Calculus III Section 15.2 Notes

Line Integral of a vector field

Line Integral of a vector field: $\int_C \mathbf{F} \cdot d\mathbf{r}$

where:

$\mathbf{F}$ is the vector field

C is the smooth curve represented by $\mathbf{r}(t)$

Example 1: Find the work done by the force field $\mathbf{F}(x, y)$

on a particle as it moves along path C while subject to the force field.

Vector Field: $\mathbf{F}(x, y) = x^2\mathbf{i} + 3y\mathbf{j}$

Path C: $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j} = x(t)\mathbf{i} + y(t)\mathbf{j}$ $\qquad 0 \le t \le 2$

Finding work done by force field = $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j}$$

$$d\mathbf{r} = (4\mathbf{i} + 3\mathbf{j})\,dt$$

Example 1 (con't):

$$\mathbf{F}(x,y)=\mathbf{F}(x(t),y(t))=\left[x(t)\right]^2\mathbf{i}+3y(t)\mathbf{j}=(4t)^2\mathbf{i}+3(3t)\mathbf{j}$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[(4t)^2\mathbf{i}+3(3t)\mathbf{j}\right]\cdot(4\mathbf{i}+3\mathbf{j})dt$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[64t^2+27t\right]dt$$

$$\text{Work}=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^2\left[64t^2+27t\right]dt=224.66667$$

Example 2: Find the work done by the force field $\mathbf{F}(x, y)$

on a particle as it moves along path C while subject to the force field.

Vector Field: $\mathbf{F}(x, y) = 3x\mathbf{i} - 5y\mathbf{j}$

Path C: $\mathbf{r}(t) = 4t\mathbf{i} + (t-3)\mathbf{j} = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad 0 \le t \le 2$

Finding work done by force field = $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = 4\mathbf{i} + 1\mathbf{j}$$

$$d\mathbf{r} = (4\mathbf{i} + \mathbf{j})\,dt$$

Example 2 (con't):

$$\mathbf{F}(x,y)=\mathbf{F}\big(x(t),y(t)\big)=3x(t)\mathbf{i}-5y(t)\,\mathbf{j}=3(4t)\mathbf{i}-5(t-3)\,\mathbf{j}$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[3(4t)\mathbf{i}-5(t-3)\,\mathbf{j}\right]\cdot(4\mathbf{i}+1\mathbf{j})\,dt$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[48t-5t+15\right]dt$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^2 \left[48t-5t+15\right]dt = 116$$

Example 3: Find the work done by the force field $\mathbf{F}(x, y, z)$

on a particle as it moves along path C while subject to the force field.

Vector Field: $\mathbf{F}(x, y) = 3x\mathbf{i} - 5y\mathbf{j} + 7z\mathbf{k}$

Path C: $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \qquad 0 \le t \le 1$

Finding work done by force field = $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$$

$$d\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k})\,dt$$

Example 3 (con't):

$$\mathbf{F}(x,y)=\mathbf{F}\left(x(t),y(t)\right)=3x\mathbf{i}-5y\mathbf{j}+7z\mathbf{k}=3(4t)\mathbf{i}-5(3t)\mathbf{j}+7\left(t^2\right)\mathbf{k}$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[3(4t)\mathbf{i}-5(3t)\mathbf{j}+7\left(t^2\right)\mathbf{k}\right]\cdot\left(4\mathbf{i}+3\mathbf{j}+2t\mathbf{k}\right)dt$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[48t-45t+14t^3\right]dt$$

$$\text{Work} = \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[48t-45t+14t^3\right]dt = 5$$

Line Integral and Mass of Thin Wire

Suppose a thin wire is in the shape of the path

$$C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}\,; \qquad a \le t \le b$$

The density of the wire at point (x, y) is $\rho(x, y)$.

Find the mass of the wire.

Suppose we cut the wire into many small segments: $s_1, s_2, s_3, \ldots$

If the segment is small enough, then the density of each small piece is uniform.

Hence, mass of ith piece (s_i) is $(\text{length of } s_i)\big(\rho(x_i, y_i)\big)$.

Therefore, mass of thin wire = $\sum_{i=1}^{\infty} (\text{length of } s_i)\big(\rho(x_i, y_i)\big)$

Line Integral and Mass of Thin Wire

Note: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$; $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$; $\mathbf{r}'(t) = \dfrac{dx}{dt}\mathbf{i} + \dfrac{dy}{dt}\mathbf{j}$;

$$\left\|\mathbf{r}'(t)\right\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Recall: Length of arc = $\displaystyle\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_a^b \left\|\mathbf{r}'(t)\right\| dt$

Mass of Thin Wire = $\displaystyle\int_C \rho(x, y)\left\|\mathbf{r}'(t)\right\| dt$

Example 4: Suppose a thin wire is in the shape of

the path $C:\mathbf{r}(t)=5t\mathbf{i}+t^2\mathbf{j}=x(t)\mathbf{i}+y(t)\mathbf{j}$; $\qquad 0\le t\le 4.$

The density of the wire at point (x,y) is $\rho(x,y)=4xy.$

Find the mass of the wire.

$$\mathbf{r}'(t)=5\mathbf{i}+2t\mathbf{j}$$

$$\left\|\mathbf{r}'(t)\right\|=\sqrt{(5)^2+(2t)^2}=\sqrt{25+4t^2}$$

$$\rho(x,y)=4xy=4x(t)\cdot y(t)=4(5t)\left(t^2\right)=20t^3$$

$$\text{Mass}=\int_C \rho(x,y)\left\|\mathbf{r}'(t)\right\|dt=\int_0^4 20t^3\sqrt{25+4t^2}\,dt=10456.38839$$

Example 5: Suppose a thin wire is in the shape of a helix.

$C: \mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$; $0 \le t \le 2\pi$.

The density of the wire at point (x, y) is $\rho(x, y) = 4xyz$.

Find the mass of the wire.

$$\mathbf{r}'(t) = -4\sin t\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$$

$$\left\|\mathbf{r}'(t)\right\| = \sqrt{(-4\sin t)^2 + (4\cos t)^2 + (1)^2} = \sqrt{16\sin^2 t + 16\cos^2 t + 1}$$

$$\rho(x, y) = 4xyz = 4x(t) \cdot y(t) \cdot z(t) =$$

$$\text{Mass} = \int_C \rho(x, y)\left\|\mathbf{r}'(t)\right\| dt = \int_0^4 15t^3\sqrt{25 + 4t^2}\, dt = 7842.29129$$