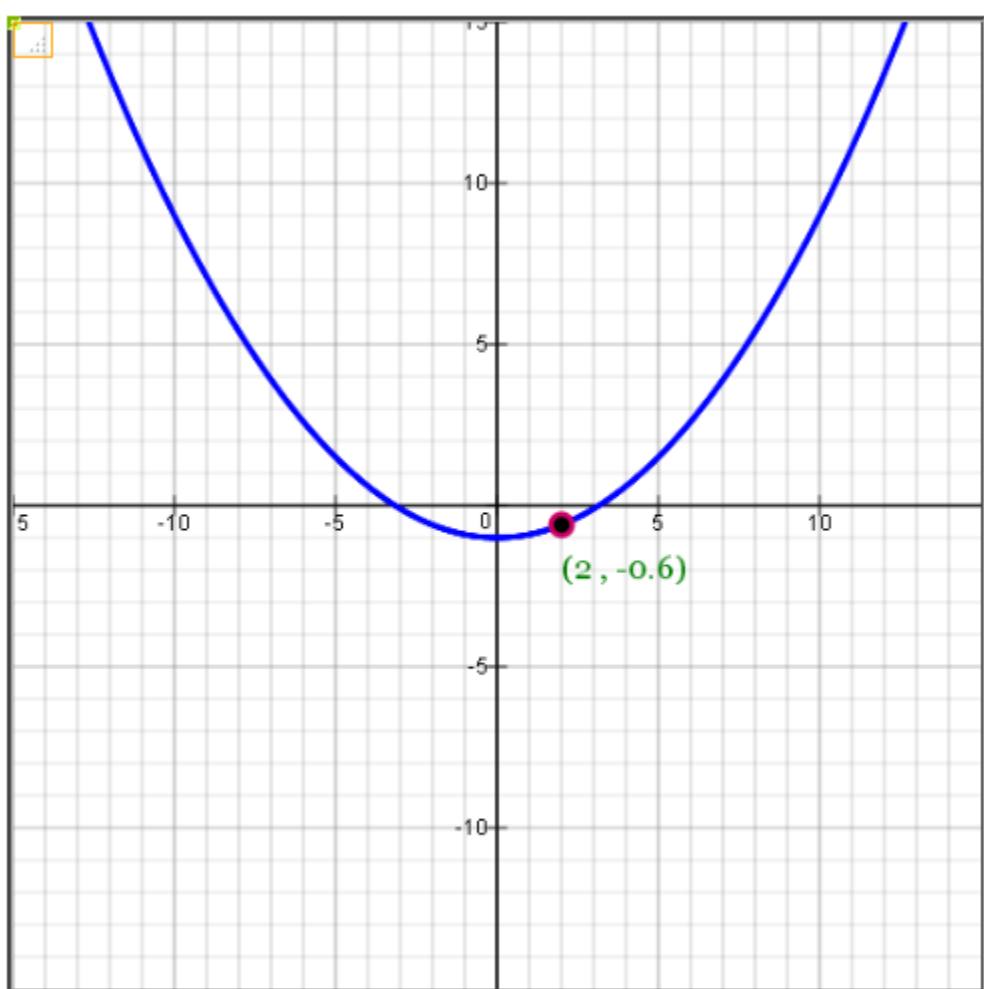


Finding Limits Graphically and Numerically

Example 1: Let $f(x) = 0.1x^2 - 1$



Let x approach 2 from the left:

$$x = 0 ; y = 0.1(0)^2 - 1 = -1$$

$$x = 0.3 ; y = 0.1(0.3)^2 - 1 = -0.991$$

$$x = 0.5 ; y = 0.1(0.5)^2 - 1 = -0.975$$

$$x = 1 ; y = -0.9$$

$$x = 1.2 ; y = -0.856$$

$$x = 1.4 ; y = -0.804$$

$$x = 1.5 ; y = -0.775$$

$$x = 1.6 ; y = -0.744$$

$$x = 1.7 ; y = -0.711000000000000001$$

$$x = 1.8 ; y = -0.675999999999999999$$

$$x = 1.9 ; y = -0.639$$

$$x = 1.99 ; y = -0.60399$$

$$x = 1.999 ; y = -0.6003999$$

$$x = 1.9999 ; y = -0.600039999$$

$$x = 1.99999 ; y = -0.60000399999$$

$$x = 1.999999 ; y = -0.6000003999999$$

$$x = 1.9999999 ; y = -0.600000039999999$$

$$x = 1.99999999 ; y = -0.600000004$$

$$x = 1.999999999 ; y = -0.6000000004$$

$$x = 1.9999999999 ; y = -0.60000000004$$

$$x = 1.99999999999 ; y = -0.600000000004$$

$$x = 1.999999999999 ; y = -0.6000000000004$$

$$x = 1.9999999999999 ; y = -0.60000000000004$$

$$x = 1.99999999999999 ; y = -0.600000000000004$$

As x approaches 2 from the left, $y = f(x) = 0.1x^2 - 1$ approaches -0.6 .

Notation: $\lim_{x \rightarrow 2^-} f(x) = -0.6$

Let x approach 2 from the right:

$$x = 4 ; y = 0.600000000000000001$$

$$x = 3.7 ; y = 0.369000000000000002$$

$$x = 3.5 ; y = 0.225000000000000001$$

$$x = 3 ; y = -0.099999999999999998$$

$$x = 2.8 ; y = -0.2160000000000000008$$

$$x = 2.6 ; y = -0.323999999999999984$$

$$x = 2.5 ; y = -0.375$$

$$x = 2.4 ; y = -0.4240000000000000004$$

$$x = 2.3 ; y = -0.4710000000000000001$$

$$x = 2.2 ; y = -0.515999999999999999$$

$$x = 2.1 ; y = -0.558999999999999999$$

$$x = 2.01 ; y = -0.59599$$

$$x = 2.001 ; y = -0.599599900000000001$$

$$x = 2.0001 ; y = -0.599959999$$

$$x = 2.00001 ; y = -0.59999599999$$

$$x = 2.000001 ; y = -0.5999995999998999$$

$$x = 2.0000001 ; y = -0.5999999599999999$$

$$x = 2.00000001 ; y = -0.599999996$$

$$x = 2.000000001 ; y = -0.5999999996$$

$$x = 2.0000000001 ; y = -0.59999999996$$

$$x = 2.00000000001 ; y = -0.599999999996$$

$$x = 2.000000000001 ; y = -0.5999999999996$$

$$x = 2.0000000000001 ; y = -0.59999999999996$$

$$x = 2.00000000000001 ; y = -0.5999999999999959$$

As x approaches 2 from the right, $y = f(x) = 0.1x^2 - 1$ approaches -0.6 .

Notation: $\lim_{x \rightarrow 2^+} f(x) = -0.6$

Summary:

As x approaches 2 from either side, $f(x)$ approaches -0.6 .

Since $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^-} f(x) = -0.6$, we say that:

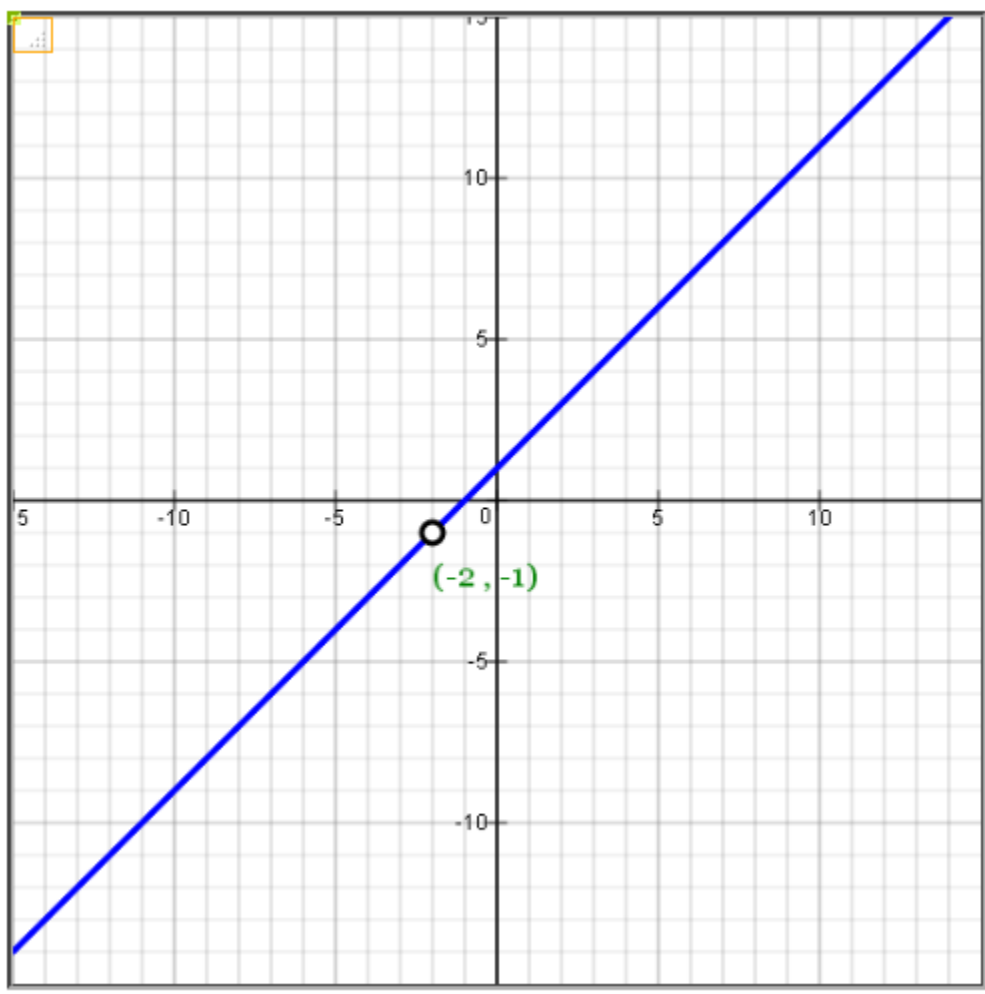
$$\lim_{x \rightarrow 2} f(x) = -0.6$$

Example 2:

$$\text{Let } f(x) = \frac{x^2 + 3x + 2}{x + 2}$$

$$\text{Note: } \frac{x^2 + 3x + 2}{x + 2} = \frac{(x + 2)(x + 1)}{(x + 2)} = x + 1$$

$$\text{Also, when } x = -2, f(x) = \frac{x^2 + 3x + 2}{x + 2} = \frac{(-2)^2 + 3(-2) + 2}{(-2) + 2} = \frac{0}{0} = \text{undefined}$$



Let x approach -2 from the left:

$$x = -4 ; y = -3$$

$$x = -3.7 ; y = -2.6999999999999997$$

$$x = -3.5 ; y = -2.5$$

$$x = -3 ; y = -2$$

$$x = -2.8 ; y = -1.8000000000000001$$

$$x = -2.6 ; y = -1.5999999999999996$$

$$x = -2.5 ; y = -1.5$$

$$x = -2.4 ; y = -1.40000000000000015$$

$$x = -2.3 ; y = -1.2999999999999996$$

$$x = -2.2 ; y = -1.2$$

$$x = -2.1 ; y = -1.09999999999999934$$

$$x = -2.01 ; y = -1.00999999999999977$$

$$x = -2.001 ; y = -1.0009999999999996957$$

$$x = -2.0001 ; y = -1.00009999999999949511$$

$$x = -2.00001 ; y = -1.000009999999999564185$$

$$x = -2.000001 ; y = -1.00000100000889004$$

$$x = -2.0000001 ; y = -1.0000000932587343$$

$$x = -2.00000001 ; y = -1$$

$$x = -2.000000001 ; y = -1$$

$$x = -2.0000000001 ; y = -1$$

$$x = -2.00000000001 ; y = -1$$

$$x = -2.000000000001 ; y = -1$$

As x approaches -2 from the left, $y = f(x) = \frac{x^2 + 3x + 2}{x + 2}$ approaches -1 .

Notation: $\lim_{x \rightarrow -2^-} f(x) = -1$

Let x approach -2 from the right:

$$x = 0 ; y = 1$$

$$x = -0.300000000000000004 ; y = 0.7$$

$$x = -0.5 ; y = 0.5$$

$$x = -1 ; y = 0$$

$$x = -1.2 ; y = -0.199999999999999962$$

$$x = -1.4 ; y = -0.399999999999999988$$

$$x = -1.5 ; y = -0.5$$

$$x = -1.6 ; y = -0.600000000000000006$$

$$x = -1.7 ; y = -0.699999999999999997$$

$$x = -1.8 ; y = -0.800000000000000009$$

$$x = -1.9 ; y = -0.899999999999999934$$

$$x = -1.99 ; y = -0.9899999999999999567$$

$$x = -1.999 ; y = -0.9989999999999999416$$

$$x = -1.9999 ; y = -0.999899999999999983873$$

$$x = -1.99999 ; y = -0.9999899999999999547638$$

$$x = -1.999999 ; y = -0.9999990001331439$$

$$x = -1.9999999 ; y = -0.9999999000799279$$

$$x = -1.99999999 ; y = -1$$

$$x = -1.999999999 ; y = -1$$

$$x = -1.9999999999 ; y = -1$$

$$x = -1.99999999999 ; y = -1$$

$$x = -1.999999999999 ; y = -1$$

As x approaches -2 from the right, $y = f(x) = \frac{x^2 + 3x + 2}{x + 2}$ approaches -1 .

Notation: $\lim_{x \rightarrow -2^+} f(x) = -1$

Summary:

As x approaches 2 from either side, $f(x)$ approaches -1 .

Since $\lim_{x \rightarrow -2^+} f(x) = \lim_{x \rightarrow -2^-} f(x) = -1$, we say that:

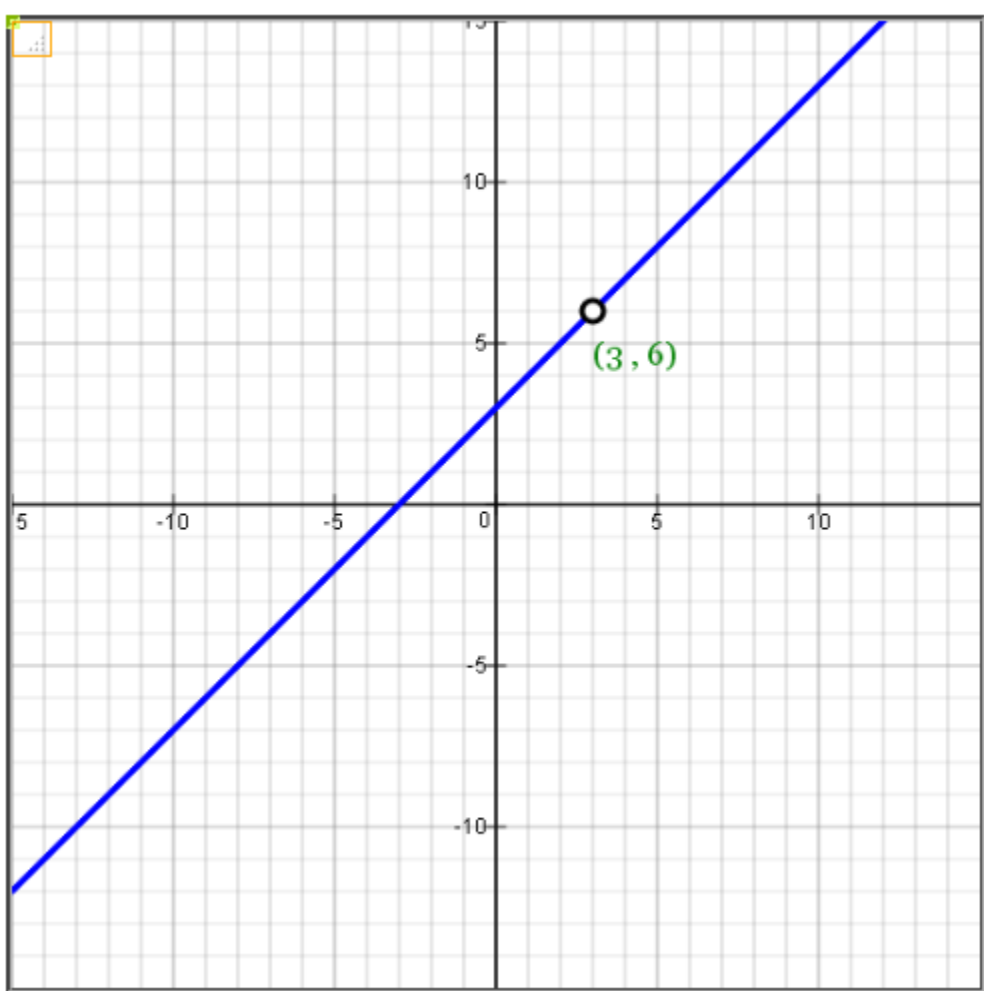
$$\lim_{x \rightarrow -2} f(x) = -1$$

Example 3:

$$\text{Let } f(x) = \frac{x^2 - 9}{x - 3}$$

$$\text{Note: } \frac{x^2 - 9}{x - 3} = \frac{(x + 3)(x - 3)}{(x - 3)} = x + 3$$

$$\text{Also, when } x = 3, f(x) = \frac{x^2 - 9}{x - 3} = \frac{(3)^2 - 9}{(3) - 3} = \frac{0}{0} = \text{undefined}$$



Let x approach 3 from the left:

$$x = 1 ; y = 4$$

$$x = 1.3 ; y = 4.3$$

$$x = 1.5 ; y = 4.5$$

$$x = 2 ; y = 5$$

$$x = 2.2 ; y = 5.2$$

$$x = 2.4 ; y = 5.4$$

$$x = 2.5 ; y = 5.5$$

$$x = 2.6 ; y = 5.6$$

$$x = 2.7 ; y = 5.7$$

$$x = 2.8 ; y = 5.8$$

$$x = 2.9 ; y = 5.8999999999999993$$

$$x = 2.99 ; y = 5.9900000000000023$$

$$x = 2.999 ; y = 5.998999999999986$$

$$x = 2.9999 ; y = 5.999900000000060$$

$$x = 2.99999 ; y = 5.99999000008799$$

$$x = 2.999999 ; y = 5.999998999911099$$

$$x = 2.9999999 ; y = 5.999999902300374$$

$$x = 2.99999999 ; y = 6$$

$$x = 2.999999999 ; y = 6$$

$$x = 2.9999999999 ; y = 6$$

$$x = 2.99999999999 ; y = 6$$

$$x = 2.999999999999 ; y = 6$$

As x approaches 3 from the left, $y = f(x) = \frac{x^2 - 9}{x - 3}$ approaches 6.

Notation: $\lim_{x \rightarrow 3^-} f(x) = 6$

Let x approach 3 from the right:

$$x = 5 ; y = 8$$

$$x = 4.7 ; y = 7.7$$

$$x = 4.5 ; y = 7.5$$

$$x = 4 ; y = 7$$

$$x = 3.8 ; y = 6.8$$

$$x = 3.6 ; y = 6.6$$

$$x = 3.5 ; y = 6.5$$

$$x = 3.4 ; y = 6.3999999999999999$$

$$x = 3.3 ; y = 6.3$$

$$x = 3.2 ; y = 6.2000000000000005$$

$$x = 3.1 ; y = 6.1000000000000007$$

$$x = 3.01 ; y = 6.0099999999999977$$

$$x = 3.001 ; y = 6.001000000000014$$

$$x = 3.0001 ; y = 6.0000999999999392$$

$$x = 3.00001 ; y = 6.0000099999991201$$

$$x = 3.000001 ; y = 6.0000010000088901$$

$$x = 3.0000001 ; y = 6.000000097699626$$

$$x = 3.00000001 ; y = 6$$

$$x = 3.000000001 ; y = 6$$

$$x = 3.0000000001 ; y = 6$$

$$x = 3.00000000001 ; y = 6$$

$$x = 3.000000000001 ; y = 6$$

As x approaches 3 from the right, $y = f(x) = \frac{x^2 - 9}{x - 3}$ approaches 6.

Notation: $\lim_{x \rightarrow 3^+} f(x) = 6$

Summary:

As x approaches 3 from either side, $f(x)$ approaches 6.

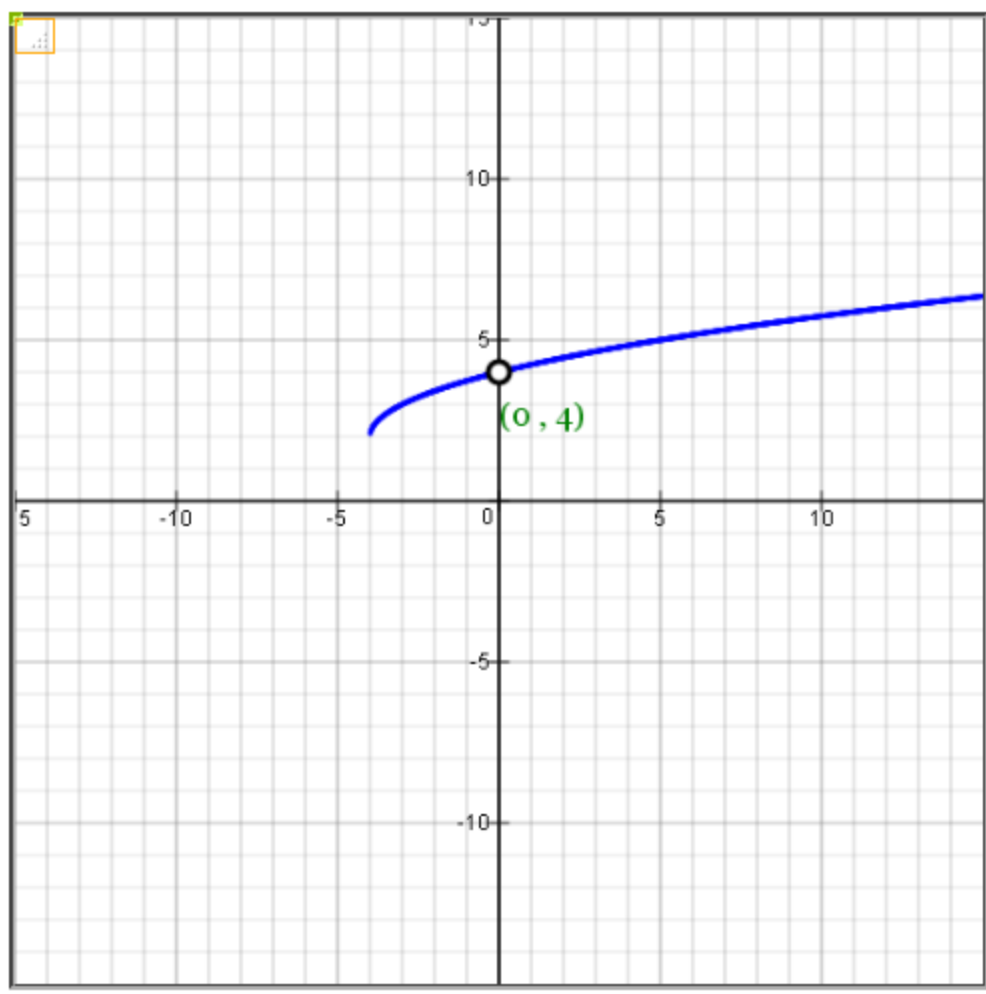
Since $\lim_{x \rightarrow 3^+} f(x) = \lim_{x \rightarrow 3^-} f(x) = 6$, we say that:

$$\lim_{x \rightarrow 3} f(x) = 6$$

Example 4:

$$\text{Let } f(x) = \frac{x}{\sqrt{x+4} - 2}$$

Note: when $x = 0$, $f(x) = \frac{x}{\sqrt{x+4} - 2} = \frac{0}{0} = \text{undefined}$



Let x approach 0 from the left:

$$x = -2 ; y = 3.414213562373096$$

$$x = -1.7 ; y = 3.5165750888103093$$

$$x = -1.5 ; y = 3.5811388300841904$$

$$x = -1 ; y = 3.732050807568876$$

$$x = -0.8 ; y = 3.788854381999831$$

$$x = -0.6 ; y = 3.8439088914585784$$

$$x = -0.5 ; y = 3.87082869338697$$

$$x = -0.4 ; y = 3.8973665961010253$$

$$x = -0.3 ; y = 3.923538406167139$$

$x = -0.2 ; y = 3.949358868961784$
 $x = -0.1 ; y = 3.9748417658131423$
 $x = -0.01 ; y = 3.9974984355438252$
 $x = -0.001 ; y = 3.999749984374003$
 $x = -0.0001 ; y = 3.9999749998259992$
 $x = -0.00001 ; y = 3.999997500108378$
 $x = -0.000001 ; y = 3.999999750750946$
 $x = -0.000001 ; y = 3.999999750750946$
 $x = -0.0000001 ; y = 3.999999988782747$
 $x = -0.00000001 ; y = 4.000000024309884$
 $x = -0.000000001 ; y = 3.9999996690385435$
 $x = -0.0000000001 ; y = 3.9999996690385435$
 $x = -0.00000000001 ; y = 3.999999669038543$

As x approaches 0 from the left, $y = f(x) = \frac{x}{\sqrt{x+4} - 2}$ approaches 4.

Notation: $\lim_{x \rightarrow 0^-} f(x) = 4$

Let x approach 0 from the right:

$$x = 2 ; y = 4.4494897427831805$$

$$x = 1.7 ; y = 4.387467277262664$$

$$x = 1.5 ; y = 4.3452078799117135$$

$$x = 1 ; y = 4.236067977499788$$

$$x = 0.8 ; y = 4.190890230020668$$

$$x = 0.6 ; y = 4.144761058952719$$

$$x = 0.5 ; y = 4.121320343559649$$

$$x = 0.4 ; y = 4.097617696340296$$

$$x = 0.3 ; y = 4.073644135332781$$

$$x = 0.2 ; y = 4.049390153191905$$

$$x = 0.1 ; y = 4.024845673131703$$

$$x = 0.01 ; y = 4.002498439449939$$

$$x = 0.001 ; y = 4.000249984373987$$

$$x = 0.0001 ; y = 4.0000249998301305$$

$$x = 0.00001 ; y = 4.000002499842338$$

$$x = 0.000001 ; y = 4.000000251683575$$

$$x = 0.0000001 ; y = 4.000000024309884$$

$$x = 0.00000001 ; y = 4.0000000379581288$$

$$x = 0.000000001 ; y = 3.9999996690385435$$

$$x = 0.0000000001 ; y = 3.9999996690385435$$

$$x = 0.00000000001 ; y = 4.000354971904864$$

As x approaches 0 from the right, $y = f(x) = \frac{x}{\sqrt{x+4} - 2}$ approaches 4.

Notation: $\lim_{x \rightarrow 0^+} f(x) = 4$

Summary:

As x approaches 0 from either side, $f(x)$ approaches 4.

Since $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = 4$, we say that:

$$\lim_{x \rightarrow 0} f(x) = 4$$

Note:

Multiply expression by conjugate of denominator and simplify:

$$\frac{x}{\sqrt{x+4}-2} = \frac{x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

Recall:

$$(\sqrt{2})^2 = (\sqrt{2})(\sqrt{2}) = \sqrt{4} = 2$$

$$(\sqrt{5})^2 = (\sqrt{5})(\sqrt{5}) = \sqrt{25} = 5$$

$$(\sqrt{x})^2 = x$$

$$(\sqrt{x+5})^2 = x+5$$

$$(\sqrt{a+b})^2 = a+b$$

$$(a-b)(a+b) = a^2 - b^2$$

$$(\sqrt{a}-b)(\sqrt{a}+b) = (\sqrt{a})^2 - b^2 = a - b^2$$

$$(\sqrt{x+4}-2)(\sqrt{x+4}+2) = (\sqrt{x+4})^2 - 2^2 = x+4-4 = x$$

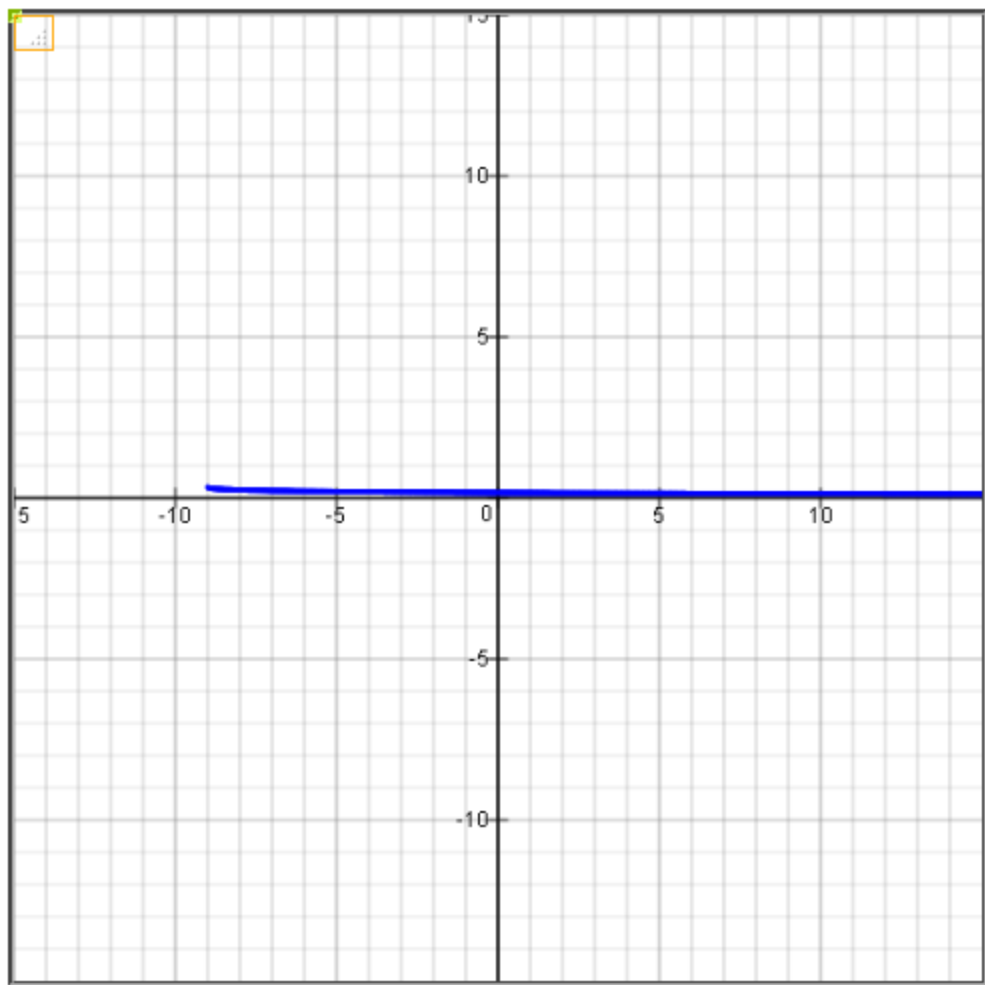
Hence,

$$\frac{x}{\sqrt{x+4}-2} = \frac{x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{(x)(\sqrt{x+4}+2)}{(x)} = \sqrt{x+4}+2$$

$$\lim_{x \rightarrow 0} \frac{x}{\sqrt{x+4}-2} = \lim_{x \rightarrow 0} (\sqrt{x+4}+2) = \sqrt{0+4}+2 = 2+2 = 4$$

Example 5: Let $f(x) = \frac{\sqrt{x+9} - 3}{x}$

Note: when $x = 0$, $f(x) = \frac{\sqrt{x+9} - 3}{x} = \frac{0}{0} = \text{undefined}$



As x approaches 0 from the left, $y = f(x) = \frac{\sqrt{x+9} - 3}{x}$ approaches 4.

Notation: $\lim_{x \rightarrow 0^-} f(x) = 4$

Let x approach 0 from the left:

$x = -2 ; y = 0.17712434446770464$
 $x = -1.7 ; y = 0.17538163692867106$
 $x = -1.5 ; y = 0.17425814164944628$
 $x = -1 ; y = 0.1715728752538097$
 $x = -0.8 ; y = 0.17054473418091187$
 $x = -0.6 ; y = 0.16954108460352021$
 $x = -0.5 ; y = 0.16904810515469926$
 $x = -0.4 ; y = 0.16856085069240967$
 $x = -0.3 ; y = 0.1680791974982506$
 $x = -0.2 ; y = 0.1676030258086736$
 $x = -0.1 ; y = 0.16713221964740566$
 $x = -0.01 ; y = 0.16671298870098994$
 $x = -0.001 ; y = 0.1666712965535666$
 $x = -0.0001 ; y = 0.1666671296307598$
 $x = -0.00001 ; y = 0.16666671296405866$
 $x = -0.000001 ; y = 0.1666666711308551$
 $x = -0.000001 ; y = 0.1666666711308551$
 $x = -0.0000001 ; y = 0.1666666671340522$
 $x = -0.00000001 ; y = 0.1666666804567285$
 $x = -0.000000001 ; y = 0.1666666804567285$
 $x = -0.0000000001 ; y = 0.1666666804567285$
 $x = -0.00000000001 ; y = 0.1666666804567285$

As x approaches 0 from the left, $y = f(x) = \frac{\sqrt{x+9} - 3}{x}$ approaches 0.16666666.

Notation: $\lim_{x \rightarrow 0^-} f(x) = 0.16666666$

Let x approach 0 from the right:

$$x = 2 ; y = 0.1583123951776999$$

$$x = 1.7 ; y = 0.15946202750542646$$

$$x = 1.5 ; y = 0.1602468994692868$$

$$x = 1 ; y = 0.16227766016837952$$

$$x = 0.8 ; y = 0.16311896062463194$$

$$x = 0.6 ; y = 0.1639777949432227$$

$$x = 0.5 ; y = 0.16441400296897601$$

$$x = 0.4 ; y = 0.16485485837794633$$

$$x = 0.3 ; y = 0.16530045465127152$$

$$x = 0.2 ; y = 0.1657508881031$$

$$x = 0.1 ; y = 0.16620625799671274$$

$$x = 0.01 ; y = 0.16662039607266976$$

$$x = 0.001 ; y = 0.16666203729398532$$

$$x = 0.0001 ; y = 0.16666620370475727$$

$$x = 0.00001 ; y = 0.1666666203714584$$

$$x = 0.000001 ; y = 0.1666666618049817$$

$$x = 0.0000001 ; y = 0.1666666671340522$$

$$x = 0.00000001 ; y = 0.1666666804567285$$

$$x = 0.000000001 ; y = 0.1666666804567285$$

$$x = 0.0000000001 ; y = 0.1666666804567285$$

$$x = 0.00000000001 ; y = 0.1666666804567285$$

As x approaches 0 from the right, $y = f(x) = \frac{\sqrt{x+9}-3}{x}$ approaches 0.16666666.

Notation: $\lim_{x \rightarrow 0^+} f(x) = 0.16666666$

Summary:

As x approaches 0 from either side, $f(x)$ approaches $0.16666666 = \frac{1}{6}$.

Since $\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = \frac{1}{6}$, we say that:

$$\lim_{x \rightarrow 0} f(x) = \frac{1}{6}$$

Note:

Multiply expression by conjugate of denominator and simplify:

$$\frac{\sqrt{x+9}-3}{x} = \frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$$

Recall:

$$(\sqrt{2})^2 = (\sqrt{2})(\sqrt{2}) = \sqrt{4} = 2$$

$$(\sqrt{5})^2 = (\sqrt{5})(\sqrt{5}) = \sqrt{25} = 5$$

$$(\sqrt{x})^2 = x$$

$$(\sqrt{x+5})^2 = x+5$$

$$(\sqrt{a+b})^2 = a+b$$

$$(a-b)(a+b) = a^2 - b^2$$

$$(\sqrt{a}-b)(\sqrt{a}+b) = (\sqrt{a})^2 - b^2 = a - b^2$$

$$(\sqrt{x+9}-3)(\sqrt{x+9}+3) = (\sqrt{x+9})^2 - 3^2 = x+9-9 = x$$

Hence,

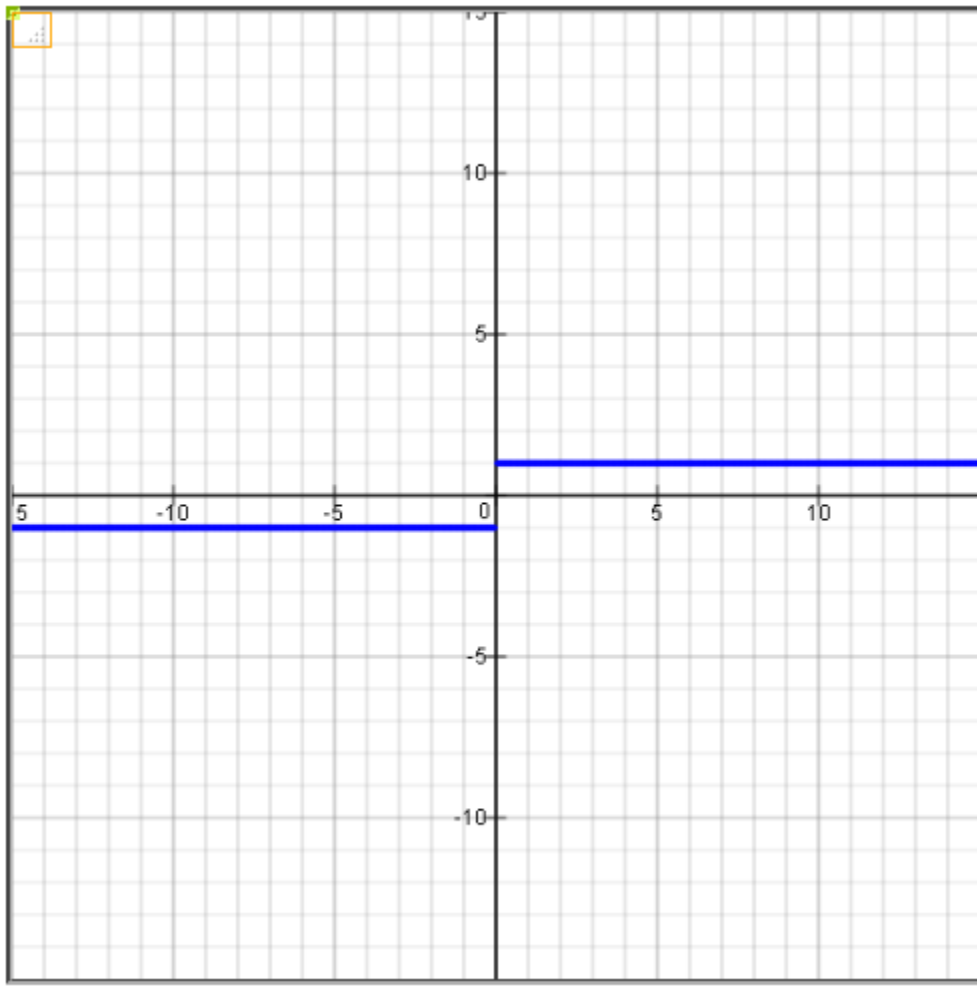
$$\frac{\sqrt{x+9}-3}{x} = \frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} = \frac{(x)}{(x)(\sqrt{x+9}+3)} = \frac{1}{\sqrt{x+9}+3}$$

$$\lim_{x \rightarrow 0} \frac{\sqrt{x+9}-3}{x} = \lim_{x \rightarrow 0} \left(\frac{1}{\sqrt{x+9}+3} \right) = \frac{1}{\sqrt{0+9}+3} = \frac{1}{6}$$

Limits That Fail To Exist

Example 6: Let $f(x) = \frac{|x|}{x}$

Note: When $x = 0$, $f(x) = \frac{|x|}{x} = \frac{0}{0} = \textit{undefined}$



Let x approach 0 from the left:

$$x = -2 ; y = |-2|/(-2) = -1$$

$$x = -1.7 ; y = |-1.7|/(-1.7) = -1$$

$$x = -1.5 ; y = |-1.5|/(-1.5) = -1$$

$$x = -1 ; y = |-1|/(-1) = -1$$

$$x = -0.8 ; y = -1$$

$$x = -0.6 ; y = -1$$

$$x = -0.5 ; y = -1$$

$$x = -0.4 ; y = -1$$

$$x = -0.3 ; y = -1$$

$$x = -0.2 ; y = -1$$

$$x = -0.1 ; y = -1$$

$$x = -0.01 ; y = -1$$

$$x = -0.001 ; y = -1$$

$$x = -0.0001 ; y = -1$$

$$x = -0.00001 ; y = -1$$

$$x = -0.000001 ; y = -1$$

$$x = -0.000001 ; y = -1$$

$$x = -0.0000001 ; y = -1$$

$$x = -0.00000001 ; y = -1$$

$$x = -0.000000001 ; y = -1$$

$$x = -0.0000000001 ; y = -1$$

$$x = -0.00000000001 ; y = -1$$

$$x = -0.000000000001 ; y = -1$$

$$x = -0.0000000000001 ; y = -1$$

As x approaches 0 from the left, $y = f(x) = \frac{|x|}{x}$ approaches -1.

Notation: $\lim_{x \rightarrow 0^-} f(x) = -1$

Let x approach 0 from the right:

$$x = 2 ; y = 1$$

$$x = 1.7 ; y = 1$$

$$x = 1.5 ; y = 1$$

$$x = 1 ; y = 1$$

$$x = 0.8 ; y = 1$$

$$x = 0.6 ; y = 1$$

$$x = 0.5 ; y = 1$$

$$x = 0.4 ; y = 1$$

$$x = 0.3 ; y = 1$$

$$x = 0.2 ; y = 1$$

$$x = 0.1 ; y = 1$$

$$x = 0.01 ; y = 1$$

$$x = 0.001 ; y = 1$$

$$x = 0.0001 ; y = 1$$

$$x = 0.00001 ; y = 1$$

$$x = 0.000001 ; y = 1$$

$$x = 0.0000001 ; y = 1$$

$$x = 0.00000001 ; y = 1$$

$$x = 0.000000001 ; y = 1$$

$$x = 0.0000000001 ; y = 1$$

$$x = 0.00000000001 ; y = 1$$

$$x = 0.000000000001 ; y = 1$$

$$x = 0.0000000000001 ; y = 1$$

$$x = 0.00000000000001 ; y = 1$$

As x approaches 0 from the right, $y = f(x) = \frac{|x|}{x}$ approaches 1.

Notation: $\lim_{x \rightarrow 0^+} f(x) = 1$

Summary:

Since $\lim_{x \rightarrow 0^+} f(x) = 1$ and $\lim_{x \rightarrow 0^-} f(x) = -1$, we say that:

$\lim_{x \rightarrow 0} f(x)$ does not exist (DNE) because limit from the left and limit from the right are not equal.

Note;

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{Definition of Absolute Value}$$

$$\frac{|x|}{x} = \begin{cases} x/x = 1 & \text{if } x > 0 \\ -x/x = -1 & \text{if } x < 0 \end{cases} \quad \text{Note: When } x = 0, \frac{|x|}{x} = \frac{0}{0} = \text{undefined}$$

$$\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -x/x = -1 \quad \text{Note: } x \text{ approaches } 0 \text{ from the left; so } x < 0$$

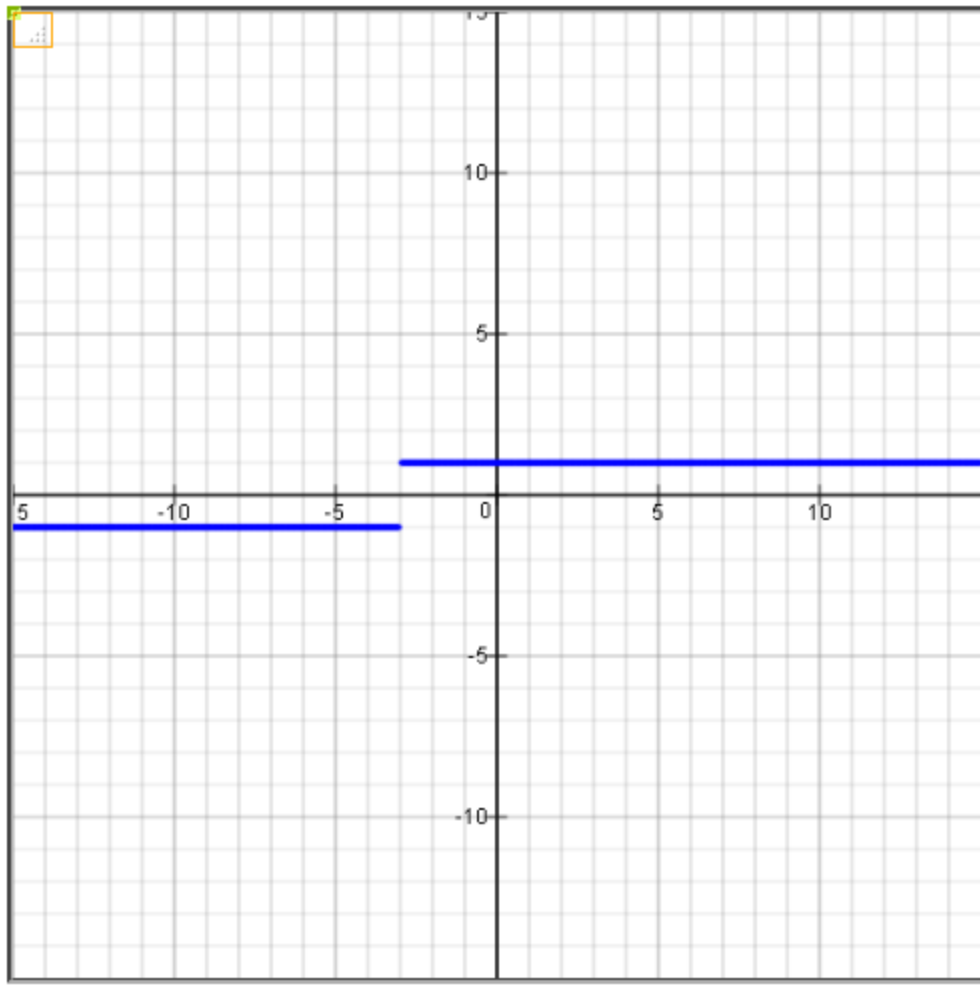
$$\lim_{x \rightarrow 0^+} \frac{|x|}{x} = x/x = 1 \quad \text{Note: } x \text{ approaches } 0 \text{ from the right; so } x > 0$$

Hence, limit from the left and limit from the right are not equal.

Therefore, as x approaches 0, $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist.

$$\text{Example 7: Let } f(x) = \frac{|x+3|}{x+3}$$

$$\text{Note: When } x = -3, f(x) = \frac{|x+3|}{x+3} = \frac{0}{0} = \text{undefined}$$



Let x approach -3 from the left:

$$x = -5 ; y = -1$$

$$x = -4.7 ; y = -1$$

$$x = -4.5 ; y = -1$$

$$x = -4 ; y = -1$$

$$x = -3.8 ; y = -1$$

$$x = -3.6 ; y = -1$$

$$x = -3.5 ; y = -1$$

$$x = -3.4 ; y = -1$$

$$x = -3.3 ; y = -1$$

$$x = -3.2 ; y = -1$$

$$x = -3.1 ; y = -1$$

$$x = -3.01 ; y = -1$$

$x = -3.001 ; y = -1$
 $x = -3.0001 ; y = -1$
 $x = -3.00001 ; y = -1$
 $x = -3.000001 ; y = -1$
 $x = -3.0000001 ; y = -1$
 $x = -3.00000001 ; y = -1$
 $x = -3.000000001 ; y = -1$
 $x = -3.0000000001 ; y = -1$
 $x = -3.00000000001 ; y = -1$
 $x = -3.000000000001 ; y = -1$
 $x = -3.0000000000001 ; y = -1$

As x approaches -3 from the left $f(x) = \frac{|x+3|}{x+3}$ approaches -1 .

Notation: $\lim_{x \rightarrow -3^-} f(x) = -1$

Let x approach -3 from the right:

$x = -1 ; y = 1$
 $x = -1.3 ; y = 1$
 $x = -1.5 ; y = 1$
 $x = -2 ; y = 1$
 $x = -2.2 ; y = 1$
 $x = -2.4 ; y = 1$

$$x = -2.5 ; y = 1$$

$$x = -2.6 ; y = 1$$

$$x = -2.7 ; y = 1$$

$$x = -2.8 ; y = 1$$

$$x = -2.9 ; y = 1$$

$$x = -2.99 ; y = 1$$

$$x = -2.999 ; y = 1$$

$$x = -2.9999 ; y = 1$$

$$x = -2.99999 ; y = 1$$

$$x = -2.999999 ; y = 1$$

$$x = -2.9999999 ; y = 1$$

$$x = -2.99999999 ; y = 1$$

$$x = -2.999999999 ; y = 1$$

$$x = -2.9999999999 ; y = 1$$

$$x = -2.99999999999 ; y = 1$$

$$x = -2.999999999999 ; y = 1$$

$$x = -2.9999999999999 ; y = 1$$

$$x = -2.99999999999999 ; y = 1$$

As x approaches -3 from the right $f(x) = \frac{|x+3|}{x+3}$ approaches 1 .

Notation: $\lim_{x \rightarrow -3^+} f(x) = 1$

Note;

$$|x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \text{ or } x \geq -3 \\ -(x+3) & \text{if } x+3 < 0 \text{ or } x < -3 \end{cases}$$

Definition of Absolute Value

$$\frac{|x+3|}{x+3} = \begin{cases} \frac{x+3}{x+3} = 1 & \text{if } x+3 \geq 0 \text{ or } x \geq -3 \\ \frac{-(x+3)}{(x+3)} = -1 & \text{if } x+3 < 0 \text{ or } x < -3 \end{cases}$$

Note: When $x = -3$, $\frac{|x+3|}{x+3} = \frac{0}{0} = \text{undefined}$

$$\lim_{x \rightarrow -3^-} \frac{|x+3|}{x+3} = -1 \quad \text{Note: } x \text{ approaches } -3 \text{ from the left; so } x < -3$$

$$\lim_{x \rightarrow -3^+} \frac{|x+3|}{x+3} = 1 \quad \text{Note: } x \text{ approaches } -3 \text{ from the right; so } x > -3$$

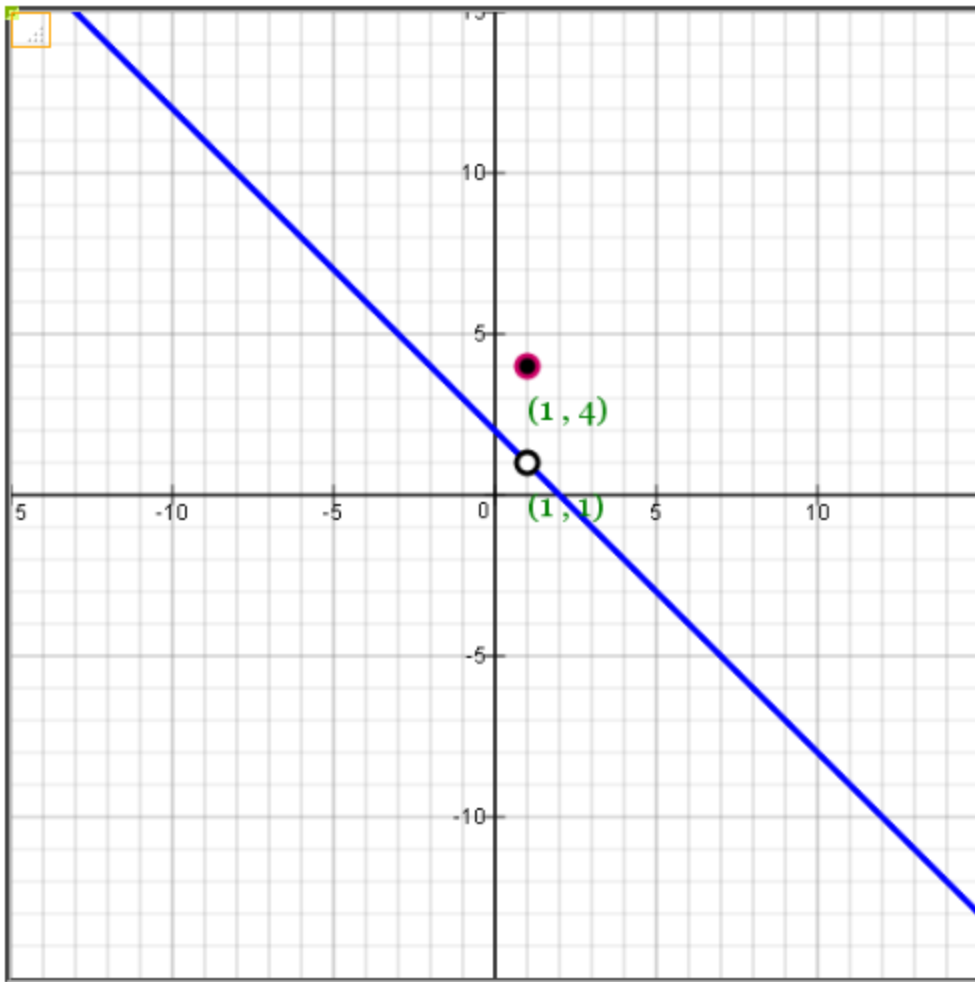
Hence, limit from the left and limit from the right are not equal.

Therefore, as x approaches -3 , $\lim_{x \rightarrow -3} \frac{|x+3|}{x+3}$ does not exist.

Example 8:

$$\text{Let } f(x) = \begin{cases} 2-x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$$

Find $\lim_{x \rightarrow 1}$.



Let x approach 1 from the left:

| x | $y = (2-x)\{x \neq 1\}$ |
|----------|-------------------------|
| -1 | 3 |
| 0 | 2 |
| 0.5 | 1.5 |
| 0.9 | 1.1 |
| 0.999 | 1.001 |
| 0.999999 | 1.0000010000000001 |

As x approaches 1 from the left $f(x)$ approaches 1.

Notation: $\lim_{x \rightarrow 1^-} f(x) = 1$

Let x approach 1 from the right:

| x | $y = (2-x)\{x \neq 1\}$ |
|-----------|-------------------------|
| 3 | -1 |
| 2 | 0 |
| 1.5 | 0.5 |
| 1.01 | 0.99 |
| 1.0001 | 0.9999 |
| 1.0000001 | 0.999999989999999999 |

As x approaches 1 from the right $f(x)$ approaches 1.

Notation: $\lim_{x \rightarrow 1^+} f(x) = 1$

Summary:

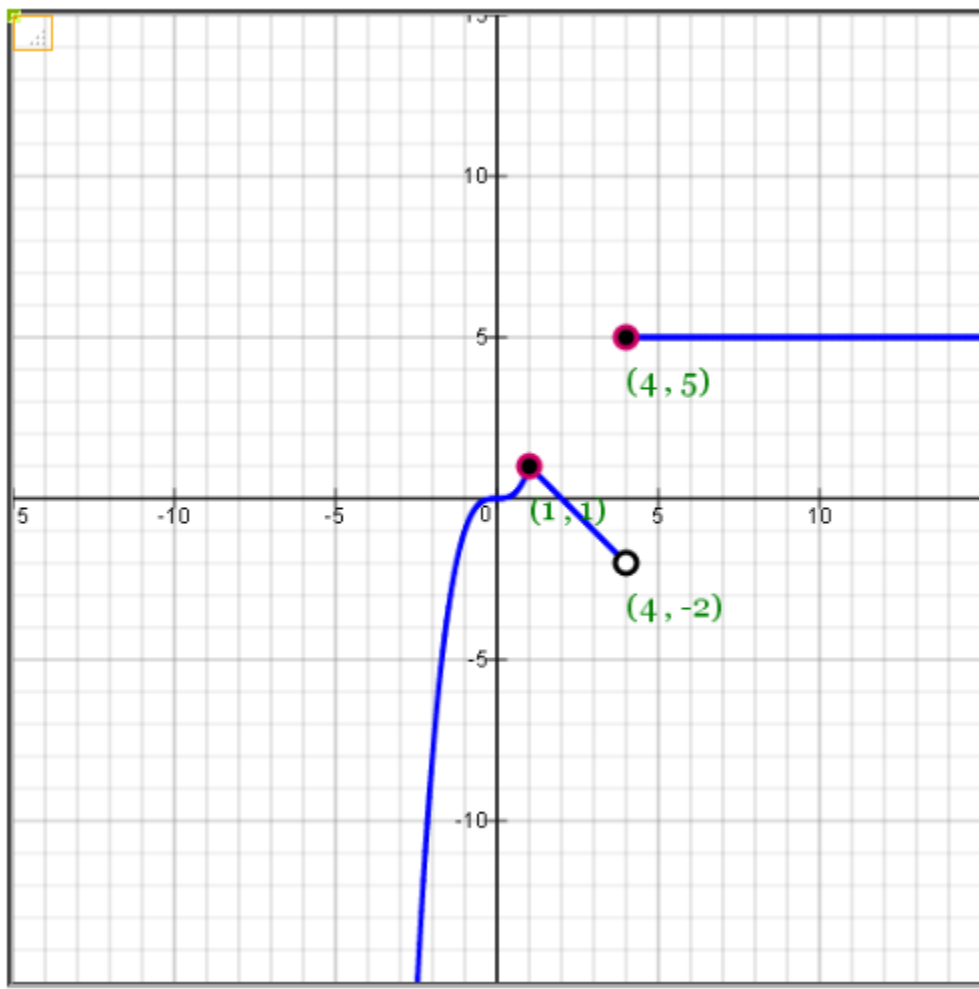
As x approaches 1 from either side, $f(x)$ approaches 1.

Since $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^-} f(x) = 1$, we say that:

$$\lim_{x \rightarrow 1} f(x) = 1$$

Example 9:

$$f(x) = \begin{cases} x^3 & \text{if } x \leq 1 \\ 2 - x & \text{if } 1 < x < 4 \\ 5 & \text{if } x \geq 4 \end{cases}$$



a) $f(4) = ?$; $f(1) = 1$

From graph, when $x = 4$, $y = 5$. Therefore, $f(4) = 5$

From graph, when $x = 1$, $y = 1$. Therefore, $f(1) = 1$

b) $\lim_{x \rightarrow 4} f(x) = ?$

From graph, as x approaches 4 from the left, $f(x)$ approaches -2.

$$\lim_{x \rightarrow 4^-} f(x) = -2$$

From graph, as x approaches 4 from the right, $f(x)$ approaches 5.

$$\lim_{x \rightarrow 4^+} f(x) = 5$$

Because limit from the left and limit from the right are not equal,

$\lim_{x \rightarrow 4} f(x)$ does not exist.

c) $\lim_{x \rightarrow 1} f(x) = ?$

From graph, as x approaches 1 from the left, $f(x)$ approaches 1.

Notation: $\lim_{x \rightarrow 1^-} f(x) = 1$

From graph, as x approaches 1 from the right, $f(x)$ approaches 1.

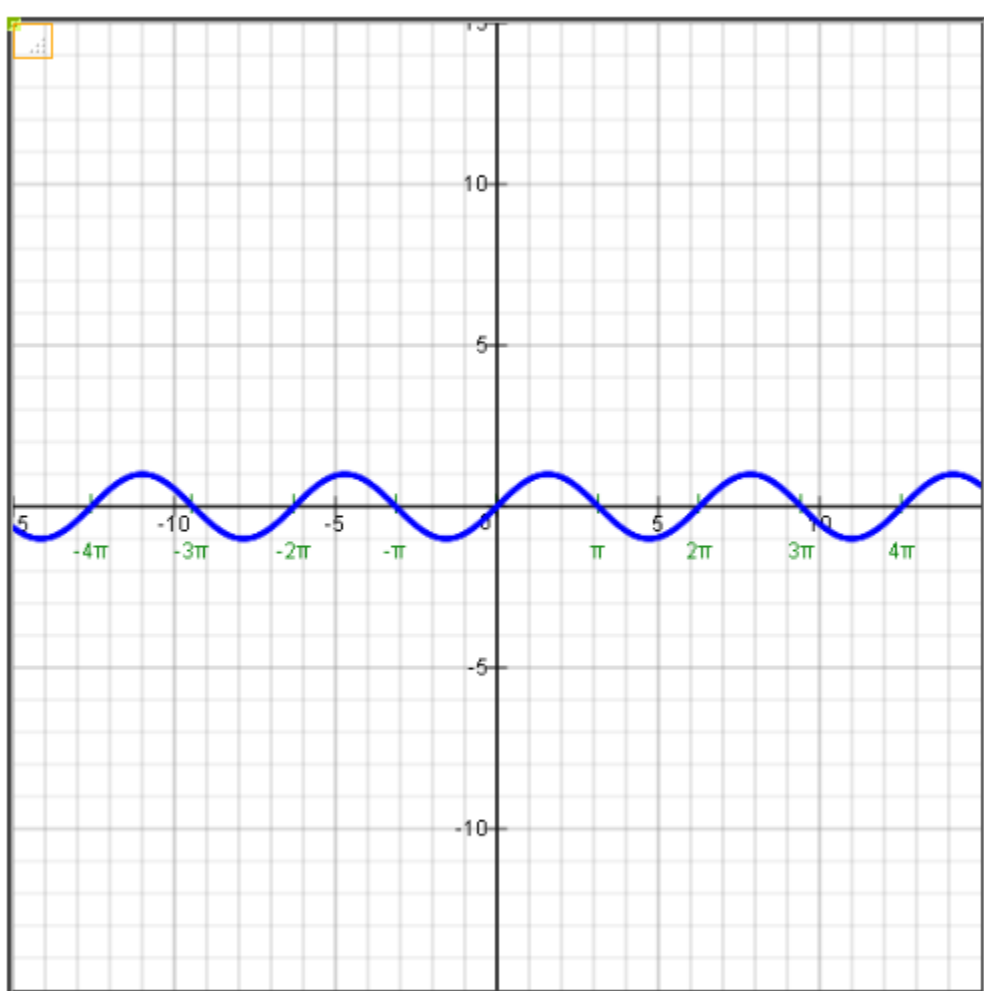
Notation: $\lim_{x \rightarrow 1^+} f(x) = 1$

Because limit from the left and limit from the right are equal,

$$\lim_{x \rightarrow 1} f(x) = 1.$$

Example 10:

Let $f(x) = \sin x$



$$a) \lim_{x \rightarrow 0} f(x) = ?$$

From graph, as x approaches 0 from the left, $f(x)$ approaches 0.

$$\text{Notation: } \lim_{x \rightarrow 0^-} f(x) = 0$$

From graph, as x approaches 0 from the right, $f(x)$ approaches 0.

$$\text{Notation: } \lim_{x \rightarrow 0^+} f(x) = 0$$

Because limit from the left and limit from the right are equal,

$$\lim_{x \rightarrow 0} f(x) = 0.$$

$$b) \lim_{x \rightarrow \pi} f(x) = ?$$

From graph, as x approaches π from the left, $f(x)$ approaches 0.

$$\text{Notation: } \lim_{x \rightarrow \pi^-} f(x) = 0$$

From graph, as x approaches π from the right, $f(x)$ approaches 0.

$$\text{Notation: } \lim_{x \rightarrow \pi^+} f(x) = 0$$

Because limit from the left and limit from the right are equal,

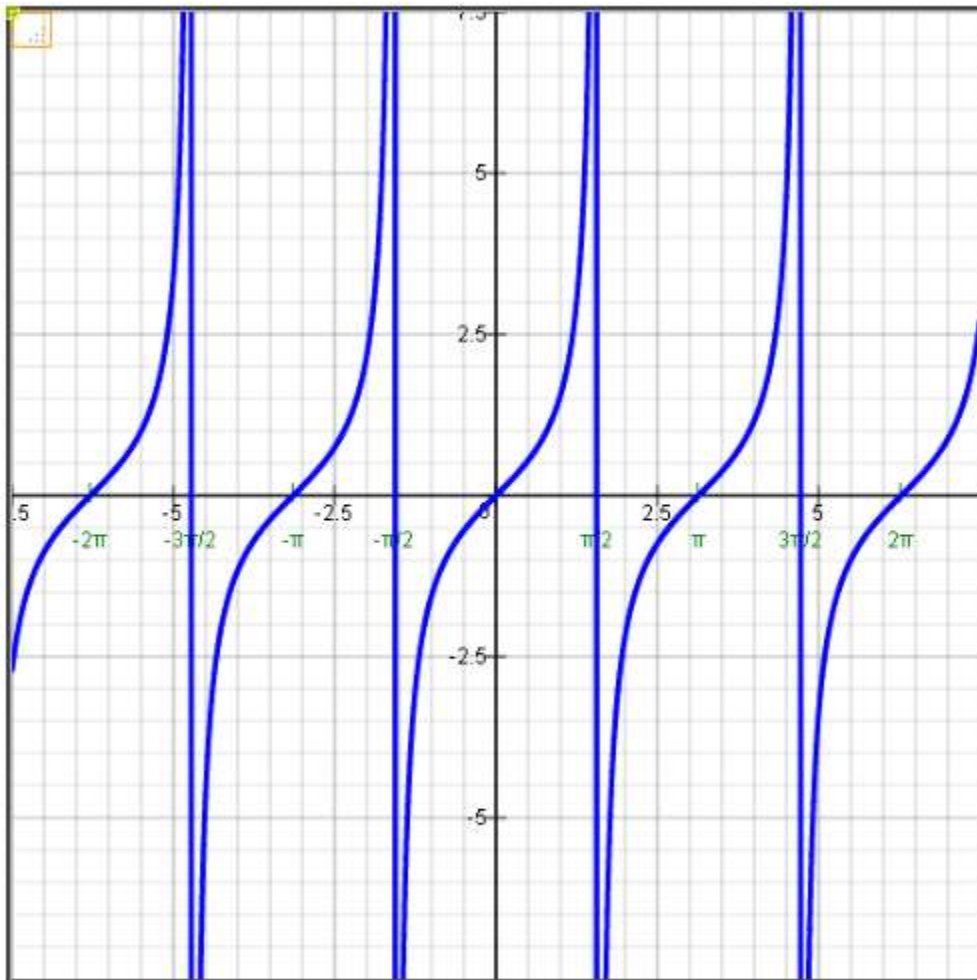
$$\lim_{x \rightarrow \pi} f(x) = 0.$$

Example 11:

Let $f(x) = \tan x$

Note: $\tan x = \frac{\sin x}{\cos x}$;

and $\cos x = 0$ when $x = \frac{-7\pi}{2}, \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$



Note: $\tan x = \frac{\sin x}{\cos x}$ is undefined when

$$x = \frac{-7\pi}{2}, \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

Therefore, we have vertical asymptotes at

$$x = \frac{-7\pi}{2}, \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \dots$$

a) $\lim_{x \rightarrow 0} f(x) = ?$

From graph, as x approaches 0 from the left, $f(x)$ approaches 0.

Notation: $\lim_{x \rightarrow 0^-} f(x) = 0$

From graph, as x approaches 0 from the right, $f(x)$ approaches 0.

Notation: $\lim_{x \rightarrow 0^+} f(x) = 0$

Because limit from the left and limit from the right are equal,

$$\lim_{x \rightarrow 0} f(x) = 0.$$

$$b) \lim_{x \rightarrow 3\pi/2} f(x) = ?$$

From graph, as x approaches $3\pi/2$ from the left, $f(x)$ approaches ∞ .

$$\text{Notation: } \lim_{x \rightarrow 3\pi/2^-} f(x) = \infty$$

From graph, as x approaches $3\pi/2$ from the right, $f(x)$ approaches $-\infty$.

$$\text{Notation: } \lim_{x \rightarrow 3\pi/2^+} f(x) = -\infty$$

Because limit from the left and limit from the right are not equal,

$\lim_{x \rightarrow 3\pi/2} f(x)$ does not exist.