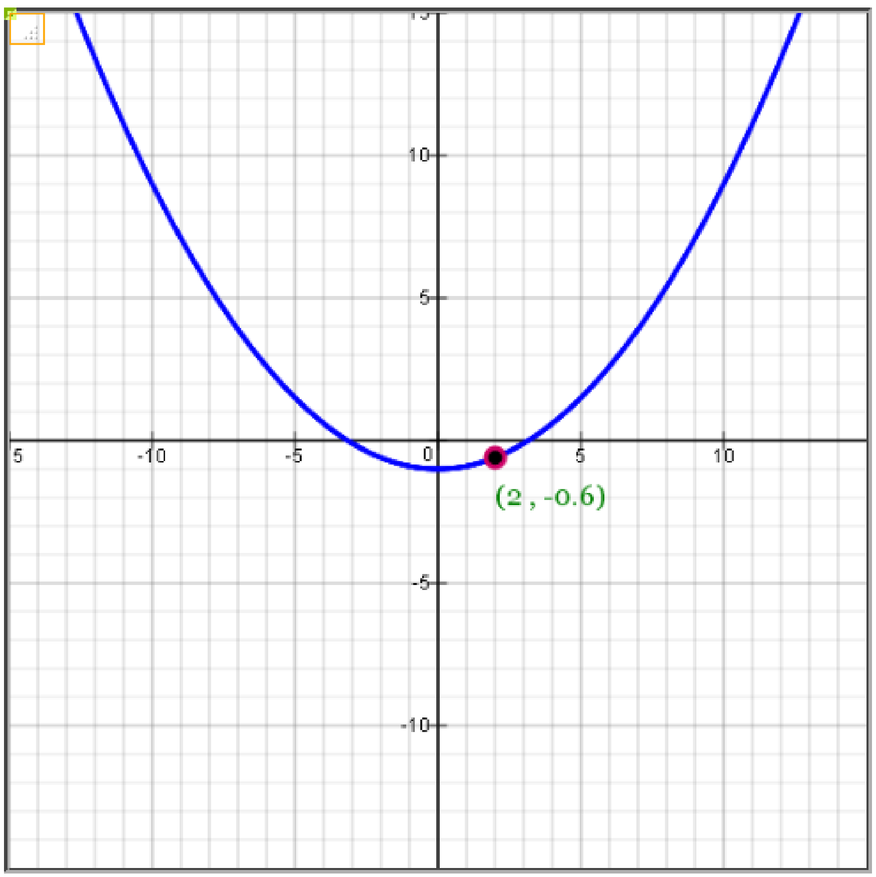
Finding Limits Graphically and Numerically

Example 1: Let $f(x) = 0.1x^2 - 1$




We will approach x = 2 from the left and right and look at the behavior of corresponding y-values.

Let x approach 2 from the left:

x = 1 ; y = -0.9

x = 1.5 ; y = -0.775

x = 1.9 ; y = -0.639

x = 1.99 ; y = -0.60399

x = 1.999 ; y = -0.6003999

x = 1.9999 ; y = -0.600039999

x = 1.99999 ; y = -0.60000399999

x = 1.999999 ; y = -0.6000003999999

x = 1.9999999 ; y = -0.600000039999999

x = 1.99999999 ; y = -0.600000004

x = 1.999999999 ; y = -0.6000000004

x = 1.9999999999 ; y = -0.60000000004

x = 1.99999999999 ; y = -0.600000000004

x = 1.999999999999 ; y = -0.6000000000004

x = 1.9999999999999 ; y = -0.60000000000004

x = 1.99999999999999 ; y = -0.600000000000004

As $x$ approaches 2 from the left, $y = f(x) = 0.1x^2 - 1$ approaches -0.6.

Notation: $\lim_{x \to 2^-} f(x) = -0.6$

Let x approach 2 from the right:

x = 3 ; y = -0.09999999999999998
x = 2.5 ; y = -0.375
x = 2.1 ; y = -0.5589999999999999
x = 2.01 ; y = -0.59599
x = 2.001 ; y = -0.5995999000000001
x = 2.0001 ; y = -0.599959999
x = 2.00001 ; y = -0.59999599999
x = 2.000001 ; y = -0.5999995999998999
x = 2.0000001 ; y = -0.599999959999999
x = 2.00000001 ; y = -0.599999996
x = 2.000000001 ; y = -0.5999999996
x = 2.0000000001 ; y = -0.59999999996
x = 2.00000000001 ; y = -0.599999999996
x = 2.000000000001 ; y = -0.5999999999996
x = 2.0000000000001 ; y = -0.59999999999996
x = 2.00000000000001 ; y = -0.5999999999999959

As $x$ approaches 2 from the right, $y = f(x) = 0.1x^2 - 1$ approaches -0.6.

Notation: $\lim_{x \to 2^+} f(x) = -0.6$

Summary:

1) As $x$ approaches $x = 2$ from the left side, $y = f(x) = 0.1x^2 - 1$ approaches -0.6.

Limit Notation: $\lim_{x \to 2^-} f(x) = -0.6$

2) As $x$ approaches $x = 2$ from the right side, $y = f(x) = 0.1x^2 - 1$ approaches -0.6.

Limit Notation: $\lim_{x \to 2^+} f(x) = -0.6$

3) As $x$ approaches 2 from either side, $f(x)$ approaches -0.6.

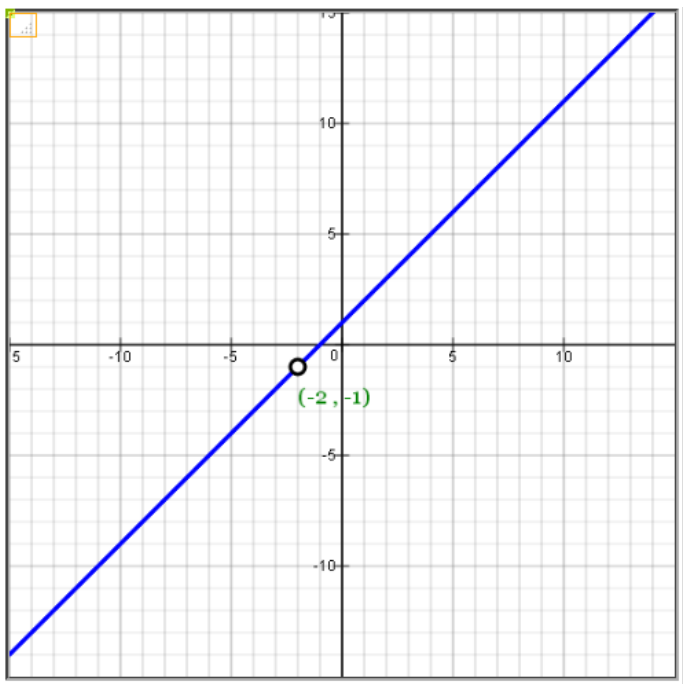
Limit Notation: $\lim_{x \to 2} f(x) = -0.6$

Example 2:

Let $f(x) = \dfrac{x^2 + 3x + 2}{x + 2}$

Note: $\dfrac{x^2 + 3x + 2}{x + 2} = \dfrac{(x+2)(x+1)}{(x+2)} = x + 1$

Also, when $x = -2$, $f(x) = \dfrac{x^2 + 3x + 2}{x + 2} = \dfrac{(-2)^2 + 3(-2) + 2}{(-2) + 2} = \dfrac{0}{0} = \text{undefined}$




We will approach x = -2 from the left and right and look at the behavior of corresponding y-values.

Let x approach -2 from the left:

x = -3 ; y = -2

x = -2.5 ; y = -1.5

x = -2.1 ; y = -1.0999999999999934

x = -2.01 ; y = -1.009999999999977

x = -2.001 ; y = -1.0009999999996957

x = -2.0001 ; y = -1.0000999999949511

x = -2.00001 ; y = -1.0000099999564185

x = -2.000001 ; y = -1.0000010000889004

x = -2.0000001 ; y = -1.0000000932587343

x = -2.00000001 ; y = -1

x = -2.000000001 ; y = -1

x = -2.0000000001 ; y = -1

x = -2.00000000001 ; y = -1

x = -2.000000000001 ; y = -1

As $x$ approaches -2 from the left, $y = f(x) = \dfrac{x^2 + 3x + 2}{x + 2}$ approaches -1.

Notation: $\lim\limits_{x \to -2^-} f(x) = -1$

Let x approach -2 from the right:

x = -1 ; y = 0

= -1.5 ; y = -0.5

x = -1.9 ; y = -0.8999999999999934

x = -1.99 ; y = -0.9899999999999567

x = -1.999 ; y = -0.998999999999416

x = -1.9999 ; y = -0.9998999999983873

x = -1.99999 ; y = -0.9999899999547638

x = -1.999999 ; y = -0.9999990001331439

x = -1.9999999 ; y = -0.9999999000799279

x = -1.99999999 ; y = -1

x = -1.999999999 ; y = -1

x = -1.9999999999 ; y = -1

x = -1.99999999999 ; y = -1

x = -1.999999999999 ; y = -1

As $x$ approaches -2 from the right, $y = f(x) = \dfrac{x^2+3x+2}{x+2}$ approaches -1.

Notation: $\lim\limits_{x \to -2^+} f(x) = -1$

Summary:

1) As $x$ approaches $x = -2$ from the left side, $y = f(x) = \dfrac{x^2 + 3x + 2}{x+2}$ approaches $-1$.

Limit Notation: $\lim_{x \to -2^-} f(x) = -1$

2) As $x$ approaches $x = -2$ from the right side, $y = f(x) = \dfrac{x^2 + 3x + 2}{x+2}$ approaches $-1$.

Limit Notation: $\lim_{x \to -2^+} f(x) = -1$

3) As $x$ approaches $-2$ from either side, $f(x)$ approaches $-1$.

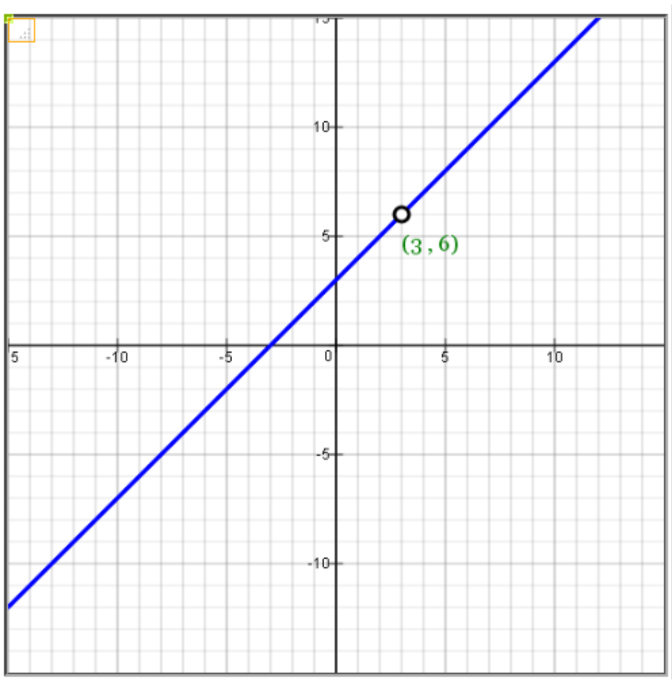
Limit Notation: $\lim_{x \to -2} f(x) = -1$

Example 3:

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$

Note: $\dfrac{x^2 - 9}{x - 3} = \dfrac{(x+3)(x-3)}{(x-3)} = x + 3$

Also, when $x = 3$, $f(x) = \dfrac{x^2 - 9}{x - 3} = \dfrac{(3)^2 - 9}{(3) - 3} = \dfrac{0}{0} = \text{undefined}$




We will approach x = 3 from the left and right and look at the behavior of corresponding y-values.

Let x approach 3 from the left:

x = 2 ; y = 5

x = 2.5 ; y = 5.5

x = 2.9 ; y = 5.8999999999999993

x = 2.99 ; y = 5.990000000000023

x = 2.999 ; y = 5.998999999999986

x = 2.9999 ; y = 5.999900000000060

x = 2.99999 ; y = 5.999990000008799

x = 2.999999 ; y = 5.999998999911099

x = 2.9999999 ; y = 5.999999902300374

x = 2.99999999 ; y = 6

x = 2.999999999 ; y = 6

x = 2.9999999999 ; y = 6

x = 2.99999999999 ; y = 6

x = 2.999999999999 ; y = 6

As $x$ approaches 3 from the left, $y = f(x) = \dfrac{x^2 - 9}{x - 3}$ approaches 6.

Notation: $\lim\limits_{x \to 3^-} f(x) = 6$

## Let x approach 3 from the right:

x = 4 ; y = 7

x = 3.5 ; y = 6.5

x = 3.1 ; y = 6.100000000000007

x = 3.01 ; y = 6.009999999999977

x = 3.001 ; y = 6.00100000000014

x = 3.0001 ; y = 6.000099999999392

x = 3.00001 ; y = 6.00000999991201

x = 3.000001 ; y = 6.000001000088901

x = 3.0000001 ; y = 6.000000097699626

x = 3.00000001 ; y = 6

x = 3.000000001 ; y = 6

x = 3.0000000001 ; y = 6

x = 3.00000000001 ; y = 6

x = 3.000000000001 ; y = 6

As $x$ approaches 3 from the right, $y = f(x) = \dfrac{x^2 - 9}{x - 3}$ approaches 6.

Notation: $\lim\limits_{x \to 3^+} f(x) = 6$

Summary:

1) As $x$ approaches $x = 3$ from the left side, $y = f(x) = \dfrac{x^2 - 9}{x - 3}$ approaches 6.

Limit Notation: $\lim_{x \to 3^-} f(x) = 6$

2) As $x$ approaches $x = 3$ from the right side, $y = f(x) = \dfrac{x^2 - 9}{x - 3}$ approaches 6.

Limit Notation: $\lim_{x \to 3^+} f(x) = 6$

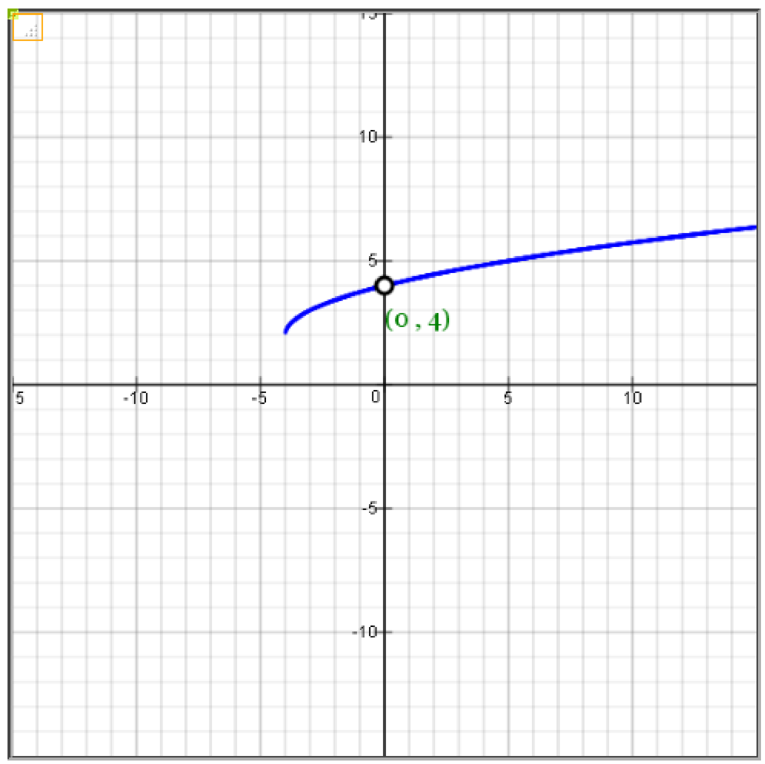
3) As $x$ approaches 3 from either side, $f(x)$ approaches 6.

Limit Notation: $\lim_{x \to 3} f(x) = 6$

Example 4:

Let $f(x) = \dfrac{x}{\sqrt{x+4}-2}$

Note: when $x = 0$, $f(x) = \dfrac{x}{\sqrt{x+4}-2} = \dfrac{0}{0} = \text{undefined}$




We will approach x = 0 from the left and right and look at the behavior of corresponding y-values.

Let x approach 0 from the left:

x = -1 ; y = 3.732050807568876

x = -0.5 ; y = 3.87082869338697

x = -0.1 ; y = 3.9748417658131423

x = -0.01 ; y = 3.9974984355438252

x = -0.001 ; y = 3.999749984374003

x = -0.0001 ; y = 3.9999749998259992

x = -0.00001 ; y = 3.999997500108378

x = -0.000001 ; y = 3.999999750750946

x = -0.000001 ; y = 3.999999750750946

x = -0.0000001 ; y = 3.999999988782747

x = -0.00000001 ; y = 4.000000024309884

x = -0.000000001 ; y = 3.9999996690385435

x = -0.0000000001 ; y = 3.9999996690385435

x = -0.00000000001 ; y = 3.999999669038543

As $x$ approaches 0 from the left, $y = f(x) = \dfrac{x}{\sqrt{x+4}-2}$ approaches 4.

Notation: $\lim\limits_{x \to 0^-} f(x) = 4$

Let x approach 0 from the right:

x = 1 ; y = 4.236067977499788

x = 0.5 ; y = 4.121320343559649

x = 0.1 ; y = 4.024845673131703

x = 0.01 ; y = 4.002498439449939

x = 0.001 ; y = 4.000249984373987

x = 0.0001 ; y = 4.0000249998301305

x = 0.00001 ; y = 4.000002499842338

x = 0.000001 ; y = 4.000000251683575

x = 0.0000001 ; y = 4.000000024309884

x = 0.00000001 ; y = 4.000000379581288

x = 0.000000001 ; y = 3.9999996690385435

x = 0.0000000001 ; y = 3.9999996690385435

x = 0.00000000001 ; y = 4.000354971904864

As $x$ approaches 0 from the right, $y = f(x) = \dfrac{x}{\sqrt{x+4}-2}$ approaches 4.

Notation: $\lim\limits_{x \to 0^+} f(x) = 4$

Summary:

1) As $x$ approaches $x = 0$ from the left side, $y = f(x) = \dfrac{x}{\sqrt{x+4} - 2}$ approaches 4.

Limit Notation: $\lim\limits_{x \to 0^-} f(x) = 4$

2) As $x$ approaches $x = 0$ from the right side, $y = f(x) = \dfrac{x}{\sqrt{x+4} - 2}$ approaches 4.

Limit Notation: $\lim\limits_{x \to 0^+} f(x) = 4$

3) As $x$ approaches 0 from either side, $f(x)$ approaches 4.

Limit Notation: $\lim\limits_{x \to 0} f(x) = 4$

Note:

Multiply expression by conjugate of denominator and simplify:

$$\frac{x}{\sqrt{x+4}-2} = \frac{x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

Recall:

$$\left(\sqrt{2}\right)^2 = \left(\sqrt{2}\right)\left(\sqrt{2}\right) = \sqrt{4} = 2; \qquad \left(\sqrt{5}\right)^2 = \left(\sqrt{5}\right)\left(\sqrt{5}\right) = \sqrt{25} = 5$$

$$\left(\sqrt{x}\right)^2 = x; \qquad \left(\sqrt{x+5}\right)^2 = x+5$$

$$\left(\sqrt{a+b}\right)^2 = a+b$$

$$(a-b)(a+b) = a^2 - b^2$$

$$(\sqrt{a}-b)(\sqrt{a}+b) = \left(\sqrt{a}\right)^2 - b^2 = a - b^2$$

$$\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right) = \left(\sqrt{x+4}\right)^2 - 2^2 = x+4-4 = x$$

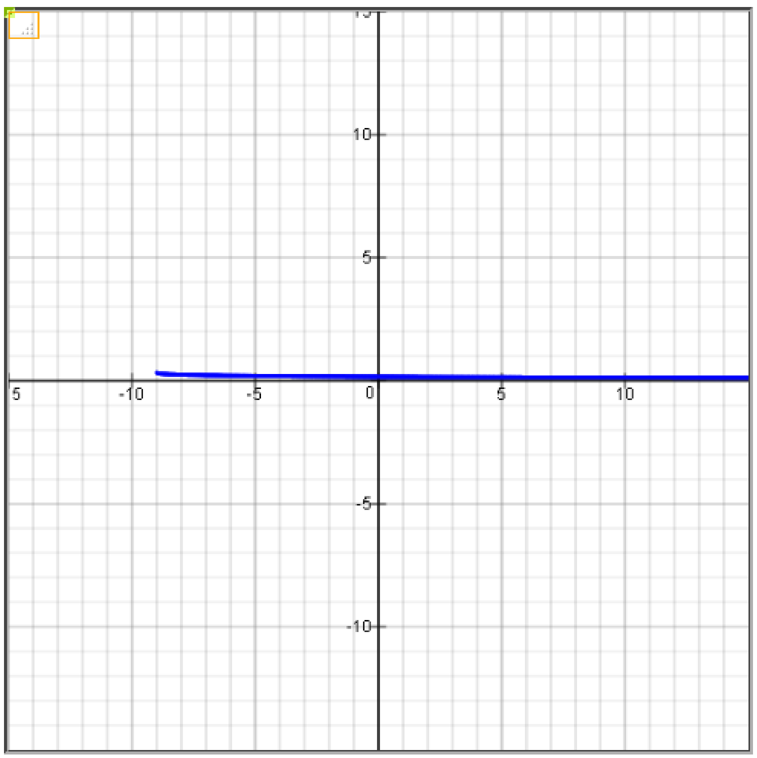
Hence,

$$\frac{x}{\sqrt{x+4}-2} = \frac{x}{\sqrt{x+4}-2} \cdot \frac{\sqrt{x+4}+2}{\sqrt{x+4}+2} = \frac{(x)\left(\sqrt{x+4}+2\right)}{(x)} = \sqrt{x+4}+2$$

$$\lim_{x\to 0} \frac{x}{\sqrt{x+4}-2} = \lim_{x\to 0}\left(\sqrt{x+4}+2\right) = \sqrt{0+4}+2 = 2+2 = 4$$

Example 5: Let $f(x) = \dfrac{\sqrt{x+9} - 3}{x}$

Note: when $x = 0$, $f(x) = \dfrac{\sqrt{x+9} - 3}{x} = \dfrac{0}{0} = \text{undefined}$




We will approach x = 0 from the left and right and look at the behavior of corresponding y-values.

## Let x approach 0 from the left:

x = -1 ; y = 0.1715728752538097

x = -0.5 ; y = 0.16904810515469926

x = -0.1 ; y = 0.16713221964740566

x = -0.01 ; y = 0.16671298870098994

x = -0.001 ; y = 0.1666712965535666

x = -0.0001 ; y = 0.1666671296307598

x = -0.00001 ; y = 0.16666671296405866

x = -0.000001 ; y = 0.1666666711308551

x = -0.000001 ; y = 0.1666666711308551

x = -0.0000001 ; y = 0.1666666671340522

x = -0.00000001 ; y = 0.1666666804567285

x = -0.000000001 ; y = 0.1666666804567285

x = -0.0000000001 ; y = 0.1666666804567285

x = -0.00000000001 ; y = 0.1666666804567285

As $x$ approaches 0 from the left, $y = f(x) = \dfrac{\sqrt{x+9}-3}{x}$ approaches 0.16666666.

Notation: $\lim\limits_{x \to 0^-} f(x) = 0.16666666$

Let x approach 0 from the right:

x = 1 ; y = 0.16227766016837952

x = 0.5 ; y = 0.16441400296897601

x = 0.1 ; y = 0.16620625799671274

x = 0.01 ; y = 0.16662039607266976

x = 0.001 ; y = 0.16666203729398532

x = 0.0001 ; y = 0.16666620370475727

x = 0.00001 ; y = 0.1666666203714584

x = 0.000001 ; y = 0.1666666618049817

x = 0.0000001 ; y = 0.1666666671340522

x = 0.00000001 ; y = 0.1666666804567285

x = 0.000000001 ; y = 0.1666666804567285

x = 0.0000000001 ; y = 0.1666666804567285

x = 0.00000000001 ; y = 0.1666666804567285

As $x$ approaches 0 from the right, $y = f(x) = \dfrac{\sqrt{x+9}-3}{x}$ approaches 0.16666666.

Notation: $\lim\limits_{x \to 0^{+}} f(x) = 0.16666666$

Summary:

1) As $x$ approaches $x = 0$ from the left side, $y = f(x) = \dfrac{\sqrt{x+9}-3}{x}$ approaches 0.16666666 = $\dfrac{1}{6}$.

Limit Notation: $\lim\limits_{x \to 0^-} f(x) = 0.16666666 = \dfrac{1}{6}$

2) As $x$ approaches $x = 0$ from the right side, $y = f(x) = \dfrac{\sqrt{x+9}-3}{x}$ approaches 0.16666666 = $\dfrac{1}{6}$.

Limit Notation: $\lim\limits_{x \to 0^+} f(x) = 0.16666666 = \dfrac{1}{6}$

3) As $x$ approaches 0 from either side, $f(x)$ approaches 0.16666666 = $\dfrac{1}{6}$.

Limit Notation: $\lim\limits_{x \to 0} f(x) = 0.16666666 = \dfrac{1}{6}$

Note:

Multiply expression by conjugate of denominator and simplify:

$$\frac{\sqrt{x+9}-3}{x}=\frac{\sqrt{x+9}-3}{x}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$$

Recall:

$$\left(\sqrt{2}\right)^2=\left(\sqrt{2}\right)\left(\sqrt{2}\right)=\sqrt{4}=2; \qquad \left(\sqrt{5}\right)^2=\left(\sqrt{5}\right)\left(\sqrt{5}\right)=\sqrt{25}=5$$

$$\left(\sqrt{x}\right)^2=x; \qquad \left(\sqrt{x+5}\right)^2=x+5$$

$$\left(\sqrt{a+b}\right)^2=a+b$$

$$(a-b)(a+b)=a^2-b^2$$

$$(\sqrt{a}-b)(\sqrt{a}+b)=\left(\sqrt{a}\right)^2-b^2=a-b^2$$

$$\left(\sqrt{x+9}-3\right)\left(\sqrt{x+9}+3\right)=\left(\sqrt{x+9}\right)^2-3^2=x+9-9=x$$

Hence,

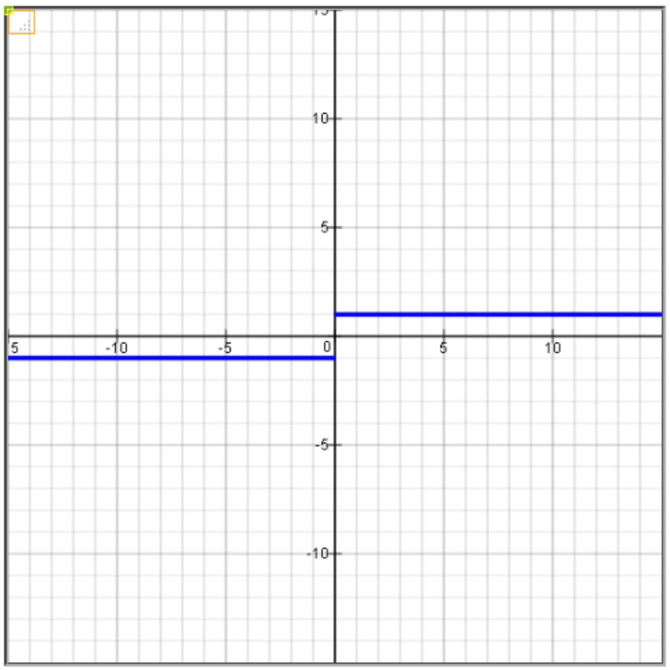
$$\frac{\sqrt{x+9}-3}{x}=\frac{\sqrt{x+9}-3}{x}\cdot\frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}=\frac{(x)}{(x)\left(\sqrt{x+9}+3\right)}=\frac{1}{\sqrt{x+9}+3}$$

$$\lim_{x\to 0}\frac{\sqrt{x+9}-3}{x}=\lim_{x\to 0}\left(\frac{1}{\sqrt{x+9}+3}\right)=\frac{1}{\sqrt{0+9}+3}=\frac{1}{6}$$

# Limits That Fail To Exist

Example 6: Let $f(x) = \frac{|x|}{x}$

Note: When x = 0, $f(x) = \frac{|x|}{x} = \frac{0}{0} = undefined$




We will approach x = 0 from the left and right and look at the behavior of corresponding y-values.

# Let x approach 0 from the left:

x = -2 ; y = |-2|/(-2) = -1

x = -1.7 ; y= |-1.7|/(-1.7)  = -1

x = -1.5 ; y = |-1.5|/(-1.5) = -1

x = -1 ; y = |-1|/(-1) = -1

x = -0.5 ; y = -1

x = -0.1 ; y = -1

x = -0.01 ; y = -1

x = -0.001 ; y = -1

x = -0.0001 ; y = -1

x = -0.00001 ; y = -1

x = -0.000001 ; y = -1

x = -0.000001 ; y = -1

x = -0.0000001 ; y = -1

x = -0.00000001 ; y = -1

x = -0.000000001 ; y = -1

x = -0.0000000001 ; y = -1

x = -0.00000000001 ; y = -1

x = -0.000000000001 ; y = -1

x = -0.0000000000001 ; y = -1

As $x$ approaches 0 from the left, $y = f(x) = \dfrac{|x|}{x}$ approaches -1.

Notation: $\lim\limits_{x \to 0^-} f(x) = -1$

Let x approach 0 from the right:

x = 1 ; y = |1|/1 = 1
x = 0.5 ; y = |0.5|/0.5 = 1
x = 0.1 ; y = 1
x = 0.01 ; y = 1
x = 0.001 ; y = 1
x = 0.0001 ; y = 1
x = 0.00001 ; y = 1
x = 0.000001 ; y = 1
x = 0.0000001 ; y = 1
x = 0.00000001 ; y = 1
x = 0.000000001 ; y = 1
x = 0.0000000001 ; y = 1
x = 0.00000000001 ; y = 1
x = 0.000000000001 ; y = 1
x = 0.0000000000001 ; y = 1
x = 0.00000000000001 ; y = 1

As $x$ approaches 0 from the right, $y = f(x) = \dfrac{|x|}{x}$ approaches 1.

Notation: $\lim\limits_{x \to 0^+} f(x) = 1$

Summary:

1) As $x$ approaches $x = 0$ from the left side, $y = f(x) = \dfrac{|x|}{x}$ approaches -1.

Limit Notation: $\lim\limits_{x \to 0^-} f(x) = -1$

2) As $x$ approaches $x = 0$ from the right side, $y = f(x) = \dfrac{|x|}{x}$ approaches 1.

Limit Notation: $\lim\limits_{x \to 0^+} f(x) = 1$

3) As $x$ approaches 0 from either side, $f(x)$ does not approach the same value.

Limit Notation: $\lim\limits_{x \to 0} f(x) = \text{Does Not Exist}$

Note:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Definition of Absolute Value

$$\frac{|x|}{x} = \begin{cases} x/x = 1 & \text{if } x > 0 \\ -x/x = -1 & \text{if } x < 0 \end{cases}$$

Note: When $x = 0$, $\frac{|x|}{x} = \frac{0}{0}$ = undefined

$$\lim_{x\to 0^-} \frac{|x|}{x} = -x/x = -1$$

Note: $x$ approaches 0 from the left; so $x < 0$

$$\lim_{x\to 0^+} \frac{|x|}{x} = x/x = 1$$

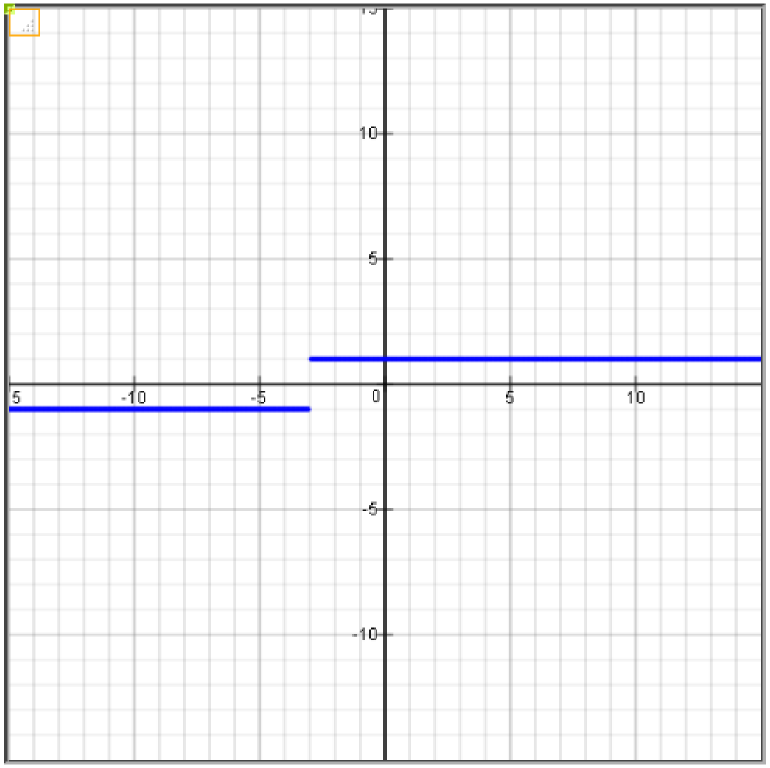
Note: $x$ approaches 0 from the right; so $x > 0$

Hence, limit from the left and limit from the right are not equal.

Therefore, as $x$ approaches 0, $\lim_{x\to 0} \frac{|x|}{x}$ does not exist.

Example 7: Let $f(x) = \dfrac{|x+3|}{x+3}$

Note: When x = -3, $f(x) = \dfrac{|x+3|}{x+3} = \dfrac{0}{0} = undefined$




We will approach x = -3 from the left and right and look at the behavior of corresponding y-values.

Let x approach -3 from the left:

x = -4 ; y = |-4+3|/(-4+3) = 1/-1 = -1
x = -3.5 ; y = -1
x = -3.1 ; y = -1
x = -3.01 ; y = -1
x = -3.001 ; y = -1
x = -3.0001 ; y = -1
x = -3.00001 ; y = -1
x = -3.000001 ; y = -1
x = -3.0000001 ; y = -1
x = -3.00000001 ; y = -1
x = -3.000000001 ; y = -1
x = -3.0000000001 ; y = -1
x = -3.00000000001 ; y = -1
x = -3.000000000001 ; y = -1
x = -3.0000000000001 ; y = -1
x = -3.00000000000001 ; y = -1

As $x$ approaches -3 from the left $f(x) = \frac{|x+3|}{x+3}$ approaches -1.

Notation: $\lim_{x \to -3^-} f(x) = -1$

Let x approach -3 from the right:

x = -2 ; y = 1

x = -2.5 ; y = 1

x = -2.9 ; y = 1

x = -2.99 ; y = 1

x = -2.999 ; y = 1

x = -2.9999 ; y = 1

x = -2.99999 ; y = 1

x = -2.999999 ; y = 1

x = -2.9999999 ; y = 1

x = -2.99999999 ; y = 1

x = -2.999999999 ; y = 1

x = -2.9999999999 ; y = 1

x = -2.99999999999 ; y = 1

x = -2.999999999999 ; y = 1

x = -2.9999999999999 ; y = 1

x = -2.99999999999999 ; y = 1

As $x$ approaches -3 from the right $f(x) = \frac{|x+3|}{x+3}$ approaches 1.

Notation: $\lim_{x \to -3^+} f(x) = 1$

Summary:

1) As $x$ approaches $x = -3$ from the left side, $y = f(x) = \dfrac{|x+3|}{x+3}$ approaches $-1$.

Limit Notation: $\lim\limits_{x \to -3^-} f(x) = -1$

2) As $x$ approaches $x = -3$ from the right side, $y = f(x) = \dfrac{|x+3|}{x+3}$ approaches 1.

Limit Notation: $\lim\limits_{x \to -3^+} f(x) = 1$

3) As $x$ approaches -3 from either side, $f(x)$ does not approach the same value.

Limit Notation: $\lim\limits_{x \to -3} f(x) = \text{Does Not Exist}$

Note:

$$|x+3| = \begin{cases} x+3 & \text{if } x+3 \geq 0 \text{ or } x \geq -3 \\ -(x+3) & \text{if } x+3 < 0 \text{ or } x < -3 \end{cases}$$

Definition of Absolute Value

$$\frac{|x+3|}{x+3} = \begin{cases} \dfrac{x+3}{x+3} = 1 & \text{if } x+3 \geq 0 \text{ or } x \geq -3 \\ \dfrac{-(x+3)}{(x+3)} = -1 & \text{if } x+3 < 0 \text{ or } x < -3 \end{cases}$$

$$\lim_{x \to -3^-} \frac{|x+3|}{x+3} = -1$$ Note: $x$ approaches -3 from the left; so $x < -3$

$$\lim_{x \to -3^+} \frac{|x+3|}{x+3} = 1$$ Note: $x$ approaches -3 from the right; so $x > -3$

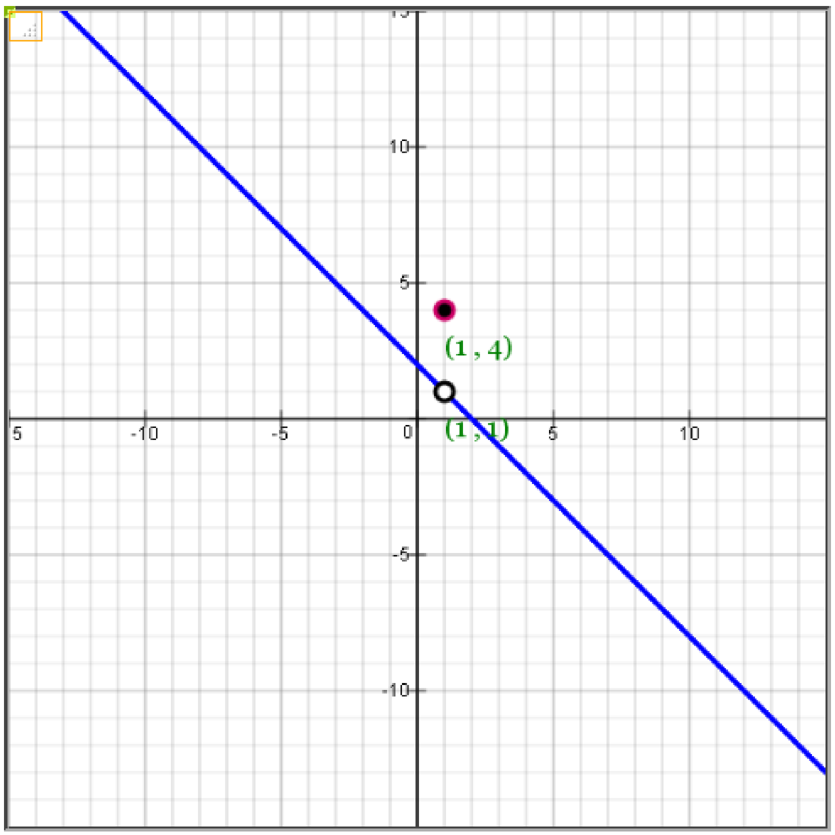
Hence, limit from the left and limit from the right are not equal.

Therefore, as $x$ approaches -3, $\lim_{x \to -3} \frac{|x+3|}{x+3}$ does not exist.

Example 8:

Let $f(x) = \begin{cases} 2 - x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$ Piecewise Function

Find $\lim\limits_{x \to 1}$.



Let x approach 1 from the left:

x = -1, y = 3 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 0, y =2 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 0.5, y = 1.5 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 0.9, y = 1.1 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 0.999, y = 1.001 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 0.999999, y = 1.000001 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

As $x$ approaches 1 from the left $f(x)$ approaches 1.

Notation: $\lim\limits_{x \to 1^-} f(x) = 1$

Let x approach 1 from the right:

x = 3, y = -1 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 2, y = 0 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 1.5, y = 0.5 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 1.01, y = 0.99 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 1.0001, y = 0.9999 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

x = 1.0000001, y = 0.999999 Note: To find $y$, we use $f(x) = 2 - x$ because $x = -1$ and $x \neq 0$

As $x$ approaches 1 from the right $f(x)$ approaches 1.

Notation: $\lim_{x \to 1^+} f(x) = 1$

Summary:

1) As $x$ approaches $x = 1$ from the left side, $y = f(x) = \begin{cases} 2 - x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$ approaches 1.

Limit Notation: $\lim\limits_{x \to -1^-} f(x) = 1$

2) As $x$ approaches $x = 1$ from the right side, $y = f(x) = \begin{cases} 2 - x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$ approaches 1.

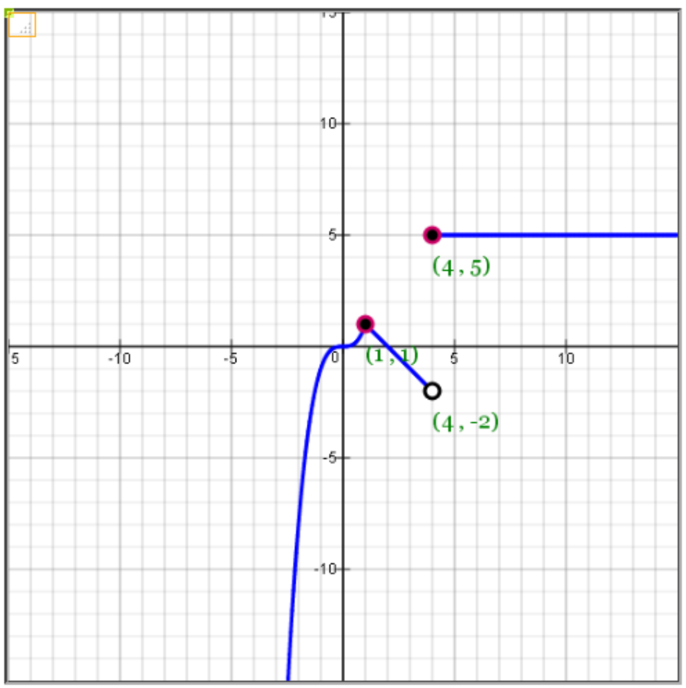
Limit Notation: $\lim\limits_{x \to 1^+} f(x) = 1$

3) As $x$ approaches 1 from either side, $f(x)$ approaches 1.

Limit Notation: $\lim\limits_{x \to 1} f(x) = 1$

Example 9:

$$f(x)=\begin{cases} x^3 & \text{if } x \le 1 \\ 2-x & \text{if } 1 < x < 4 \\ 5 & \text{if } x \ge 4 \end{cases}$$



a) Find $f(4) = ?$; and $f(1) = ?$

From graph, when $x = 4,\ y = 5$. Therefore, $f(4) = 5$

From graph, when $x = 1,\ y = 1$. Therefore, $f(1) = 1$

b) $\lim\limits_{x \to 4} f(x) = ?$

From graph, as $x$ approaches 4 from the left, $f(x)$ approaches -2.

$\lim\limits_{x \to 4^-} f(x) = -2$

From graph, as $x$ approaches 4 from the right, $f(x)$ approaches 5.

$\lim\limits_{x \to 4^+} f(x) = 5$

Because limit from the left and limit from the right are not equal,

$\lim\limits_{x \to 4} f(x)$ does not exist.

c) Find $\lim_{x \to 1} f(x) = ?$

From graph, as $x$ approaches 1 from the left, $f(x)$ approaches 1.

Notation: $\lim_{x \to 4^-} f(x) = 1$

From graph, as $x$ approaches 1 from the right, $f(x)$ approaches 1.

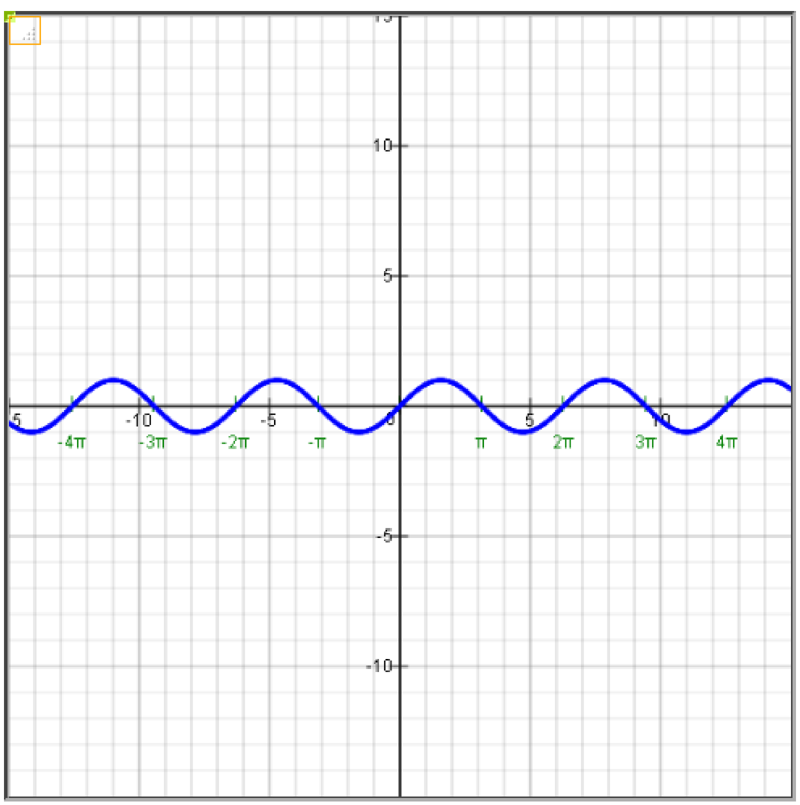
Notation: $\lim_{x \to 4^+} f(x) = 5$

Because limit from the left and limit from the right are equal,

$\lim_{x \to 1} f(x) = 1.$

Example 10:

Let $f(x) = \sin x$



a) Find $\lim_{x \to 0} f(x) = ?$

From graph, as $x$ approaches 0 from the left, $f(x)$ approaches 0.

Notation: $\lim_{x \to 0^-} f(x) = 0$

From graph, as $x$ approaches 0 from the right, $f(x)$ approaches 0.

Notation: $\lim_{x \to 0^+} f(x) = 0$

Because limit from the left and limit from the right are equal,

$\lim_{x \to 0} f(x) = 0.$

b) Find $\lim_{x \to \pi} f(x) = ?$

From graph, as $x$ approaches $\pi$ from the left, $f(x)$ approaches 0.

Notation: $\lim_{x \to \pi^-} f(x) = 0$

From graph, as $x$ approaches $\pi$ from the right, $f(x)$ approaches 0.

Notation: $\lim_{x \to \pi^+} f(x) = 0$

Because limit from the left and limit from the right are equal,

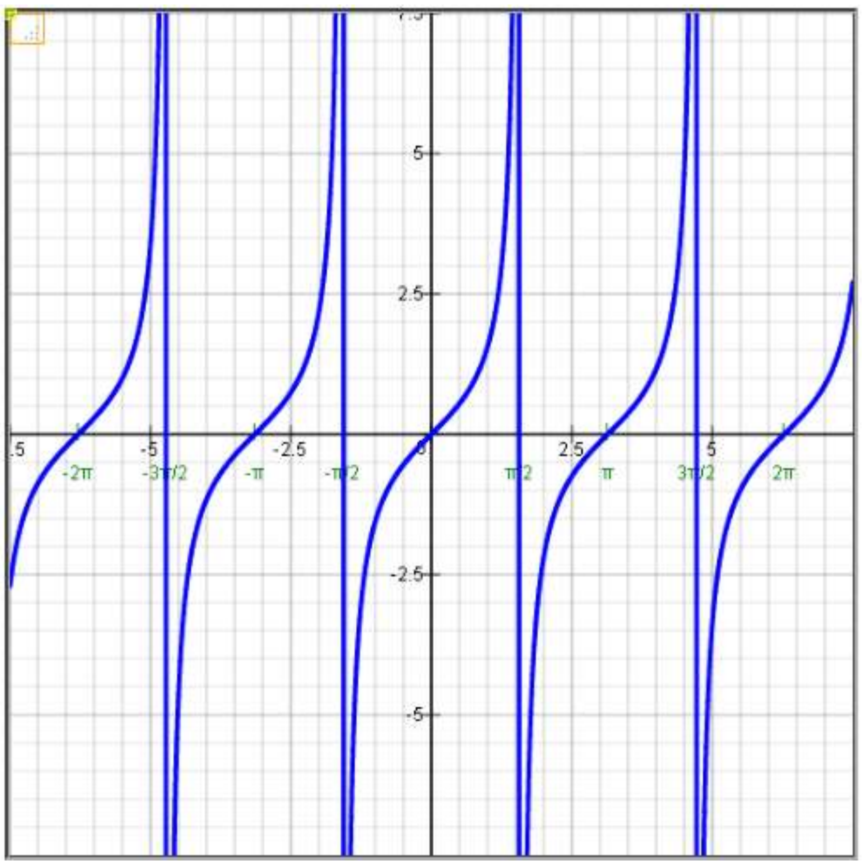
$\lim_{x \to \pi} f(x) = 0.$

Example 11:

Let $f(x) = \tan x$

Note: $\tan x = \dfrac{\sin x}{\cos x}$;

and $\cos x = 0$ when $x = \dfrac{-7\pi}{2}, \dfrac{-5\pi}{2}, \dfrac{-3\pi}{2}, \dfrac{-\pi}{2}, \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}, \ldots$



Note: $\tan x = \dfrac{\sin x}{\cos x}$ is undefined when

$$x = \frac{-7\pi}{2}, \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

Therefore, we have vertical asymptotes at

$$x = \frac{-7\pi}{2}, \frac{-5\pi}{2}, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \ldots$$

a) Find $\lim_{x \to 0} f(x) = ?$

From graph, as $x$ approaches 0 from the left, $f(x)$ approaches 0. Notation: $\lim_{x \to 0^-} f(x) = 0$

From graph, as $x$ approaches 0 from the right, $f(x)$ approaches 0. Notation: $\lim_{x \to 0^+} f(x) = 0$

Because limit from the left and limit from the right are equal, $\lim_{x \to 0} f(x) = 0$.

b) Find $\lim_{x \to 3\pi/2} f(x) = ?$

From graph, as $x$ approaches $3\pi/2$ from the left, $f(x)$ approaches $\infty$.

Notation: $\lim_{x \to 3\pi/2^-} f(x) = \infty$

From graph, as $x$ approaches $3\pi/2$ from the right, $f(x)$ approaches $-\infty$.

Notation: $\lim_{x \to 3\pi/2^+} f(x) = -\infty$

Because limit from the left and limit from the right are not equal,

$\lim_{x \to 3\pi/2} f(x)$ does not exist.