Calculus III   Section 15.1 Notes

Newton's Law of Gravitation is an example of a Vector Field:

Newton's law of universal **gravitation** states that a particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.

Sourse: https://en.wikipedia.org/wiki/Newton%27s_law_of_universal_gravitation

Newton's Law of Gravitation:

$$\mathbf{F} = \frac{G m_1 m_2}{r^2}$$

where:

F = the force between the masses;

G = the gravitational constant = $6.674 \times 10^{-11}\ \mathrm{N} \cdot (\mathrm{m/kg})^2$

m_1 = mass of particle 1;

m_2 = mass of particle 2;

r = distance between the centers of the masses.

F = gravitational force between the sun and the earth

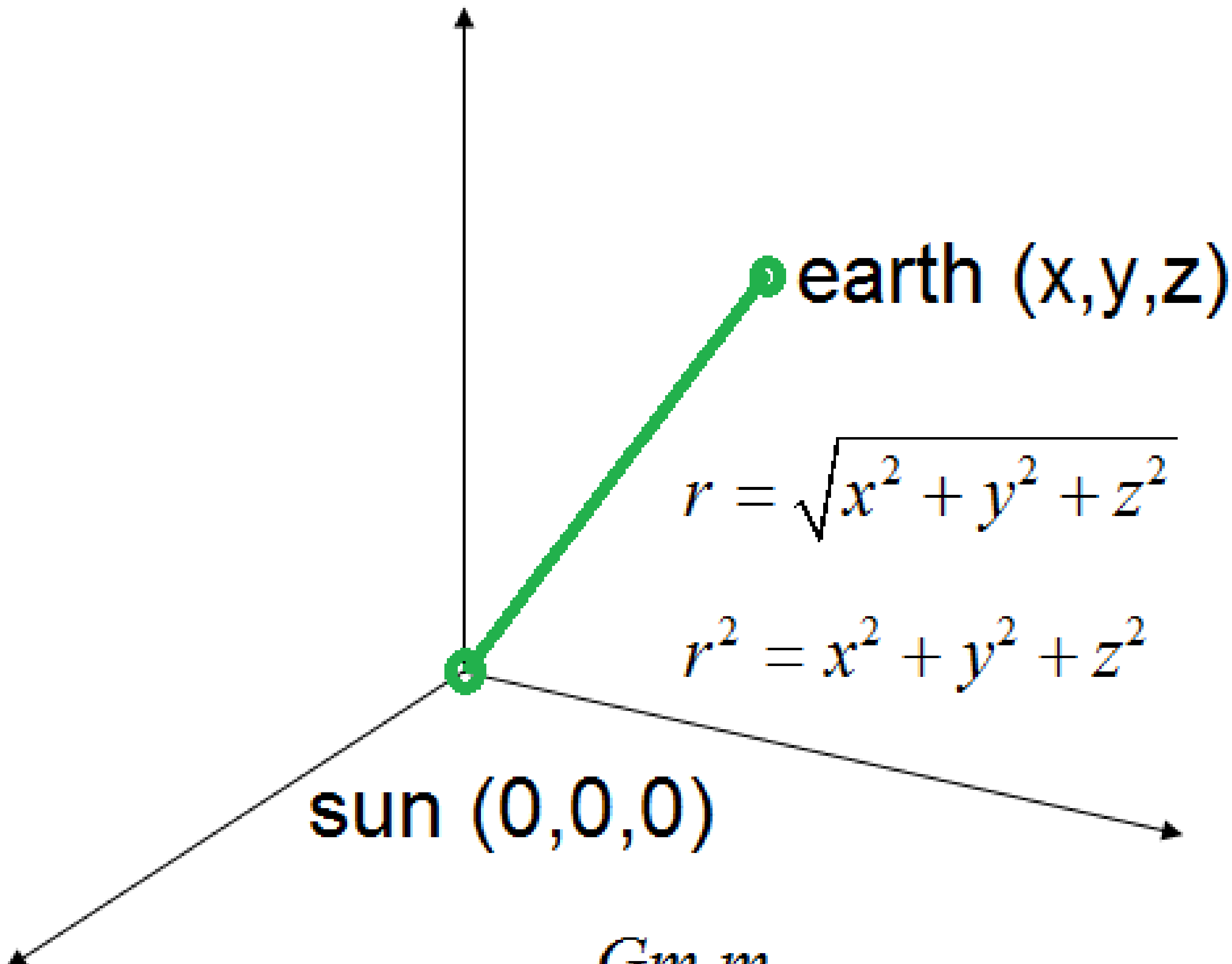
G = the gravitational constant = $6.674\times10^{-11}\ \mathrm{N}\cdot(\mathrm{m/kg})^2$

m_1 = Mass of earth = 5.97219×10^{24}

m_2 = Mass of sun = 1.9891×10^{30}

r = distance between earth and sun = 1.5×10^{11}

$$\mathbf{F} = \frac{Gm_1m_2}{r^2} = 3.52\times10^{22}\ N$$



$$r = \sqrt{x^2 + y^2 + z^2}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\mathbf{F}(x, y, z) = \frac{-Gm_1m_2}{x^2 + y^2 + z^2}\mathbf{u}$$

$\mathbf{u}$ = unit vector in the direction from the origin to (x, y, z)

The gradient of the function $f(x, y) = 3x^2 y + 4xy^2$ is an example of a vector field.

$$\mathbf{F}(x, y) = \nabla f(x, y) = \langle f_x, f_y \rangle = \langle 6xy, 8xy \rangle$$

$$\mathbf{F}(0,0) = \langle 0,0 \rangle = 0\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{F}(1,1) = \langle 6,8 \rangle = 6\mathbf{i} + 8\mathbf{j}$$

$$\mathbf{F}(1,2) = \langle 6,16 \rangle = 6\mathbf{i} + 16\mathbf{j}$$

$$\mathbf{F}(2,2) = \langle 12,16 \rangle = 12\mathbf{i} + 16\mathbf{j}$$

Example 1: Conservative Vector Field

Let $\mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$

Determine whether the vector field is conservative.

Let $M = 2y^3$ and $N = 6xy^2$

$$\frac{\partial M}{\partial y} = 6y^2; \qquad \frac{\partial N}{\partial x} = 6y^2$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the vector field $\mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$

is a conservative vector field.

Example 1 (con't): Since $\mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$ is conservative,

find a potential function $f\left(x, y\right)$.

Meaning, find $f\left(x, y\right)$ such that $\nabla f(x, y) = \mathbf{F}(x, y)$.

Hence, $\nabla f(x, y) = \left\langle f_x, f_y \right\rangle = \mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$

$f_x = 2y^3; \qquad f_y = 6xy^2$

$f = \int f_x dx = \int 2y^3 dx = 2y^3 x + C_1$

$f = \int f_y dy = \int 6xy^2 dy = 6x \cdot \dfrac{y^3}{3} + C_2 = 2xy^3 + C_2$

Therefore, let $f\left(x, y\right) = 2xy^3 + K$

Note: When $f\left(x, y\right) = 2xy^3 + K, \nabla f(x, y) = \left\langle f_x, f_y \right\rangle = \left\langle 2y^3, 6xy^2 \right\rangle$

Example 2: Let $\mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$

Is $\mathbf{F}(x, y)$ a conservative vector field?

Let $M = 5xe^x$; Let $N = 7ye^y$

$$\frac{\partial M}{\partial y} = 0; \qquad \frac{\partial N}{\partial x} = 0$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the vector field $\mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$

is a conservative vector field.

Example 2 (con't): Since $\mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$ is conservative,

find a potential function $f\left(x, y\right)$.

Meaning, find $f\left(x, y\right)$ such that $\nabla f(x, y) = \mathbf{F}(x, y)$.

Hence, $\nabla f(x, y) = \left\langle f_x, f_y \right\rangle = \mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$

$f_x = 5xe^x; \qquad f_y = 7ye^y$

$f = \int f_x dx = \int 5xe^x dx = 5xe^x - 5e^x + C_1$

$f = \int f_y dy = \int 7ye^y dy = 7ye^y - 7e^y + C_2$

Therefore, let $f\left(x, y\right) = 5xe^x - 5e^x + 7ye^y - 7e^y + K$

Note: When $f\left(x, y\right) = 5xe^x - 5e^x + 7ye^y - 7e^y + K$,

$\nabla f(x, y) = \left\langle f_x, f_y \right\rangle = \left\langle 5xe^x + 5e^x - 5e^x, 7ye^y + 7e^y - 7e^y \right\rangle = \left\langle 5xe^x, 7ye^y \right\rangle$

Example 3: Let $\mathbf{F}(x, y) = 4xy^3\mathbf{i} + \left(1 + 6x^2y^2\right)\mathbf{j}$

Is $\mathbf{F}(x, y)$ a conservative vector field?

Let $M = 4xy^3$; Let $N = 1 + 6x^2y^2$

$$\frac{\partial M}{\partial y} = 12xy^2; \quad \frac{\partial N}{\partial x} = 12xy^2;$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the vector field $\mathbf{F}(x, y) = 4xy^3\mathbf{i} + \left(1 + 6x^2y^2\right)\mathbf{j}$

is a conservative vector field.

Example 3 (con't): Since Let $\mathbf{F}(x,y) = 4xy^3\mathbf{i} + \left(1+6x^2y^2\right)\mathbf{j}$ is conservative,

find a potential function $f\left(x,y\right)$.

Meaning, find $f\left(x,y\right)$ such that $\nabla f(x,y) = \mathbf{F}(x,y)$.

Hence, $\nabla f(x,y) = \left\langle f_x, f_y \right\rangle = \mathbf{F}(x,y) = 4xy^3\mathbf{i} + \left(1+6x^2y^2\right)\mathbf{j}$

$f_x = 4xy^3; \qquad f_y = 1+6x^2y^2$

$$f = \int f_x dx = \int 4xy^3 dx = 4y^3 \cdot \frac{x^2}{2} + C_1 = 2x^2y^3 + C_1$$

$$f = \int f_y dy = \int \left(1+6x^2y^2\right) dy = y + 6x^2 \cdot \frac{y^3}{3} + C_2 = y + 2x^2y^3 + C_2$$

Therefore, let $f\left(x,y\right) = y + 2x^2y^3 + K$

Note: When $f\left(x,y\right) = y + 2x^2y^3 + K$,

$\nabla f(x,y) = \left\langle f_x, f_y \right\rangle = \left\langle 4xy^3, 6x^2y^2 \right\rangle$

Conservative Vector Fields in Space

Let $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}, \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

Example 4:

Let $\mathbf{F}(x, y, z) = x^2 z\mathbf{i} + (y^2 x + z)\mathbf{j} + (y + 2z)\mathbf{k}$

Determine if $\mathbf{F}(x, y, z)$ is conservative.

Let $M = x^2 z,\quad N = y^2 x + z,\quad$ and $P = y + 2z$

$Note:\mathbf{F}(x, y, z)$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z},\quad \frac{\partial P}{\partial x} = \frac{\partial M}{\partial z},\quad \text{and}\quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

$$\frac{\partial P}{\partial y} = 1;\qquad \frac{\partial N}{\partial z} = 1;$$

$$\frac{\partial P}{\partial x} = 0;\qquad \frac{\partial M}{\partial z} = 2$$

$$\frac{\partial N}{\partial x} = y^2;\qquad \frac{\partial M}{\partial y} = 0$$

Therefore, $\mathbf{F}(x, y, z)$ is not conservative.

Definition of Curl of a Vector Field

Let $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$

$$\text{Curl } \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \right)\mathbf{i} - \left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} \right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\mathbf{k}$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

Theorem: Let $\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$

$\mathbf{F}(x, y, z)$ is conservative if and only if Curl $\mathbf{F} = 0$.

Example 5:

Let $\mathbf{F}(x, y, z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

Find curl $\mathbf{F}(x, y, z)$ and determine if $\mathbf{F}(x, y, z)$ is conservative.

Let $M = x^2, \quad N = y^2, \quad$ and $\; P = z^2$

$Note: \mathbf{F}(x, y, z)$ is conservative if and only if curl $\mathbf{F}(x, y, z) = 0$

$$\frac{\partial P}{\partial y} = 0; \qquad \frac{\partial N}{\partial z} = 0; \quad \text{hence } \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} = 0$$

$$\frac{\partial P}{\partial x} = 0; \qquad \frac{\partial M}{\partial z} = 0; \quad \text{hence } \frac{\partial P}{\partial x} - \frac{\partial M}{\partial z} = 0$$

$$\frac{\partial N}{\partial x} = 0; \qquad \frac{\partial M}{\partial y} = 0; \quad \text{hence } \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$$

$$\text{curl } \mathbf{F}(x, y, z) = \left\langle \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}, -\left(\frac{\partial P}{\partial x} - \frac{\partial M}{\partial z}\right), \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right\rangle = \langle 0, 0, 0 \rangle$$

Therefore, $\mathbf{F}(x, y, z)$ is conservative

Example 5 (con't):

Let $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

Find the potential function $f(x,y,z)$ for $\mathbf{F}(x,y,z)$.

Note: $f(x,y,z)$ is a function such that $\mathbf{F}(x,y,z) = \nabla f(x,y,z)$

Let $M = x^2, \quad N = y^2, \quad$ and $P = z^2$

Hence, $\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \mathbf{F}(x,y,z) = \langle x^2, y^2, z^2 \rangle$

$f_x = x^2, \qquad f_y = y^2, \qquad f_z = z^2$

$$f(x,y,z) = \int f_x dx = \int M dx = \int x^2 dx = \frac{1}{3}x^3 + p(y,z)$$

$$f(x,y,z) = \int f_y dy = \int N dx = \int y^2 dx = \frac{1}{3}y^3 + q(x,z)$$

$$f(x,y,z) = \int f_z dx = \int P dx = \int z^2 dx = \frac{1}{3}z^3 + r(x,y)$$

Let $p(y,z) = q(x,z) = r(x,y) = K$

Therefore $f(x,y,z) = \frac{1}{3}x^3 + \frac{1}{3}y^3 + \frac{1}{3}z^3 + K$

Definition of Divergence of a Vector Field

Let $\mathbf{F}(x,y,z) = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

Example 6: Let $\mathbf{F}(x,y) = 4xy^3\mathbf{i} + \left(1+6x^2y^2\right)\mathbf{j} + z^3\mathbf{k}$

Find div $\mathbf{F}(x,y,z)$.

Let $M = 4xy^3$; Let $N = 1+6x^2y^2$; Let $P = z^3$

$$\frac{\partial M}{\partial x} = 4y^3; \qquad \frac{\partial N}{\partial y} = 6x^2; \qquad \frac{\partial P}{\partial z} = 3z^2$$

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} = 4y^3 + 6x^2 + 3z^2$$