Section 14.6

Example 1: Find $\int_0^3 \int_0^5 \int_0^2 (6x + 8y + 4z)\, dxdzdy$

Evaluate $\int_0^2 (6x + 8y + 4z)\, dx = \left[\frac{6x^2}{2} + 8yx + 4zx \right]_0^2$

$= 12 + 16y + 8z$

$$\int_0^5 \int_0^2 (6x + 8y + 4z)\, dxdz = \int_0^5 (12 + 16y + 8z)\, dz = \left[12z + 16yz + \frac{8z^2}{2} \right]_0^5$$

$= 60 + 80y + 100 = 160 + 80y$

$$\int_0^3 \int_0^5 \int_0^2 (6x + 8y + 4z)\, dxdzdy = \int_0^3 (160 + 80y)\, dy = \left[160y + \frac{80y^2}{2} \right]_0^3 = 840$$

Example 2: Evaluate: $\int_0^2 \int_0^1 \int_0^x \left(2ye^{-x^2}\right) dydzdx$

Evaluate $\int_0^x \left(2ye^{-x^2}\right) dy = \left[2e^{-x^2} \cdot \frac{y^2}{2}\right]_0^x = e^{-x^2} x^2$

Evaluate $\int_0^1 \int_0^x \left(2ye^{-x^2}\right) dydz = \int_0^1 e^{-x^2} x^2 dz = \left[e^{-x^2} x^2 z\right]_0^1 = e^{-x^2} x^2$

Hence $\int_0^2 \int_0^1 \int_0^x \left(2ye^{-x^2}\right) dydzdx = \int_0^2 e^{-x^2} x^2 dx = 0.4227250$

Mass of three-dimensional region R

Let $\rho(x, y, z)$ be the density function;

Let V = Volume of Region R = $\int\int\int 1 dV = \int\int\int 1 dxdydz$

$$m = mass = \ (\text{density})(\text{volume}) = \int\int\int \rho(x, y, z)\, dxdydz$$

Center of mass of three-dimensional region R

Let P(x, y, z) be a point in the three dimensional region R.

Then x is the distance from P to the yz-plane,

y is the distance from P to the xz-plane,

and z is the distance from P to the xy-plane

Let M_{yz} = first moment about the yz-plane = (mass)(distance between point in R to the yz-plane)

$$M_{yz} = \iiint x\rho(x,y,z)\,dxdydz$$

Let M_{xz} = first moment about the xz-plane = (mass)(distance between point in R to the xz-plane)

$$M_{xz} = \iiint y\rho(x,y,z)\,dxdydz$$

Let M_{xy} = first moment about the xy-plane = (mass)(distance between point in R to the xy-plane)

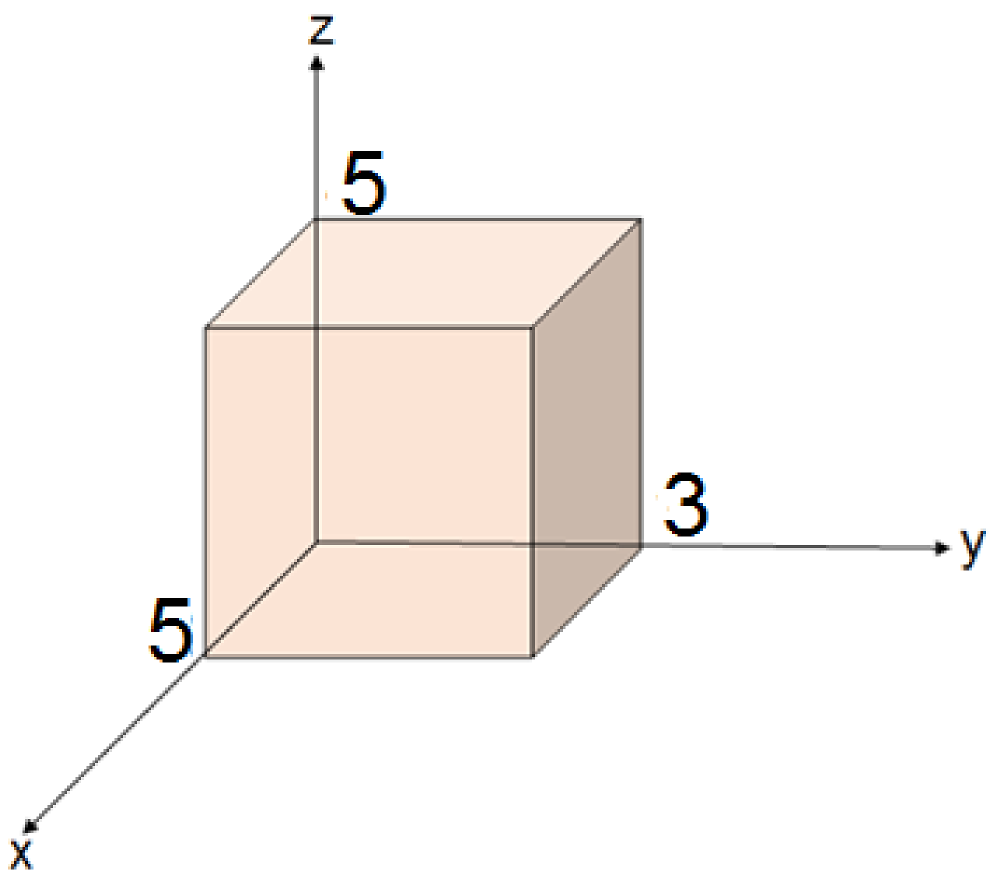
$$M_{xy} = \iiint z\rho(x,y,z)\,dxdydz$$

Center of Mass= $\left(\bar{x} = \dfrac{M_{yz}}{m}, \quad \bar{y} = \dfrac{M_{xz}}{m}, \quad \bar{z} = \dfrac{M_{xy}}{m} \right)$

Example 3: Find the mass and center of mass of the solid

bouded by the following graphs with the following density.

$x = 0;\quad x = 5;\quad y = 0;\quad y = 3;\quad z = 0;\quad z = 5;\quad \rho(x, y, z) = 8xyz$



Example 3 (con't):

$$m = mass = \int_0^5\int_0^3\int_0^5 1\rho(x,y,z)\,dxdydz = \int_0^5\int_0^3\int_0^5 8xyz\,dxdydz = \int_0^5\int_0^3\left(\left[8yz\cdot\frac{x^2}{2}\right]_0^5\right)dydz$$

$$= \int_0^5\int_0^3 (100yz)\,dydz = \int_0^5\left(\left[100z\cdot\frac{y^2}{2}\right]_0^3\right)dx = \int_0^5 [450z]dz \;=\; \left[450\cdot\frac{z^2}{2}\right]_0^5 = 5625$$

$$M_{yz} = \text{ first moment about the } yz\text{-plane} = \int_0^5\int_0^3\int_0^5 x\rho(x,y,z)\,dxdydz$$

$$M_{yz} = \int_0^5\int_0^3\int_0^5 8x^2yz\,dxdydz = \int_0^5\int_0^3\left(\left[8yz\cdot\frac{x^3}{3}\right]_0^5\right)dydz \;=\; \int_0^5\int_0^3\left(\frac{1000}{3}yz\right)dydz$$

$$= \int_0^5\left(\left[\frac{1000}{3}z\cdot\frac{y^2}{2}\right]_0^3\right)dx = \int_0^5 [1500z]dz \;=\; \left[1500\cdot\frac{z^2}{2}\right]_0^5 = 18750$$

$$M_{xz} = \text{ first moment about the } xz\text{-plane} = \int_0^5\int_0^3\int_0^5 y\rho(x,y,z)\,dxdydz$$

$$M_{xz} = \int_0^5\int_0^3\int_0^5 8xy^2z\,dxdydz = \int_0^5\int_0^3\left(\left[8y^2z\cdot\frac{x^2}{2}\right]_0^5\right)dydz$$

$$= \int_0^5\int_0^3\left(100y^2z\right)dydz = \int_0^5\left(\left[100z\cdot\frac{y^3}{3}\right]_0^3\right)dx = \int_0^5[900z]dz = \left[900\cdot\frac{z^2}{2}\right]_0^5 = 11250$$

$$M_{xy} = \text{ first moment about the } xy\text{-plane} = \int_0^5\int_0^3\int_0^5 z\rho(x,y,z)\,dxdydz$$

$$M_{xy} = \int_0^5\int_0^3\int_0^5 8xyz^2\,dxdydz = \int_0^5\int_0^3\left(\left[8yz^2\cdot\frac{x^2}{2}\right]_0^5\right)dydz$$

$$= \int_0^5\int_0^3\left(100yz^2\right)dydz = \int_0^5\left(\left[100z^2\cdot\frac{y^2}{2}\right]_0^3\right)dx = \int_0^5\left[450z^2\right]dz = \left[450\cdot\frac{z^3}{3}\right]_0^5 = 18750$$

Example 3 (con't):

$m = 5625; \quad M_{yz} = 18750; \quad M_{xz} = 11250; \quad M_{xy} = 18750$

Center of Mass: $\left(\bar{x} = \dfrac{M_{yz}}{m}, \quad \bar{y} = \dfrac{M_{xz}}{m}, \quad \bar{z} = \dfrac{M_{xy}}{m} \right)$

Center of Mass: $\left(\bar{x} = \dfrac{18750}{5625}, \quad \bar{y} = \dfrac{11250}{5625}, \quad \bar{z} = \dfrac{18750}{5625} \right)$

Center of Mass: $\left(\bar{x} = 10/3, \quad \bar{y} = 2, \quad \bar{z} = 10/3 \right)$