

Newton's Law of Gravitation is an example of a Vector Field:

**Newton's law** of universal **gravitation** states that a particle attracts every other particle in the universe using a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between their centers.

Source: [https://en.wikipedia.org/wiki/Newton%27s\\_law\\_of\\_universal\\_gravitation](https://en.wikipedia.org/wiki/Newton%27s_law_of_universal_gravitation)

## Newton's Law of Gravitation:

$$\mathbf{F} = \frac{Gm_1m_2}{r^2}$$

where:

$F$  = the force between the masses;

$G$  = the gravitational constant =  $6.674 \times 10^{-11} \text{ N} \cdot (\text{m}/\text{kg})^2$

$m_1$  = mass of particle 1;

$m_2$  = mass of particle 2;

$r$  = distance between the centers of the masses.

**F** = gravitational force between the sun and the earth

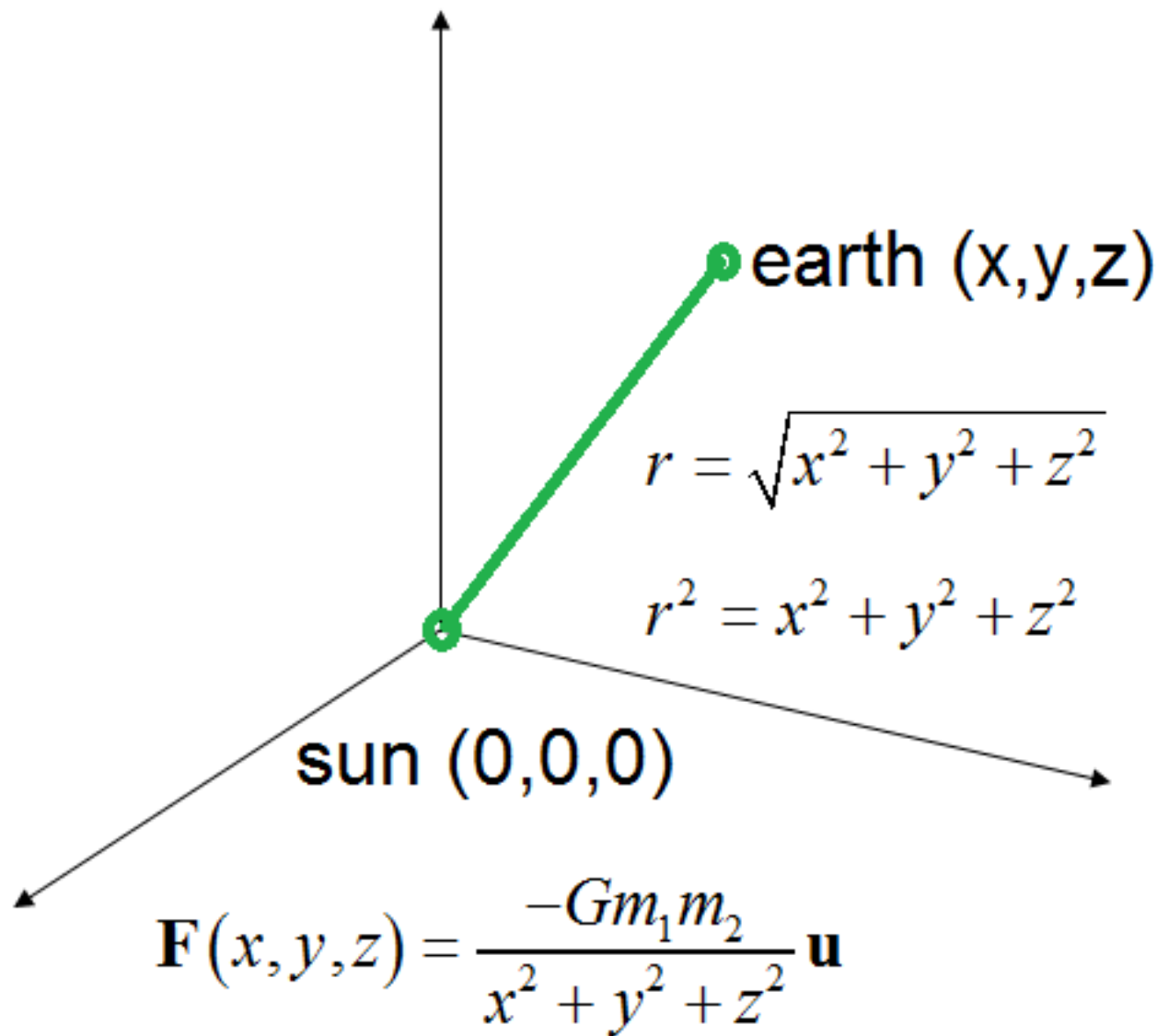
**G** = the gravitational constant =  $6.674 \times 10^{-11} \text{ N} \cdot (\text{m}/\text{kg})^2$

$m_1$  = Mass of earth =  $5.97219 \times 10^{24}$

$m_2$  = Mass of sun =  $1.9891 \times 10^{30}$

$r$  = distance between earth and sun =  $1.5 \times 10^{11}$

$$\mathbf{F} = \frac{Gm_1m_2}{r^2} = 3.52 \times 10^{22} \text{ N}$$



$\mathbf{u}$  = unit vector in the direction from the origin to  $(x, y, z)$

The gradient of the function  $f(x, y) = 3x^2y + 4xy^2$  is an example of a vector field.

$$\mathbf{F}(x, y) = \nabla f(x, y) = \langle f_x, f_y \rangle = \langle 6xy, 8xy \rangle$$

$$\mathbf{F}(0, 0) = \langle 0, 0 \rangle = 0\mathbf{i} + 0\mathbf{j}$$

$$\mathbf{F}(1, 1) = \langle 6, 8 \rangle = 6\mathbf{i} + 8\mathbf{j}$$

$$\mathbf{F}(1, 2) = \langle 6, 16 \rangle = 6\mathbf{i} + 16\mathbf{j}$$

$$\mathbf{F}(2, 2) = \langle 12, 16 \rangle = 12\mathbf{i} + 16\mathbf{j}$$

## Example 1: Conservative Vector Field

$$\text{Let } \mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$$

Determine whether the vector field is conservative.

$$\text{Let } M = 2y^3 \quad \text{and} \quad N = 6xy^2$$

$$\frac{\partial M}{\partial y} = 6y^2; \quad \frac{\partial N}{\partial x} = 6y^2$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , the vector field  $\mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$

is a conservative vector field.

Example 1 (con't): Since  $\mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$  is conservative, find a potential function  $f(x, y)$ .

Meaning, find  $f(x, y)$  such that  $\nabla f(x, y) = \mathbf{F}(x, y)$ .

Hence,  $\nabla f(x, y) = \langle f_x, f_y \rangle = \mathbf{F}(x, y) = 2y^3\mathbf{i} + 6xy^2\mathbf{j}$

$$f_x = 2y^3; \quad f_y = 6xy^2$$

$$f = \int f_x dx = \int 2y^3 dx = 2y^3 x + C_1$$

$$f = \int f_y dy = \int 6xy^2 dy = 6x \cdot \frac{y^3}{3} + C_2 = 2xy^3 + C_2$$

Therefore, let  $f(x, y) = 2xy^3 + K$

Note: When  $f(x, y) = 2xy^3 + K$ ,  $\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 2y^3, 6xy^2 \rangle$

Example 2: Let  $\mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$

Is  $\mathbf{F}(x, y)$  a conservative vector field?

Let  $M = 5xe^x$ ;      Let  $N = 7ye^y$

$$\frac{\partial M}{\partial y} = 0; \quad \frac{\partial N}{\partial x} = 0$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , the vector field  $\mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$

is a conservative vector field.



Example 2 (con't): Since  $\mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$  is conservative, find a potential function  $f(x, y)$ .

Meaning, find  $f(x, y)$  such that  $\nabla f(x, y) = \mathbf{F}(x, y)$ .

Hence,  $\nabla f(x, y) = \langle f_x, f_y \rangle = \mathbf{F}(x, y) = 5xe^x\mathbf{i} + 7ye^y\mathbf{j}$

$$f_x = 5xe^x; \quad f_y = 7ye^y$$

$$f = \int f_x dx = \int 5xe^x dx = 5xe^x - 5e^x + C_1$$

$$f = \int f_y dy = \int 7ye^y dy = 7ye^y - 7e^y + C_2$$

Therefore, let  $f(x, y) = 5xe^x - 5e^x + 7ye^y - 7e^y + K$

Note: When  $f(x, y) = 5xe^x - 5e^x + 7ye^y - 7e^y + K$ ,

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 5xe^x + 5e^x - 5e^x, 7ye^y + 7e^y - 7e^y \rangle = \langle 5xe^x, 7ye^y \rangle$$

Example 3: Let  $\mathbf{F}(x, y) = 4xy^3\mathbf{i} + (1 + 6x^2y^2)\mathbf{j}$

Is  $\mathbf{F}(x, y)$  a conservative vector field?

Let  $M = 4xy^3$ ;    Let  $N = 1 + 6x^2y^2$

$$\frac{\partial M}{\partial y} = 12xy^2; \quad \frac{\partial N}{\partial x} = 12xy^2;$$

Since  $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ , the vector field  $\mathbf{F}(x, y) = 4xy^3\mathbf{i} + (1 + 6x^2y^2)\mathbf{j}$

is a conservative vector field.

Example 3 (con't): Since Let  $\mathbf{F}(x, y) = 4xy^3\mathbf{i} + (1 + 6x^2y^2)\mathbf{j}$  is conservative, find a potential function  $f(x, y)$ .

Meaning, find  $f(x, y)$  such that  $\nabla f(x, y) = \mathbf{F}(x, y)$ .

Hence,  $\nabla f(x, y) = \langle f_x, f_y \rangle = \mathbf{F}(x, y) = 4xy^3\mathbf{i} + (1 + 6x^2y^2)\mathbf{j}$

$$f_x = 4xy^3; \quad f_y = 1 + 6x^2y^2$$

$$f = \int f_x dx = \int 4xy^3 dx = 4y^3 \cdot \frac{x^2}{2} + C_1 = 2x^2y^3 + C_1$$

$$f = \int f_y dy = \int (1 + 6x^2y^2) dy = y + 6x^2 \cdot \frac{y^3}{3} + C_2 = y + 2x^2y^3 + C_2$$

Therefore, let  $f(x, y) = y + 2x^2y^3 + K$

Note: When  $f(x, y) = y + 2x^2y^3 + K$ ,

$$\nabla f(x, y) = \langle f_x, f_y \rangle = \langle 4xy^3, 6x^2y^2 \rangle$$

