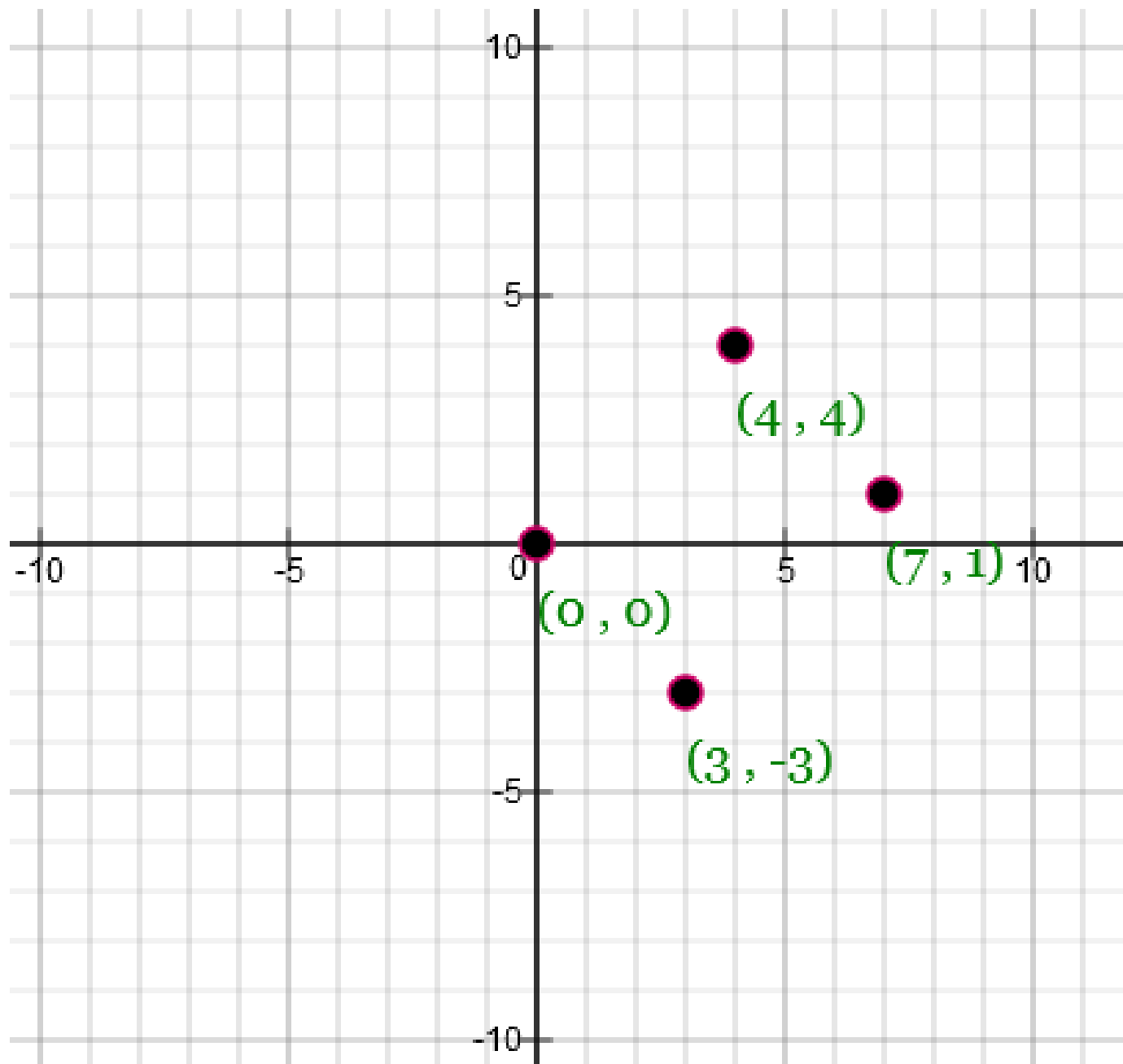


1) Let R be the rectangular region bounded by the following vertices $(0,0)$ $(3,-3)$ $(4,4)$ $(7, 1)$.

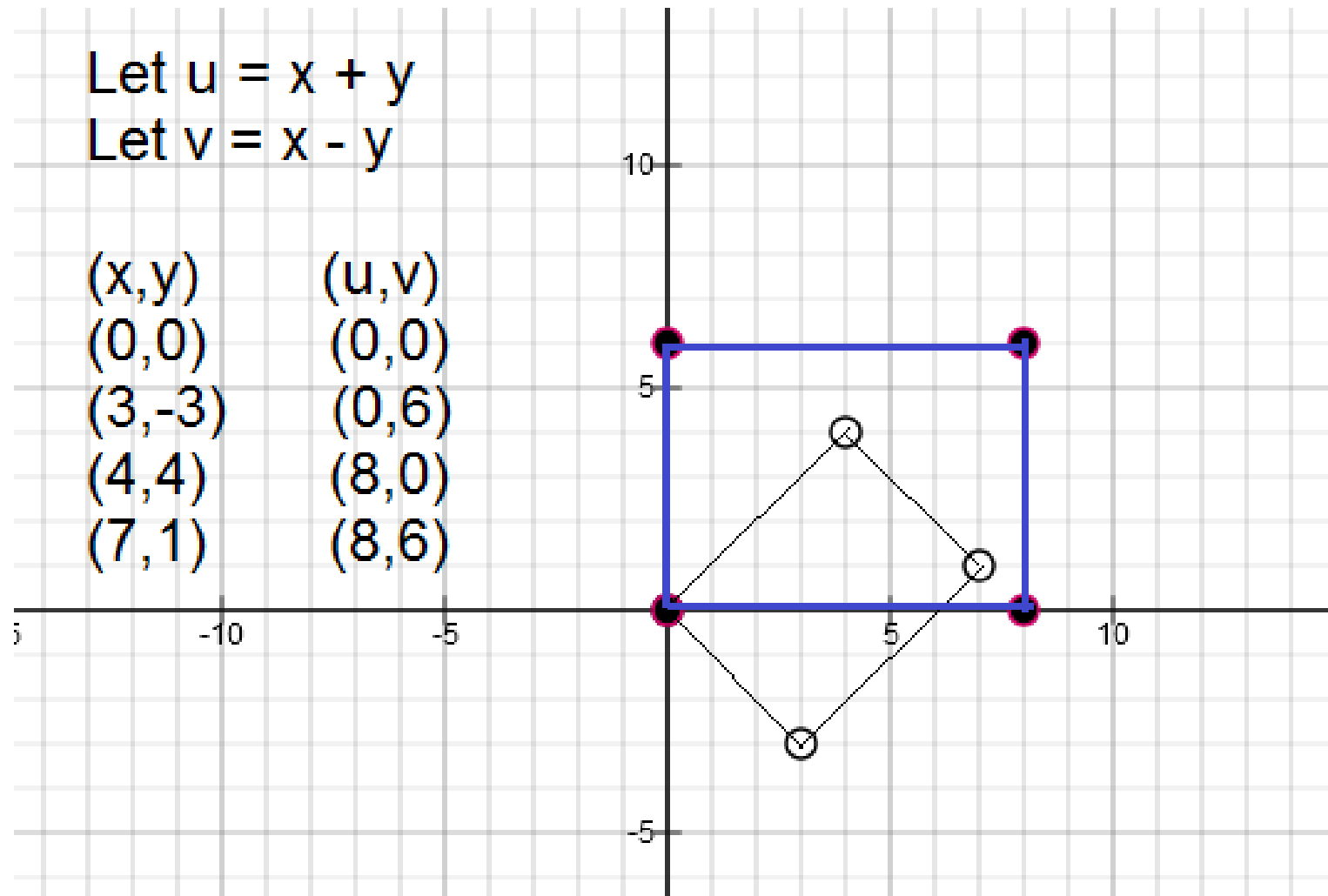
Find the volume of the solid between the surface $z = f(x, y)$ and region R .

$$\text{Let } z = f(x, y) = x^2 + y$$

$$\text{Volume} = \iint_R f(x, y) dA = \underline{\quad? \quad}$$



We will now make a change of variables so that we end up with a standard rectangular region R.



New region R has the following vertices: $(0,0)$, $(0,6)$, $(8,0)$, $(8,6)$

Finding change of variables to simplify region R .

$$\text{Let } u = x + y \quad (\text{equ 1}); \quad \text{Let } v = x - y \quad (\text{equ 2})$$

$$(\text{equ 1}) + (\text{equ 2}): u + v = 2x \Rightarrow x = \frac{1}{2}(u + v)$$

$$(\text{equ 1}) - (\text{equ 2}): u - v = 2y \Rightarrow y = \frac{1}{2}(u - v)$$

$$\frac{\partial x}{\partial u} = \frac{1}{2}; \quad \frac{\partial x}{\partial v} = \frac{1}{2}; \quad \frac{\partial y}{\partial u} = \frac{1}{2}; \quad \frac{\partial y}{\partial v} = \frac{-1}{2}$$

Jacobian of x and y :

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}$$

$$z = f(x, y) = x^2 + y = \left(\frac{1}{2}(u + v) \right)^2 + \frac{1}{2}(u - v)$$

$$= \frac{1}{4}u^2 + uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v$$

$$\text{Volume} = \iint_R f(x, y) dA = \iint_R f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv$$

$$\text{Volume} = \iint_R \left(\frac{1}{4}u^2 + uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v \right) \left| \frac{-1}{2} \right| dudv$$

$$\text{Volume} = \frac{1}{2} \int_0^6 \int_0^8 \left(\frac{1}{4}u^2 + uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v \right) dudv$$

$$\text{Evaluating } \int_0^8 \left(\frac{1}{4}u^2 + uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v \right) du$$

$$= \left[\frac{1}{4} \frac{u^3}{3} + v \cdot \frac{u^2}{2} + \frac{1}{4}v^2u + \frac{1}{2} \frac{u^2}{2} - \frac{1}{2}vu \right]_0^8$$

$$= \frac{512}{12} + 32v + 2v^2 + 16 - 4v$$

$$\text{Volume} = \frac{1}{2} \int_0^6 \int_0^8 \left(\frac{1}{4}u^2 + uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v \right) dudv$$

$$= \frac{1}{2} \int_0^6 \left(\frac{512}{12} + 32v + 2v^2 + 16 - 4v \right) dv = \frac{1}{2} (1000) = 500$$

Example 2: Let R be the triangular region bounded by the following vertices $(0,0)$ $(4,0)$ $(8,4)$.

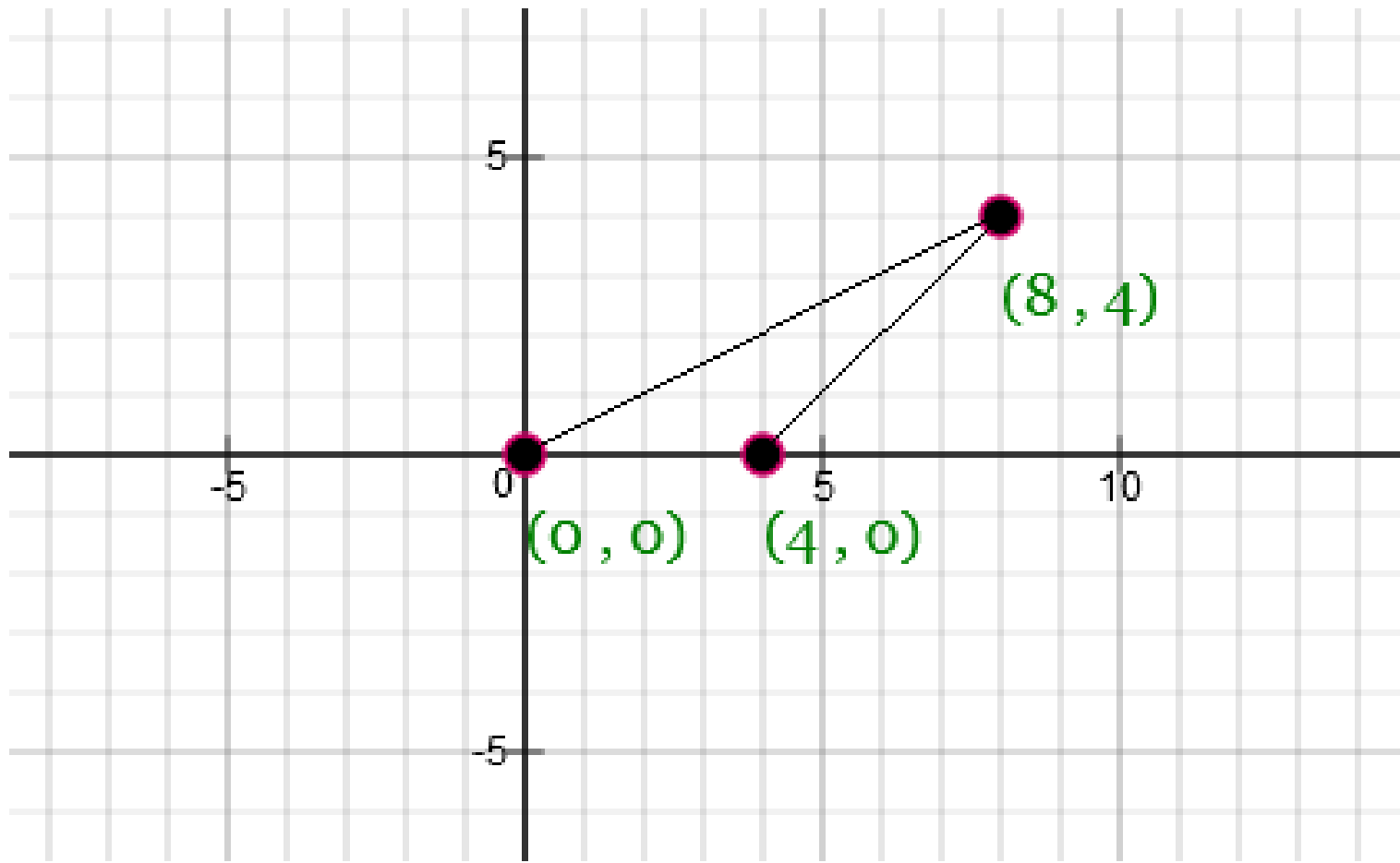
Find the volume of the solid between the surface

$z = f(x, y)$ and region R .

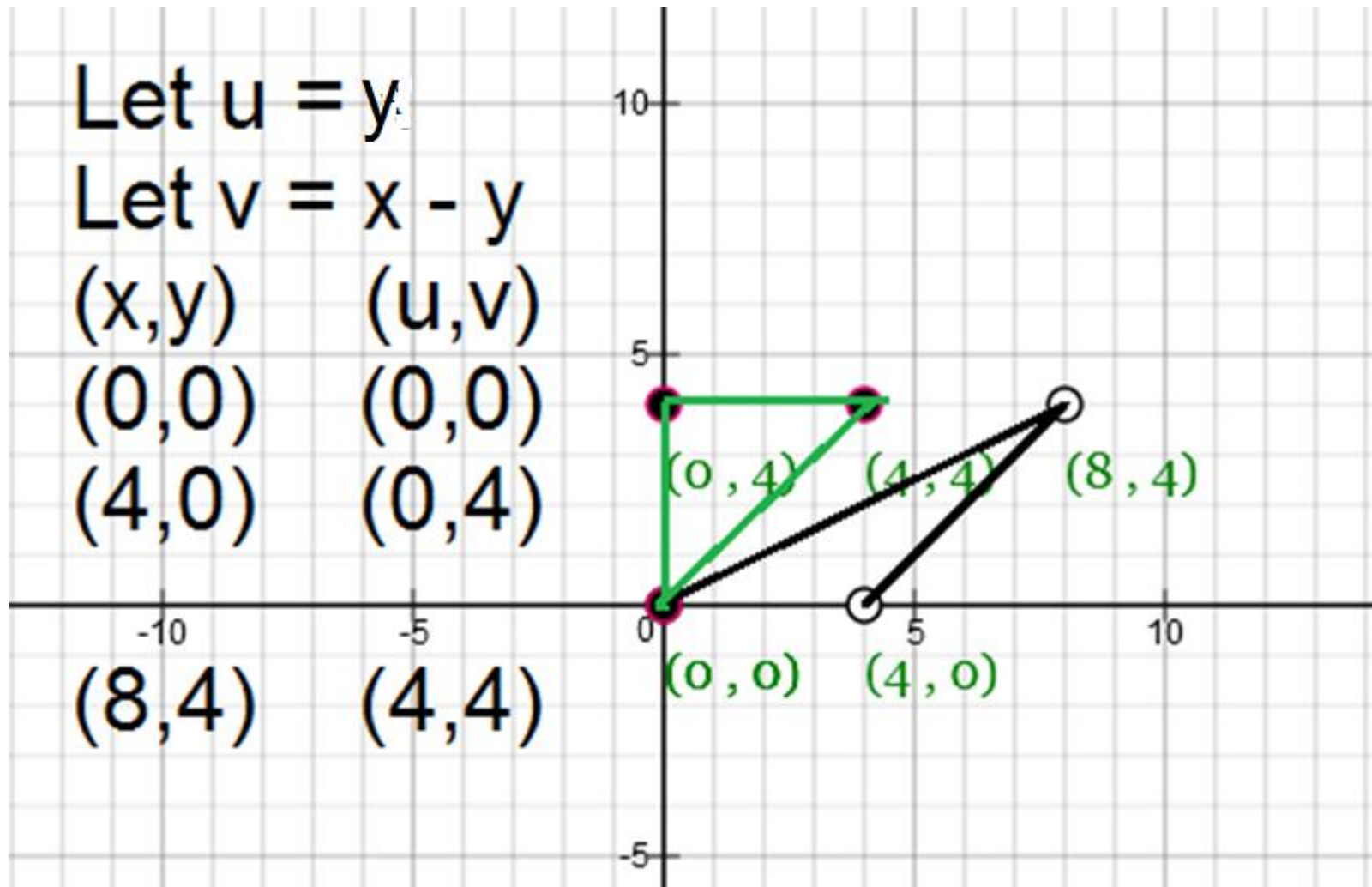
Let $z = f(x, y) = \sin(xy)$

Volume = $\iint_R f(x, y) dA = \underline{\quad ? \quad}$

Let $u = y; \quad v = x - y$



We will make a change of variables so that we end up with a standard triangular region R .



New triangular region has vertices: $(0, 0)$, $(0, 4)$, $(4, 4)$

Finding change of variables to simplify region R .

$$\text{Let } u = y \quad (\text{equ 1})$$

$$\text{Let } v = x - y \quad (\text{equ 2})$$

$$(\text{equ 2}): x = v + y \Rightarrow x = v + u$$

$$\frac{\partial x}{\partial u} = 1; \quad \frac{\partial x}{\partial v} = 1; \quad \frac{\partial y}{\partial u} = 1; \quad \frac{\partial y}{\partial v} = 0$$

Jacobian of x and y :

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

$$\text{Let } z = f(x, y) = \sin(xy) = \sin((u+v)(u)) = \sin(u^2 + uv)$$

$$\text{Volume} = \iint_R f(x, y) dA = \iint_R f(g(u, v), h(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dv du$$

$$\text{Volume} = \iint_R \sin(u^2 + uv) |-1| dv du = \int_0^4 \int_{v=u}^{v=4} \sin(u^2 + uv) dv du$$

$$\text{Evaluating } \int_{v=4}^{v=u} \sin(u^2 + uv) dv = \left[\frac{-1}{u} \cos(u^2 + uv) \right]_u^4$$

$$= \frac{-1}{u} \cos(u^2 + 4u) - \frac{-1}{u} \cos(u^2 + u^2)$$

$$= \frac{-1}{u} \cos(u^2 + 4u) + \frac{1}{u} \cos(2u^2)$$

$$\text{Volume} = \int_0^4 \int_{v=u}^{v=4} \sin(u^2 + uv) dv du =$$

$$= \int_0^4 \left[\frac{-1}{u} \cos(u^2 + 4u) + \frac{1}{u} \cos(2u^2) + \right] du = 1.33$$

Calculus I Review:

Find $\int \sin(a + bx) dx$.

Let $u = a + bx$

$$du = b dx \quad \Rightarrow \quad \frac{1}{b} du = dx$$

$$\int \sin(a + bx) dx = \int \sin(u) \frac{1}{b} du = \frac{1}{b} [-\cos u]$$

$$= -\frac{1}{b} \cos(a + bx)$$