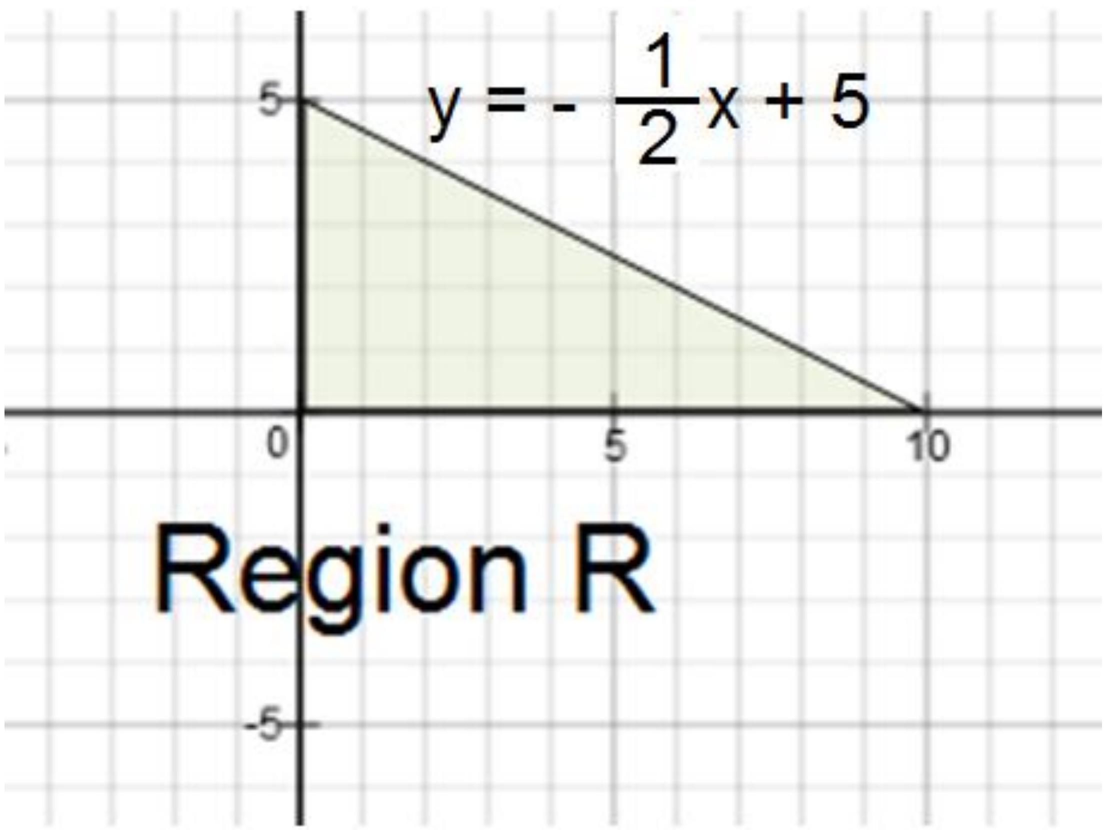
Hence, $\displaystyle\int_0^5 \int_0^{10} 8xy\, dx\, dy = \int_0^5 400y\, dy = 400 \left[\frac{y^2}{2} \right]_0^5 = 500$

Example 2: Find the mass (m) of the lamina corresponding to region R

with the given desisty function ρ.

$\rho(x, y) = 14xy.$

$m = \iint_R \rho(x, y)dA = $ ___?___



$$m = \iint_R \rho(x,y)\,dA = \iint_R 14kxy\,dy\,dx = \int_0^{10} \int_{y=0}^{y=-0.5x+5} 14xy\,dy\,dx$$

Evaluate $\displaystyle\int_{y=0}^{y=-0.5x+5} 14xy\,dy = 14x \int_0^{-0.5x+5} y\,dy = 14x\left[\frac{y^2}{2}\right]_0^{-0.5x+5}$

$$= 14x\left[\frac{(-0.5x+5)^2}{2}\right] - 4kx\left[\frac{(0)}{2}\right] = 14x\left[\frac{(-0.5x+5)^2}{2}\right]$$

Hence, $\displaystyle\int_0^{10} \int_{y=0}^{y=-0.5x+5} 14xy\,dy\,dx = \int_0^{10} 14x\left[\frac{(-0.5x+5)^2}{2}\right]dx$

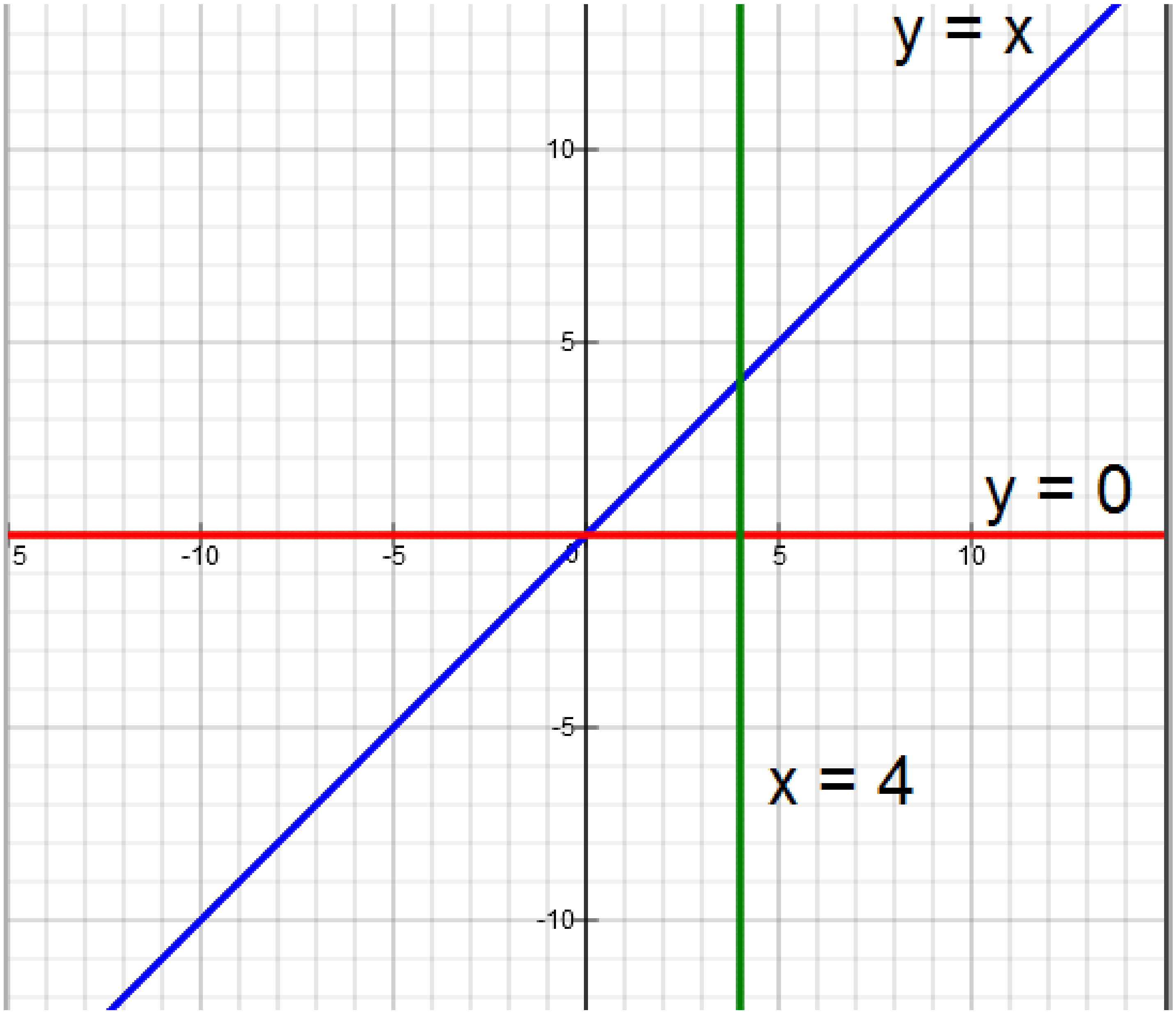
$$= \int_0^{10} 7x(-0.5x+5)^2\,dx = 1458.333333333$$

Example 3: Find the mass and the center of mass of the lamina

bounded by the following graphs.

$y = x; \quad y = 0; \quad x = 4;$

Density Function: $\rho(x, y) = 8x.$



y = x
y = 0
x = 4
10
5
-5
-10
5
-10
-5
0
5
10

mass of lamina: $m = \iint_R \rho(x,y)\,dy\,dx = \int_0^4 \int_{y=0}^{y=x} 8x\,dy\,dx$

Evaluate inside integral: $\int_{y=0}^{y=x} 8x\,dy = \left[8xy\right]_0^x = 8x^2$

$$m = \int_0^4 \int_{y=0}^{y=x} 8x\,dy\,dx = \int_0^4 8x^2\,dx = \left[8 \cdot \frac{x^3}{3}\right]_0^4 = \frac{512}{3}$$

moment of mass with respect to *x*-axis:

$$M_x = \iint_R y\rho(x,y)\,dA = \int_0^4 \int_{y=0}^{y=x} 8xy\,dy\,dx$$

Evaluate inside integral: $\displaystyle\int_{y=0}^{y=x} 8xy\,dy = \left[8x \cdot \frac{y^2}{2}\right]_0^x = 4x^3$

$$M_x = \int_0^4 \int_{y=0}^{y=x} 8xy\,dy\,dx = \int_0^4 4x^3\,dx = \left[4 \cdot \frac{x^4}{4}\right]_0^4 = 256$$

moment of mass with respect to y-axis:

$$M_y = \iint_R x\rho(x,y)\,dA = \int_0^4 \int_{y=0}^{y=x} 8xx\,dy\,dx$$

Evaluate inside integral: $\displaystyle\int_{y=0}^{y=x} 8xx\,dy = \int_{y=0}^{y=x} 8x^2\,dy = \left[8x^2 y\right]_0^x = 8x^3$

$$M_y = \int_0^4 \int_{y=0}^{y=x} 8xx\,dy\,dx = \int_0^4 8x^3\,dx = 8\left[\frac{x^4}{4}\right]_0^4 = 512$$

Center of Mass: $\displaystyle\bar{x} = \frac{M_y}{m} = \frac{512}{\left(\frac{512}{3}\right)} = 3$ $\qquad\displaystyle\bar{y} = \frac{M_x}{m} = \frac{256}{\left(\frac{512}{3}\right)} = 1.5$

Moments of Inertia (rotational inertia):

For a point mass the **moment of inertia** is the mass times the square of perpendicular distance to the rotation axis, $I = mr^2$.

Illustration of Moments of Inertia:

https://www.youtube.com/watch?v=fHDB7PMUdZE

https://www.youtube.com/watch?v=W9fPGXyvrVM

I_x = moment of inertia with respect to x-axis of a lamina of variable density

$$I_x = \iint_R y^2 \rho(x,y)\,dA \;;$$

Note: mass $= \iint_R \rho(x,y)\,dA$

I_y = moment of inertia with respect to y-axis of a lamina of variable density

$$I_y = \iint_R x^2 \rho(x,y)\,dA$$

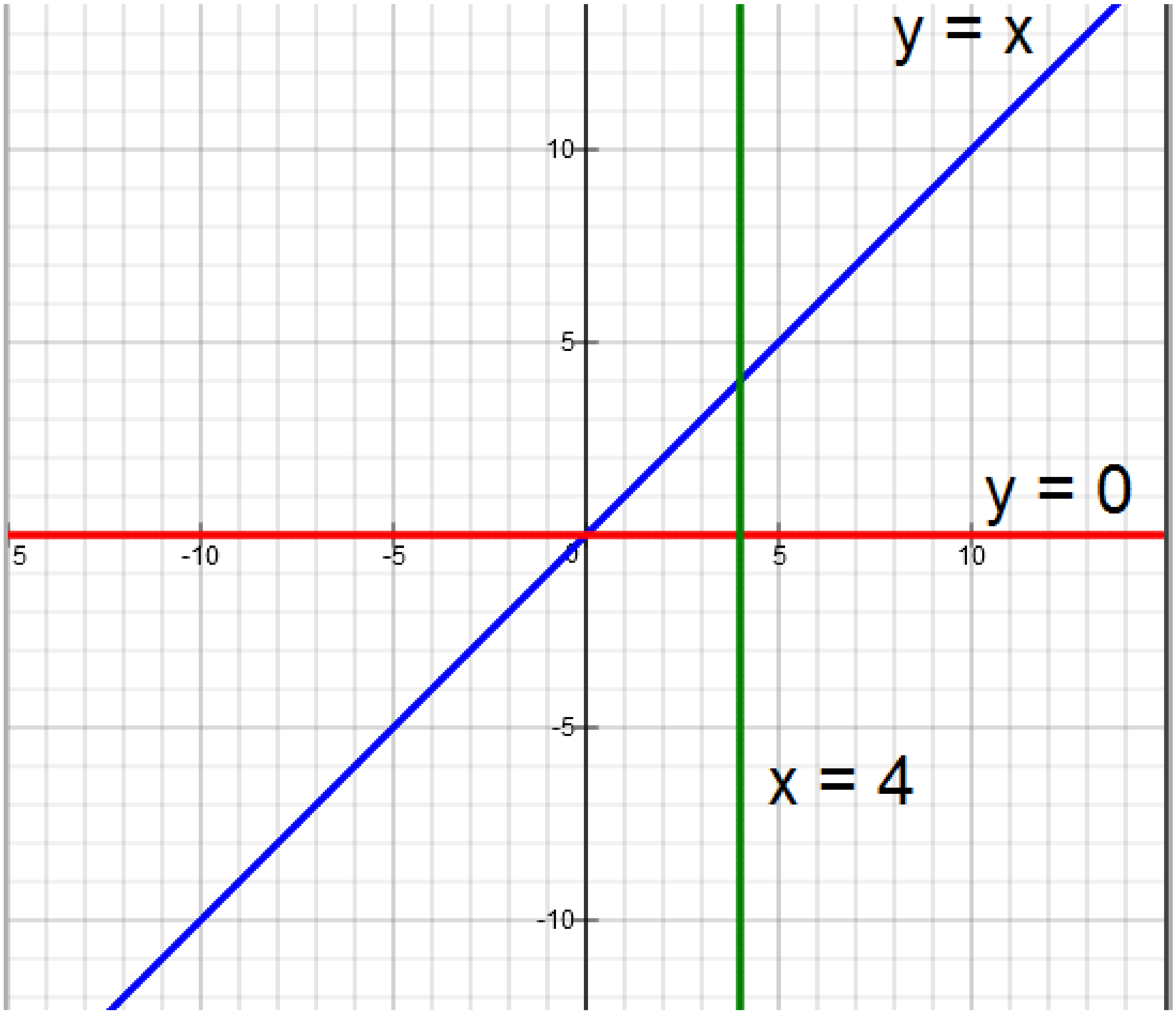
Note: mass $= \iint_R \rho(x,y)\,dA$

Example 4: Find the moments of inertia of the lamina

bounded by the following graphs

$$y = x; \quad y = 0; \quad x = 4;$$

Density Function: $\rho(x, y) = 8x.$



y = x
y = 0
x = 4
10
5
-5
-10
5
-10
-5
0
5
10

Example 4: Find the moments of inertia of the lamina

bounded by the following graphs

$$y = x; \quad y = 0; \quad x = 4;$$

Density Function: $\rho(x, y) = 8x.$

$I_x =$ moment of inertia with respect to x-axis $= \iint_R y^2 \rho(x, y) dA$

$$I_x = \int_0^4 \int_{y=0}^{y=x} y^2 8x dy dx$$

Evaluate inside integral: $\int_{y=0}^{y=x} y^2 8x dy = \left[8x \cdot \frac{y^3}{3} \right]_0^x = \frac{8}{3} x^4$

$$I_x = \int_0^4 \int_{y=0}^{y=x} y^2 8x dy dx = I_x = \int_0^4 \frac{8}{3} x^4 dx = \left[\frac{8}{3} \frac{x^5}{5} \right]_0^4 = \frac{8}{15} (1024) = 546.133333$$

$$I_x = \text{moment of inertia with respect to } y\text{-axis} = \iint_R x^2 \rho(x, y)\,dA$$

$$I_x = \int_0^4 \int_{y=0}^{y=x} x^2 8x\,dy\,dx = \int_0^4 \int_{y=0}^{y=x} 8x^3\,dy\,dx$$

Evaluate inside integral: $\displaystyle\int_{y=0}^{y=x} 8x^3\,dy = \left[8x^3 y\right]_0^x = 8x^4$

$$I_x = \int_0^4 \int_{y=0}^{y=x} 8x^3\,dy\,dx = I_x = \int_0^4 8x^4\,dx = \left[8\frac{x^5}{5}\right]_0^4 = \frac{8(1024)}{5} = 1638.4$$