Calculus III Section 15.2 Notes

Types of Integral:

a) Area = $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i, y_i)\Delta x_i = \int_a^b f(x)dx$ Integrate over the interval $[a,\ b]$

b) Volume = $\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i, y_i)\Delta y_i \Delta x_i = \iint_R f(x, y)dA$ Integrate over region R

c) $\int_C f(x, y)ds = \lim_{\|\Delta\|\to 0}\sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$ Integrate over curve C

Also, application of Line Integral of a vector field: $\int_C \mathbf{F}\cdot d\mathbf{r}$

where: $\mathbf{F}$ is the vector field; and C is the smooth curve represented by $\mathbf{r}(t)$

Recall: From Calculus II:

Arc Length $s = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt$

Hence, $\frac{ds}{dt} = \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}$ or $ds = \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\, dt$

Definition of $\int_C f(x, y)ds$ and $\int_C f(x, y, z)ds$:

$$\int_C f(x, y)ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i = \int_a^b f\left(x(t), y(t)\right) ds = \int_a^b f\left(x(t), y(t)\right) \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$\int_C f(x, y, z)ds = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(x_i, y_i, z_i)\Delta s_i = \int_a^b f\left(x(t), y(t), z(t)\right) \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt$$

Example 1: Evaluate the line integral over the given path C.

$$\int_C (x+y)\,ds \quad \text{and} \quad C : \mathbf{r}(t) = \left\langle t, t^2 \right\rangle \quad 0 \le t \le 2$$

Solution :

$$C : \mathbf{r}(t) = \left\langle t, t^2 \right\rangle = \left\langle x(t), y(t) \right\rangle \quad \Rightarrow \quad x(t) = t; \quad y(t) = t^2$$

$$\mathbf{r}'(t) = \left\langle x'(t), y'(t) \right\rangle = \left\langle 1, 2t \right\rangle$$

$$\left\| \mathbf{r}'(t) \right\| = \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2} = \sqrt{(1)^2 + (2t)^2} = \sqrt{1+4t^2}$$

$$\int_C f(x,y)\,ds = \int_a^b f\left(x(t), y(t)\right)\sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2}\,dt$$

$$= \int_0^2 \left(x+y\right)\sqrt{1+4t^2}\,dt$$

$$= \int_0^2 \left(t+t^2\right)\sqrt{1+4t^2}\,dt = 14.228908855676$$

Example 2: Evaluate the line integral over the given path C.

$$\int_C (x+y+z)\,ds \quad \text{and} \quad C: \mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle \quad 0 \le t \le 2$$

Solution:

$$C: \mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle = \left\langle x(t), y(t), z(t) \right\rangle \quad \Rightarrow \quad x(t) = t; \quad y(t) = t^2; \quad z(t) = t^3$$

$$\mathbf{r}'(t) = \left\langle x'(t), y'(t), z'(t) \right\rangle = \left\langle 1, 2t, 3t^2 \right\rangle$$

$$\left\|\mathbf{r}'(t)\right\| = \sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + \left[z'(t)\right]^2} = \sqrt{(1)^2 + (2t)^2 + (3t^2)^2} = \sqrt{1+4t^2+9t^4}$$

$$\int_C f(x,y)\,ds = \int_a^b f\left(x(t), y(t), z(t)\right)\sqrt{\left[x'(t)\right]^2 + \left[y'(t)\right]^2 + \left[z'(t)\right]^2}\,dt$$

$$= \int_0^2 (x+y+z)\sqrt{1+4t^2+9t^4}\,dt$$

$$= \int_0^2 \left(t+t^2+t^3\right)\sqrt{1+4t^2+9t^4}\,dt = 69.386482685899$$

Example 3: Evaluate the line integral over the given path C.

$$\int_C (x+y+z)\,ds \quad \text{and} \quad C:\mathbf{r}(t)=\left\langle t,t^2,t^3\right\rangle \quad 0\le t\le 2$$

Solution:

$$C:\mathbf{r}(t)=\left\langle t,t^2,t^3\right\rangle=\left\langle x(t),y(t),z(t)\right\rangle \quad \Rightarrow \quad x(t)=t;\quad y(t)=t^2;\quad z(t)=t^3$$

$$\mathbf{r}'(t)=\left\langle x'(t),y'(t),z'(t)\right\rangle=\left\langle 1,2t,3t^2\right\rangle$$

$$\left\|\mathbf{r}'(t)\right\|=\sqrt{\left[x'(t)\right]^2+\left[y'(t)\right]^2+\left[z'(t)\right]^2}=\sqrt{(1)^2+\left(2t\right)^2+(3t^2)^2}=\sqrt{1+4t^2+9t^4}$$

$$\int_C f(x,y)\,ds=\int_a^b f\left(x(t),y(t),z(t)\right)\sqrt{\left[x'(t)\right]^2+\left[y'(t)\right]^2+\left[z'(t)\right]^2}\,dt$$

$$=\int_0^2 (x+y+z)\sqrt{1+4t^2+9t^4}\,dt$$

$$=\int_0^2 \left(t+t^2+t^3\right)\sqrt{1+4t^2+9t^4}\,dt=69.386482685899$$

Example 4: Find the work done by the force field $\mathbf{F}(x,y)$ on a particle as it moves along path C while subject to the force field.

Vector Field: $\mathbf{F}(x,y)=x^2\mathbf{i}+3y\mathbf{j}$ and Path C: $\mathbf{r}(t)=4t\mathbf{i}+3t\mathbf{j}=x(t)\mathbf{i}+y(t)\mathbf{j}$ $0\le t\le 2$

Solution:

Work done by force field = $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$\frac{d\mathbf{r}}{dt}=\mathbf{r}'(t)=4\mathbf{i}+3\mathbf{j} \quad\Rightarrow\quad d\mathbf{r}=(4\mathbf{i}+3\mathbf{j})\,dt$$

$$\mathbf{F}(x,y)=\mathbf{F}(x(t),y(t))=[x(t)]^2\,\mathbf{i}+3y(t)\,\mathbf{j}=(4t)^2\,\mathbf{i}+3(3t)\,\mathbf{j}$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[(4t)^2\,\mathbf{i}+3(3t)\,\mathbf{j}\right]\cdot(4\mathbf{i}+3\mathbf{j})\,dt$$

$$\mathbf{F}\cdot d\mathbf{r}=\left[64t^2+27t\right]dt$$

$$\text{Work}=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_0^2\left[64t^2+27t\right]dt=224.66667$$

Example 5: Find the work done by the force field $\mathbf{F}(x, y)$

on a particle as it moves along path C while subject to the force field.

Vector Field: $\mathbf{F}(x, y) = 3x\mathbf{i} - 5y\mathbf{j}$ and Path C: $\mathbf{r}(t) = 4t\mathbf{i} + (t-3)\mathbf{j} = x(t)\mathbf{i} + y(t)\mathbf{j} \quad 0 \le t \le 2$

Solution:

Work done by force field = $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = 4\mathbf{i} + 1\mathbf{j} \quad \Rightarrow \quad d\mathbf{r} = (4\mathbf{i} + \mathbf{j})\,dt$$

$$\mathbf{F}(x, y) = \mathbf{F}(x(t), y(t)) = 3x(t)\mathbf{i} - 5y(t)\mathbf{j} = 3(4t)\mathbf{i} - 5(t-3)\mathbf{j}$$

$$\mathbf{F} \cdot d\mathbf{r} = \left[3(4t)\mathbf{i} - 5(t-3)\mathbf{j}\right] \cdot (4\mathbf{i} + 1\mathbf{j})\,dt$$

$$\mathbf{F} \cdot d\mathbf{r} = \left[48t - 5t + 15\right]dt$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 \left[48t - 5t + 15\right]dt = 116$$

Example 6: Find the work done by the force field $\mathbf{F}(x, y, z)$

on a particle as it moves along path C while subject to the force field.

Vector Field: $\mathbf{F}(x, y) = 3x\mathbf{i} - 5y\mathbf{j} + 7z\mathbf{k}$

Path C : $\mathbf{r}(t) = 4t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k} = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ $\qquad 0 \le t \le 1$

Solution:

Work done by force field = $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = 4\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k}$$

$$d\mathbf{r} = (4\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k})\,dt$$

$$\mathbf{F}(x, y) = \mathbf{F}(x(t), y(t)) = 3x\mathbf{i} - 5y\mathbf{j} + 7z\mathbf{k} = 3(4t)\mathbf{i} - 5(3t)\mathbf{j} + 7(t^2)\mathbf{k}$$

$$\mathbf{F} \cdot d\mathbf{r} = \left[3(4t)\mathbf{i} - 5(3t)\mathbf{j} + 7(t^2)\mathbf{k}\right] \cdot (4\mathbf{i} + 3\mathbf{j} + 2t\mathbf{k})\,dt$$

$$\mathbf{F} \cdot d\mathbf{r} = \left[48t - 45t + 14t^3\right]dt$$

$$\text{Work} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \left[48t - 45t + 14t^3\right]dt = 5$$

Line Integral and Mass of Thin Wire:

Suppose a thin wire is in the shape of the path $C : \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} \qquad a \le t \le b$

The density of the wire at point (x, y) is $\rho(x, y)$.

Find the mass of the wire.

Solution:

Suppose we cut the wire into many small segments: $s_1, s_2, s_3, \ldots$

If the segment is small enough, then the density of each small piece is uniform.

Hence, mass of *i*th piece (s_i) = $(\text{length of } s_i)(\rho(x_i, y_i))$.

Therefore, mass of thin wire = $\sum_{i=1}^{\infty} (\text{length of } s_i)(\rho(x_i, y_i)) =$

Line Integral and Mass of Thin Wire

Note: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$; $\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$; $\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}$;

$$\|\mathbf{r}'(t)\| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Recall: Length of arc = $\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt$

Mass of Thin Wire = $\int_C \rho(x, y) \|\mathbf{r}'(t)\|\, dt$

Example 7: Suppose a thin wire is in the shape of

the path $C : \mathbf{r}(t) = 5t\mathbf{i} + t^2\mathbf{j} = x(t)\mathbf{i} + y(t)\mathbf{j}$; $\quad 0 \le t \le 4.$

The density of the wire at point (x, y) is $\rho(x, y) = 4xy.$

Find the mass of the wire.

Solution :

$\mathbf{r}'(t) = 5\mathbf{i} + 2t\mathbf{j}$

$\|\mathbf{r}'(t)\| = \sqrt{(5)^2 + (2t)^2} = \sqrt{25 + 4t^2}$

$\rho(x, y) = 4xy = 4x(t) \cdot y(t) = 4(5t)(t^2) = 20t^3$

Mass = $\int_C \rho(x, y)\|\mathbf{r}'(t)\|\,dt = \int_0^4 20t^3\sqrt{25 + 4t^2}\,dt = 10456.38839$

Example 8: Suppose a thin wire is in the shape of a helix.

$C:\mathbf{r}(t)=4\cos t\mathbf{i}+4\sin t\mathbf{j}+t\mathbf{k}=x(t)\mathbf{i}+y(t)\mathbf{j}+z(t)\mathbf{k}\ ;\quad 0\le t\le 2\pi.$

The density of the wire at point (x,y) is $\rho(x,y)=4x^2y^2z.$

Find the mass of the wire.

Solution :

$\mathbf{r}'(t)=-4\sin t\mathbf{i}+4\cos t\mathbf{j}+\mathbf{k}$

$\|\mathbf{r}'(t)\|=\sqrt{(-4\sin t)^2+(4\cos t)^2+(1)^2}=\sqrt{16\sin^2 t+16\cos^2 t+1}=\sqrt{16(\sin^2 t+\cos^2 t)+1}=\sqrt{17}$

$\rho(x,y)=4x^2y^2z=4x(t)\cdot y(t)\cdot z(t)=4(4\cos t)^2(4\sin t)^2\cdot t$

$\text{Mass}=\int_C \rho(x,y)\|\mathbf{r}'(t)\|dt=\int_0^{2\pi}4(4\cos t)^2(4\sin t)^2\cdot t\sqrt{17}dt=3315.998194152087$