1) Let R be the rectagular region bounded by the following

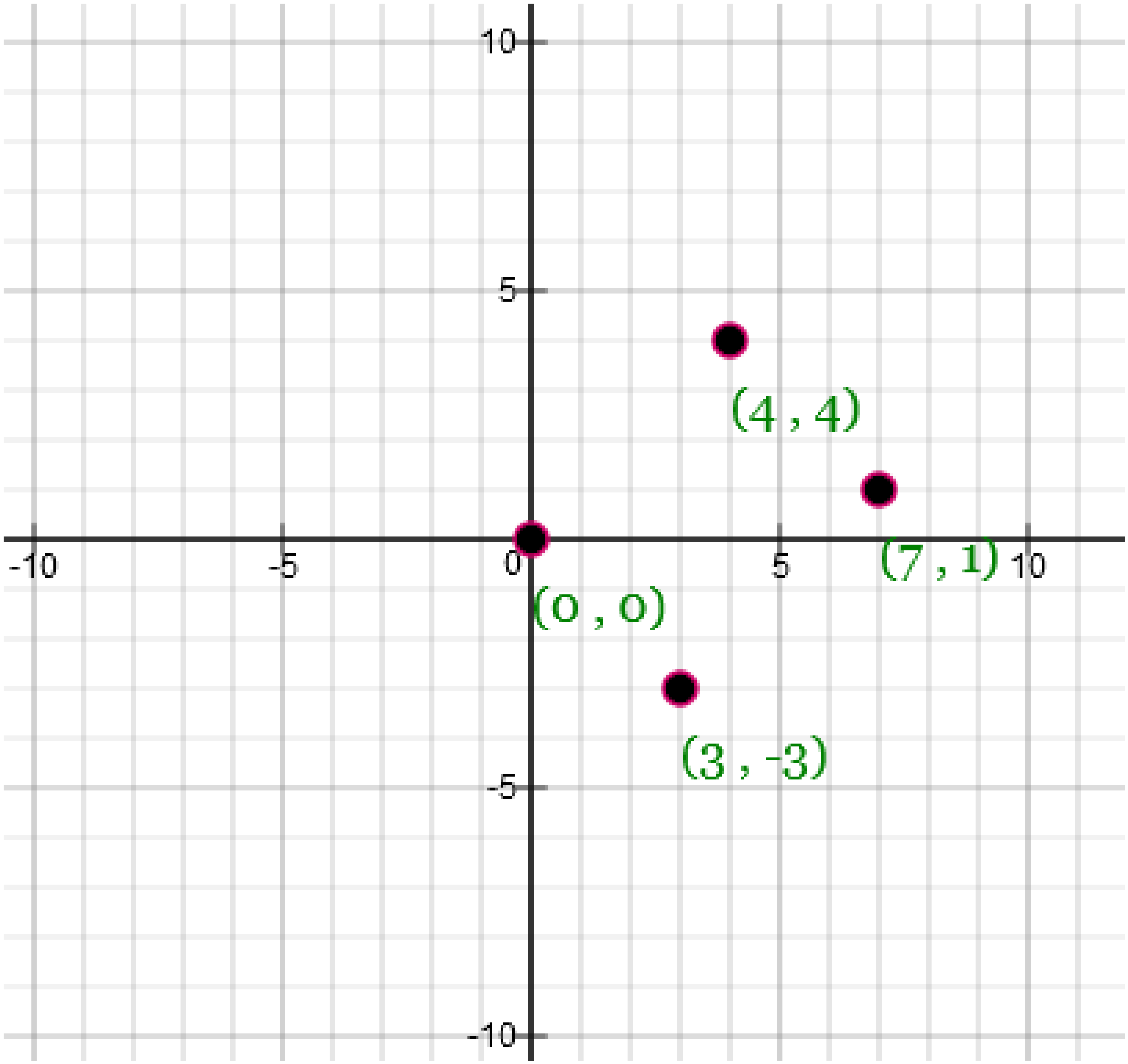
vertices (0,0) (3,-3) (4,4) (7, 1).

Find the volume of the solid between the surface $z = f(x, y)$

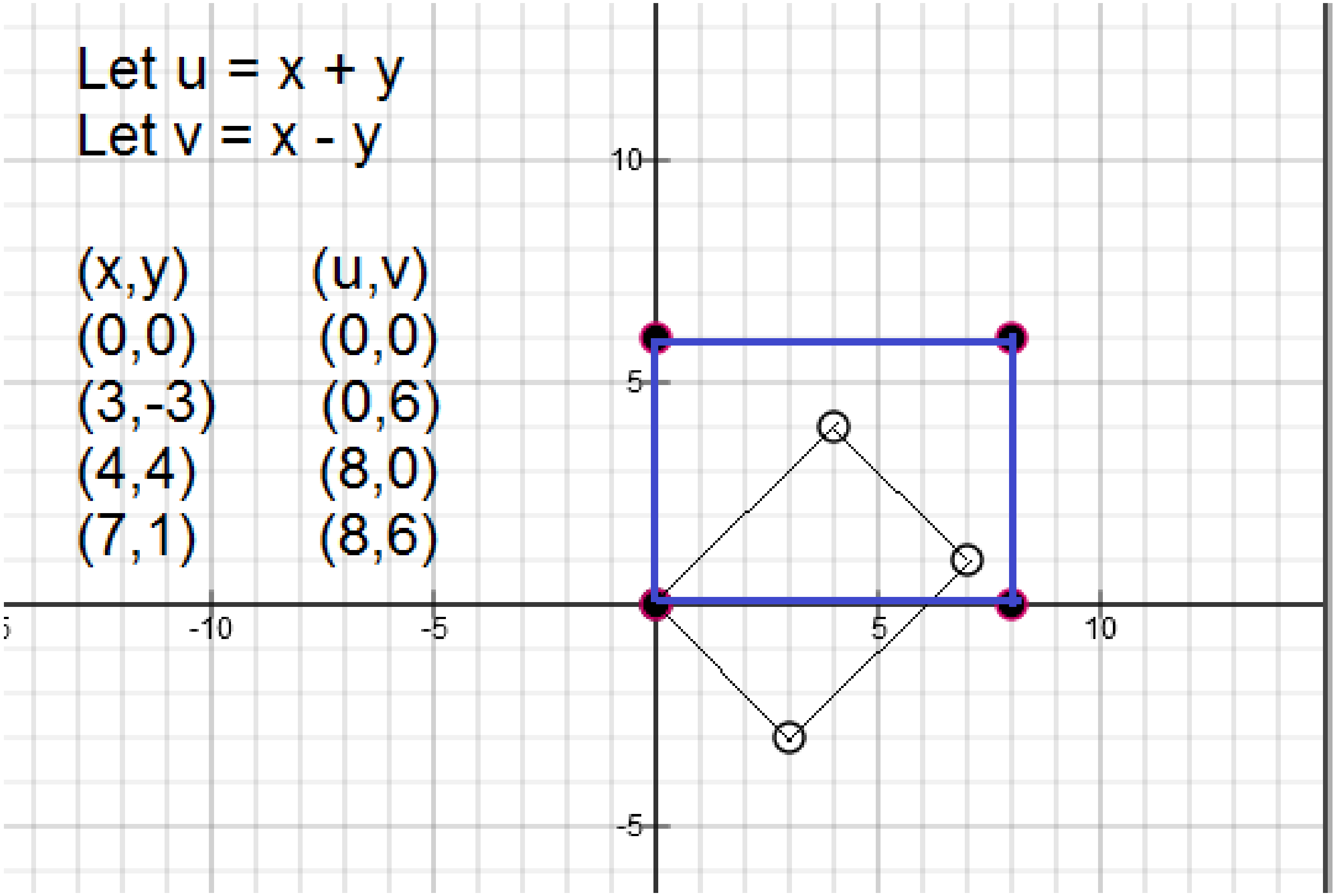
and region R.

Let $z = f(x, y) = x^2 + y$

Volume $= \iint_R f(x, y)\,dA =$ __?___


10
5
-5
-10
-10
-5
0
5
10
(4 , 4)
(7 , 1)
(0 , 0)
(3 , -3)

We will now make a change of variables so that we end up with a standard rectangular region R.



New region R has the following vertices: (0,0), (0,6), (8,0), (8,6)

Finding change of variables to simplify region R.

Let $u = x + y \quad (\text{equ } 1)$; Let $v = x - y \quad (\text{equ } 2)$

$$(\text{equ } 1) + (\text{equ } 2):\; u + v = 2x \Rightarrow x = \frac{1}{2}(u+v)$$

$$(\text{equ } 1) - (\text{equ } 2):\; u - v = 2y \Rightarrow y = \frac{1}{2}(u-v)$$

$$\frac{\partial x}{\partial u} = \frac{1}{2}; \quad \frac{\partial x}{\partial v} = \frac{1}{2}; \quad \frac{\partial y}{\partial u} = \frac{1}{2}; \quad \frac{\partial y}{\partial v} = \frac{-1}{2}$$

Jacobian of x and y:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{-1}{2} \end{vmatrix} = \frac{-1}{4} - \frac{1}{4} = \frac{-1}{2}$$

$$z = f(x,y) = x^2 + y = \left(\frac{1}{2}(u+v)\right)^2 + \frac{1}{2}(u-v)$$

$$= \ = \frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v$$

$$\text{Volume} = \iint_R f(x,y)\,dA = \iint_R f\left(g(u,v), h(u,v)\right)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\text{Volume} = \iint_R \left(\frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v\right)\left|\frac{-1}{2}\right| du\,dv$$

$$\text{Volume} = \frac{1}{2}\int_0^6\int_0^8 \left(\frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v\right) du\,dv$$

Evaluating $\int_0^8 \left(\frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v \right) du$

$$= \left[\frac{1}{4}\frac{u^3}{3} + \frac{1}{2}v \cdot \frac{u^2}{2} + \frac{1}{4}v^2 u + \frac{1}{2}\frac{u^2}{2} - \frac{1}{2}vu \right]_0^8$$

$$= \frac{512}{12} + \ 16\,v + 2v^2 + 16 - 4v$$

$$\text{Volume} = \frac{1}{2}\int_0^6 \int_0^8 \left(\frac{1}{4}u^2 + \frac{1}{2}uv + \frac{1}{4}v^2 + \frac{1}{2}u - \frac{1}{2}v \right) du dv$$

$$= \frac{1}{2}\int_0^6 \left(\frac{512}{12} + 16v + 2v^2 + 16 - 4v \right) dv = 356$$

Example 2: Let R be the triangular region bounded

by the following vertices (0,0) (4,0)   (8,4).

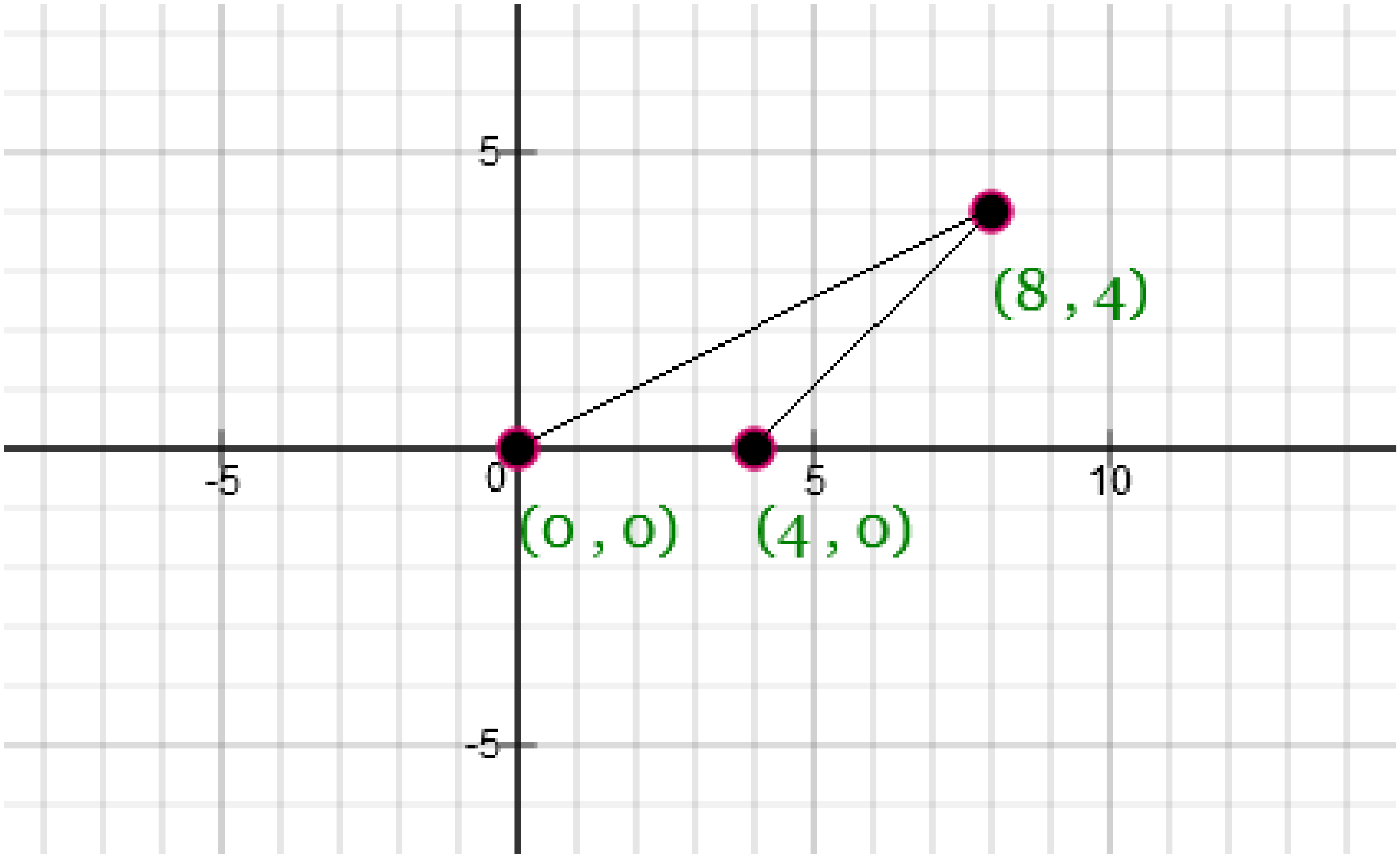
Find the volume of the solid between the surface

$z = f(x, y)$ and region R.

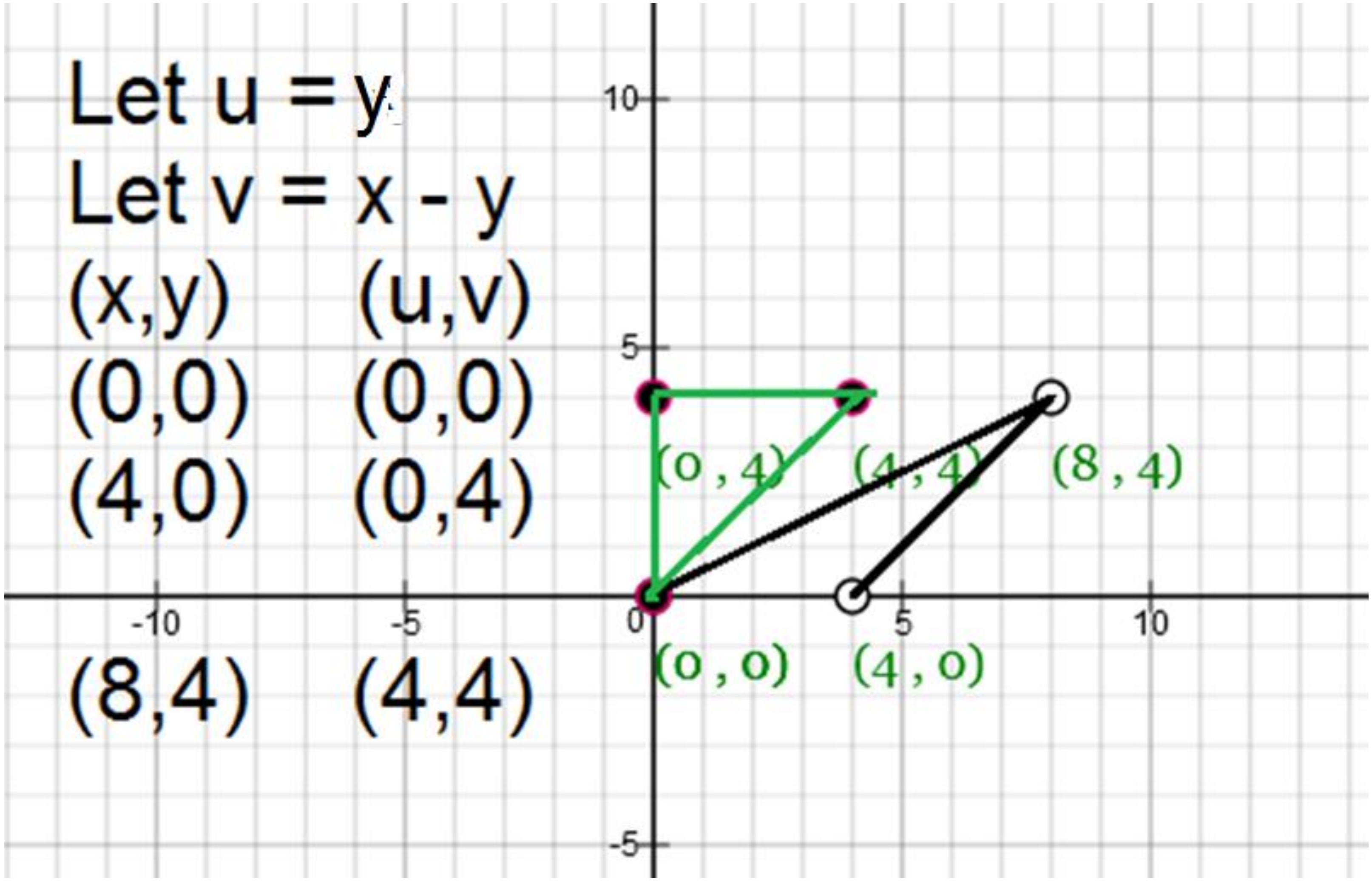
Let $z = f(x, y) = \sin(xy)$

$$\text{Volume} = \iint_R f(x, y)\,dA = \underline{\quad ? \quad}$$

Let $u = y; \quad v = x - y$


5
-5
0
5
10
-5
(8 , 4)
(0 , 0)
(4 , 0)

We will make a change of variables so that we end up with a standard triangular region R.



New triangular region has vertices: (0,0), (0,4), (4,4)

Finding change of variables to simplify region R.

Let $u = y \quad (\text{equ } 1)$

Let $v = x - y \quad (\text{equ } 2)$

$(\text{equ } 2): x = v + y \Rightarrow x = v + u$

$$\frac{\partial x}{\partial u} = 1; \quad \frac{\partial x}{\partial v} = 1; \quad \frac{\partial y}{\partial u} = 1; \quad \frac{\partial y}{\partial v} = 0$$

Jacobian of x and y:

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 0 - 1 = -1$$

Let $z = f\left(x,y\right) = \sin\left(xy\right) = \sin\left(\left(u+v\right)\left(u\right)\right) = \sin\left(u^2+uv\right)$

$$\text{Volume} = \iint_R f\left(x,y\right)dA = \iint_R f\left(g(u,v),h(u,v)\right)\left|\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}\right|dvdu$$

$$\text{Volume} = \iint_R \sin\left(u^2+uv\right)\left|-1\right|dvdu = \int_0^4\int_{v=u}^{v=4}\sin\left(u^2+uv\right)dvdu$$

$$\text{Evaluating } \int_{v=4}^{v=u}\sin\left(u^2+uv\right)dv = \left[\frac{-1}{u}\cos\left(u^2+uv\right)\right]_u^4$$

$$= \frac{-1}{u}\cos\left(u^2+4u\right) - \frac{-1}{u}\cos\left(u^2+u^2\right)$$

$$= \frac{-1}{u}\cos\left(u^2+4u\right) + \frac{1}{u}\cos\left(2u^2\right)$$

$$\text{Volume} = \int_0^4 \int_{v=u}^{v=4} \sin\left(u^2 + uv\right) dv\,du =$$

$$= \int_0^4 \left[\frac{-1}{u} \cos\left(u^2 + 4u\right) + \frac{1}{u} \cos\left(2u^2\right) + \right] du = 1.33$$

Calculus I Review:

Find $\int \sin(a+bx)\,dx.$

Let $u = a + bx$

$$du = b\,dx \quad \Rightarrow \quad \frac{1}{b}du = dx$$

$$\int \sin(a+bx)\,dx = \int \sin(u)\frac{1}{b}du = \frac{1}{b}\left[-\cos u\right]$$

$$= -\frac{1}{b}\cos(a+bx)$$