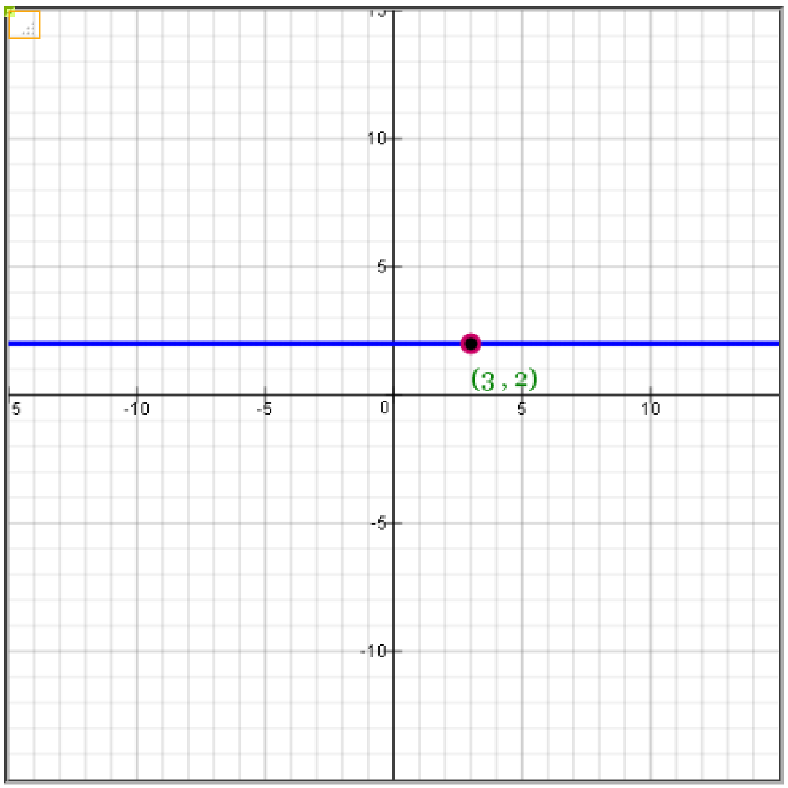
Finding Limits Analytically

Example 1: Constant Function

Let f(x) = 2



Let x approach 3 from the left:

As x approaches 3 from the left, $y = f(x)$ approaches 2

Notation: $\lim\limits_{x \to 3^-} f(x) = 2$

Let x approach 3 from the right:

As x approaches 3 from the right, $y = f(x)$ approaches 2.

Notation: $\lim\limits_{x \to 3^+} f(x) = 2$

Summary:

As x approaches 3 from either side, $f(x)$ approaches 2

Since $\lim\limits_{x \to 3^+} f(x) = \lim\limits_{x \to 3^-} f(x) = 2;$

we say that: $\lim\limits_{x \to 3} f(x) = 2$

For constant function $f(x) = a$

$$\lim_{x \to 1} f(x) = a$$

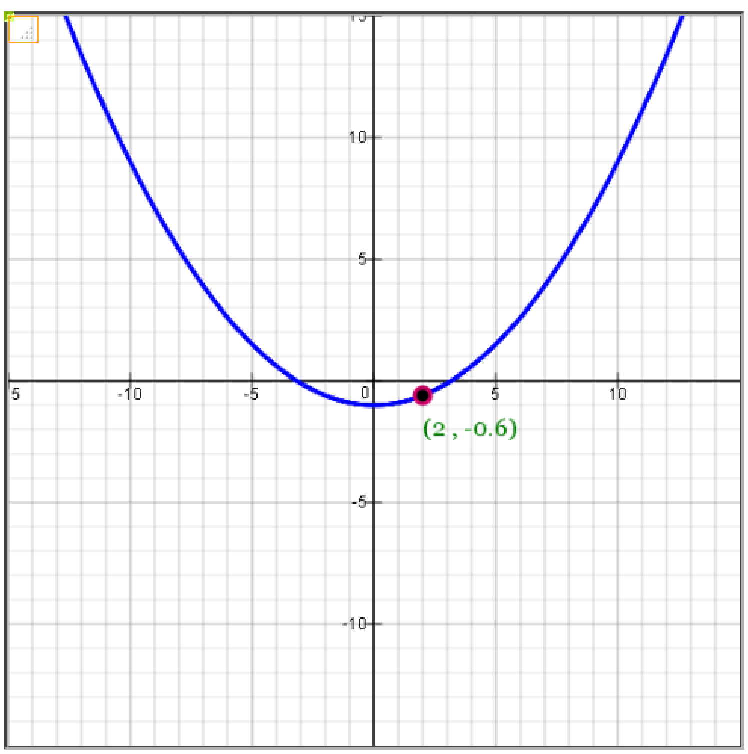
$$\lim_{x \to 2} f(x) = a$$

$$\lim_{x \to 4} f(x) = a$$

$$\lim_{x \to c} f(x) = a$$

Example 2: Polynomial Function

Let $f(x) = 0.1x^2 - 1$



Let x approach 2 from the left:

As x approaches 2 from the left, $y = f(x)$ approaches -0.6.

Notation: $\lim_{x \to 2^-} f(x) = -0.6$

Let x approach 2 from the right:

As x approaches 2 from the right, $y = f(x) = 0.1x^2 - 1$ approaches -0.6.

Notation: $\lim_{x \to 2^+} f(x) = -0.6$

Summary:

As x approaches 2 from either side, $f(x)$ approaches -0.6.

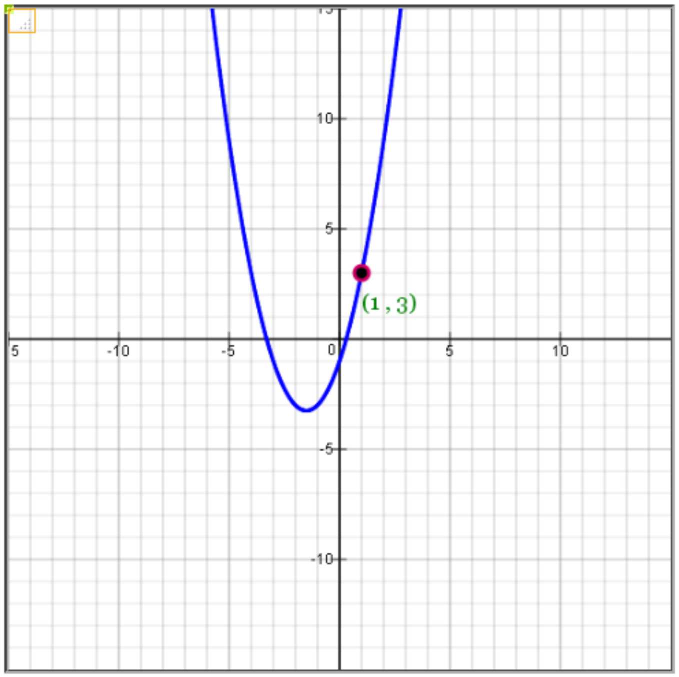
$$\lim_{x \to 2} f(x) = -0.6$$

In general, if $f(x)$ is a polynomial function, then

$$\lim_{x \to c} f(x) = f(c)$$

Example 3: Polynomial Function

Let $f(x) = x^2 + 3x - 1$



Let x approach 1 from the left:

As x approaches 1 from the left, $y = f(x)$ approaches 3.

Notation: $\lim\limits_{x \to 1^-} f(x) = 3$

Let x approach 1 from the right:

As x approaches 1 from the right, $y = f(x)$ approaches 3.

Notation: $\lim_{x \to 1^+} f(x) = 3$

Summary:

As x approaches 1 from either side, $f(x)$ approaches 3.

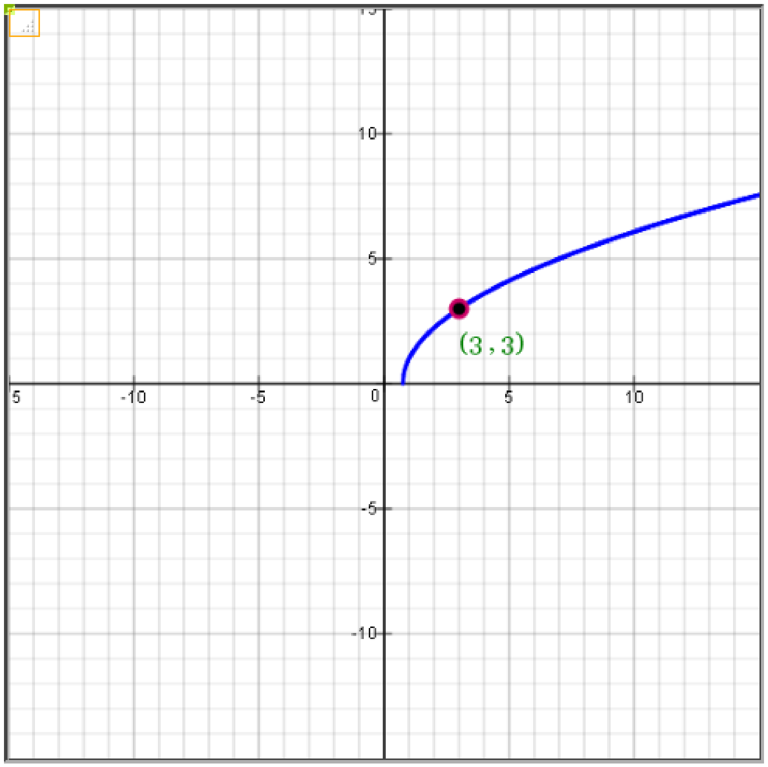
Hence $\lim_{x \to 1} f(x) = 3$

Also, since $f(x) = x^2 + 3x - 1$ is a polynomial function,

$$\lim_{x \to 1} f(x) = (1)^2 + 3(1) - 1 = 3$$

Example 4: Square Root Function and Limit of Composite Function Theorem

Let $f(x) = \sqrt{4x-3}$



$f(x)=\sqrt{4x-3}$

Note: From Graph we have $\lim_{x\to 3^-} f(x)=3$ and $\lim_{x\to 3^+} f(x)=3$;

thus, $\lim_{x\to 3} f(x)=3$

Using Limit of Composite Function Theorem:

$$\lim_{x\to 3} f(x)= \lim_{x\to 3}\sqrt{4x-3}=\sqrt{\lim_{x\to 3}(4x-3)}=\sqrt{(4(3)-3)}=3$$

Example 5: Square Root Function and Limit of Composite Function Theorem

Let $f(x) = \sqrt[3]{4x - 6}$

Let $f(x) = \sqrt[3]{4x - 6}$

Using Limit of Composite Function Theorem:

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \sqrt[3]{4x - 6} = \sqrt[3]{\lim_{x \to 3}(4x - 6)} = \sqrt[3]{4(3) - 6} = \sqrt[3]{6}$$

Example 6: Square Root Function and Limit of Composite Function Theorem

Let $f(x) = \sqrt[3]{4x-6}$

Let $f(x) = \sqrt[5]{4x-6}$

Using Limit of Composite Function Theorem:

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \sqrt[5]{4x-6} = \sqrt[5]{\lim_{x\to 2}(4x-6)} = \sqrt[5]{4(2)-6} = \sqrt[5]{2}$$

Example 7: Limit of Product of Functions

Let $f(x) = (2x-4)(6x-1)$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(2x-4)\cdot(6x-1) = \left[\lim_{x\to 2}(2x-4)\right]\cdot\left[\lim_{x\to 2}(6x-1)\right] = (0)(11) = 0$$

Example 8: Limit of Product of Functions

Let $f(x) = (4x-4)\sqrt{3x-1}$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(4x-4)\sqrt{3x-1} = \left[\lim_{x\to 2}(4x-4)\right]\cdot\left[\lim_{x\to 2}\sqrt{3x-1}\right] = (4)\sqrt{5} = 4\sqrt{5}$$

Example 9: Limit of Product of Functions

Let $f(x) = (3x-4)\cos x$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(3x-4)\cos x = \left[\lim_{x\to 2}(3x-4)\right]\cdot\left[\lim_{x\to 2}\cos x\right] = (2)\cos 2 = \text{-}0.8322936730942848$$

Example 10: Limit of Quotient of Functions

Let $f(x) = \dfrac{2x+1}{4x-5}$

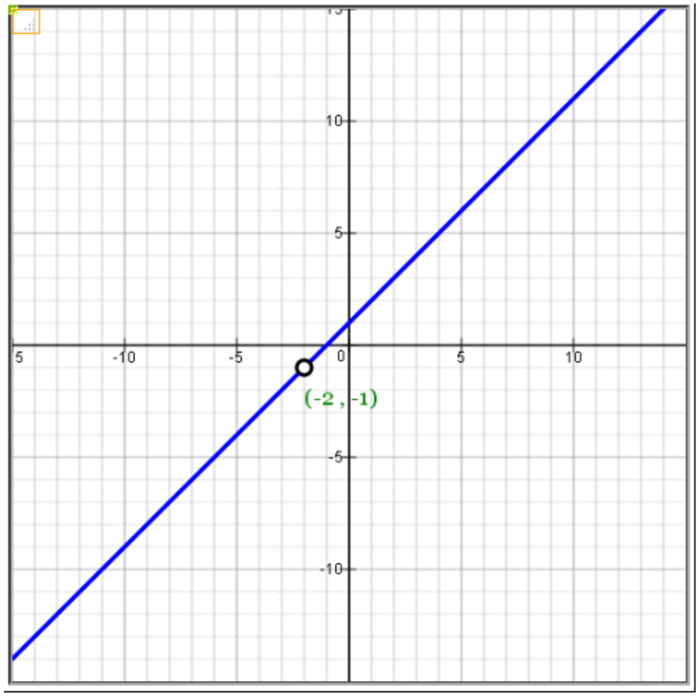
$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\frac{2x+1}{4x-5}\right) = \frac{\lim_{x\to 2}(2x+1)}{\lim_{x\to 2}(4x-5)} = \frac{5}{3}$$

Example 11:

Let $f(x) = \dfrac{x^2 + 3x + 2}{x + 2}$

Note: $\dfrac{x^2 + 3x + 2}{x + 2} = \dfrac{(x+2)(x+1)}{(x+2)} = x + 1$

Also, when $x = -2$, $f(x) = \dfrac{x^2 + 3x + 2}{x + 2} = \dfrac{(-2)^2 + 3(-2) + 2}{(-2) + 2} = \dfrac{0}{0} = \text{undefined}$



Note: $\lim_{x \to -2} f(x) = \lim_{x \to -2} \frac{x^2 + 3x + 2}{x + 2} = \frac{\lim_{x \to -2}\left(x^2 + 3x + 2\right)}{\lim_{x \to -2}\left(x + 2\right)} = \frac{0}{0}$

Hence, we need to look at limit from the left and limit from the right.

As x approaches -2 from the left, $y = f(x) = \frac{x^2 + 3x + 2}{x + 2}$ approaches -1.

Notation: $\lim_{x \to -2^-} f(x) = -1$

As x approaches -2 from the right, $y = f(x) = \frac{x^2 + 3x + 2}{x + 2}$ approaches -1.

Notation: $\lim_{x \to -2^+} f(x) = -1$

Summary:

As x approaches 2 from either side, $f(x)$ approaches -1.

Since $\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2^-} f(x) = -1,$ we say that: $\lim\limits_{x\to -2} f(x) = -1$

Also,

Let $f(x) = \dfrac{x^2+3x+2}{x+2}$

Note: $\dfrac{x^2+3x+2}{x+2} = \dfrac{(x+2)(x+1)}{(x+2)} = x+1$

$$\lim_{x\to -2} f(x) = \lim_{x\to -2} \frac{x^2+3x+2}{x+2} = \lim_{x\to -2} \frac{(x+2)(x+1)}{(x+2)} = \lim_{x\to -2}(x+1) = -1$$

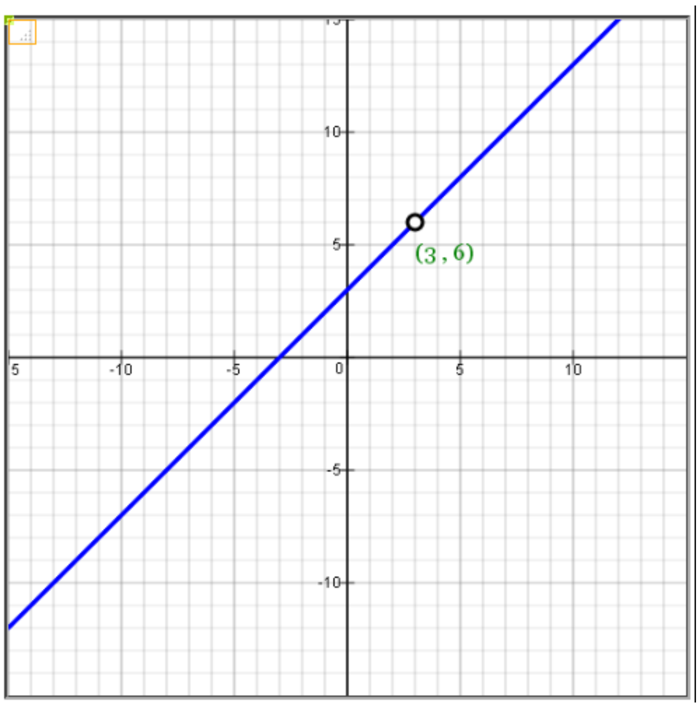
For quotient of functions, try to factor numerator and denominator (if possible) before finding limit.

Example 12:

Let $f(x) = \dfrac{x^2 - 9}{x - 3}$

Note: $\dfrac{x^2 - 9}{x - 3} = \dfrac{(x+3)(x-3)}{(x-3)} = x + 3$

Also, when $x = 3$, $f(x) = \dfrac{x^2 - 9}{x - 3} = \dfrac{(3)^2 - 9}{(3) - 3} = \dfrac{0}{0} = \text{undefined}$



Note:

$$f(x) = \frac{x^2 - 9}{x - 3} = \frac{(x-3)(x+3)}{(x-3)} = x + 3.$$

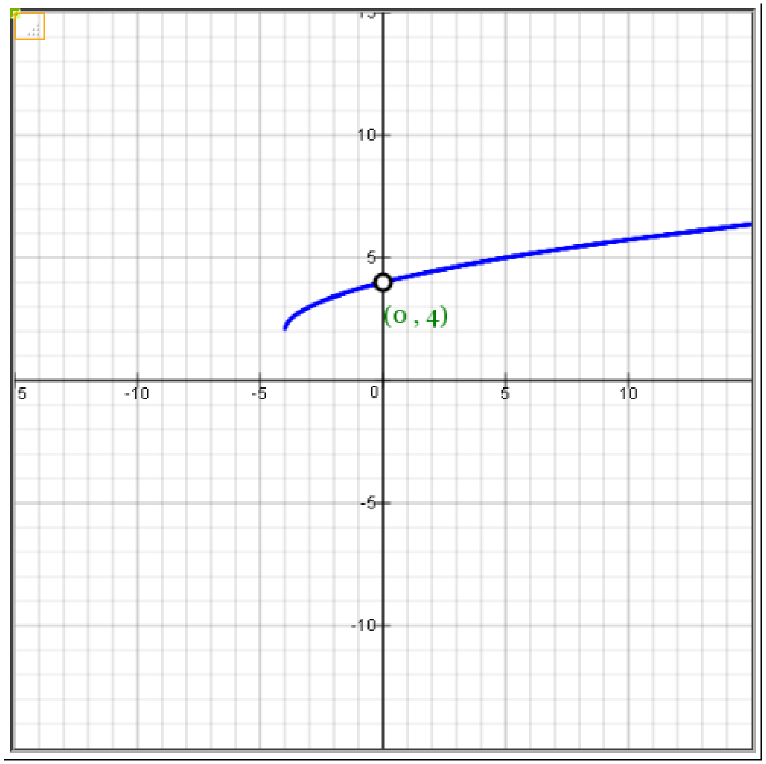
$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \lim_{x \to 3} \frac{(x-3)(x+3)}{(x-3)} = \lim_{x \to 3}(x+3) = 6$$

Example 13: Function with radical expression in numerator or denominator

Let $f(x) = \dfrac{x}{\sqrt{x+4}-2}$

Find $\lim_{x \to 0} f(x)$.

Note: when $x = 0$, $f(x) = \dfrac{x}{\sqrt{x+4}-2} = \dfrac{0}{0} = \text{undefined}$



Note: The conjugate of $\sqrt{x+4}-2$ is $\sqrt{x+4}+2$

Multiply expression by conjugate of denominator and simplify:

$$\frac{x}{\sqrt{x+4}-2}=\frac{x}{\sqrt{x+4}-2}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}$$

Recall:

$$\left(\sqrt{2}\right)^2=\left(\sqrt{2}\right)\left(\sqrt{2}\right)=\sqrt{4}=2; \qquad \left(\sqrt{5}\right)^2=\left(\sqrt{5}\right)\left(\sqrt{5}\right)=\sqrt{25}=5$$

$$\left(\sqrt{x}\right)^2=x; \qquad \left(\sqrt{x+5}\right)^2=x+5$$

$$\left(\sqrt{a+b}\right)^2=a+b$$

$$(a-b)(a+b)=a^2-b^2$$

$$(\sqrt{a}-b)(\sqrt{a}+b)=\left(\sqrt{a}\right)^2-b^2=a-b^2$$

$$\left(\sqrt{x+4}-2\right)\left(\sqrt{x+4}+2\right)=\left(\sqrt{x+4}\right)^2-2^2=x+4-4=x$$

Hence,

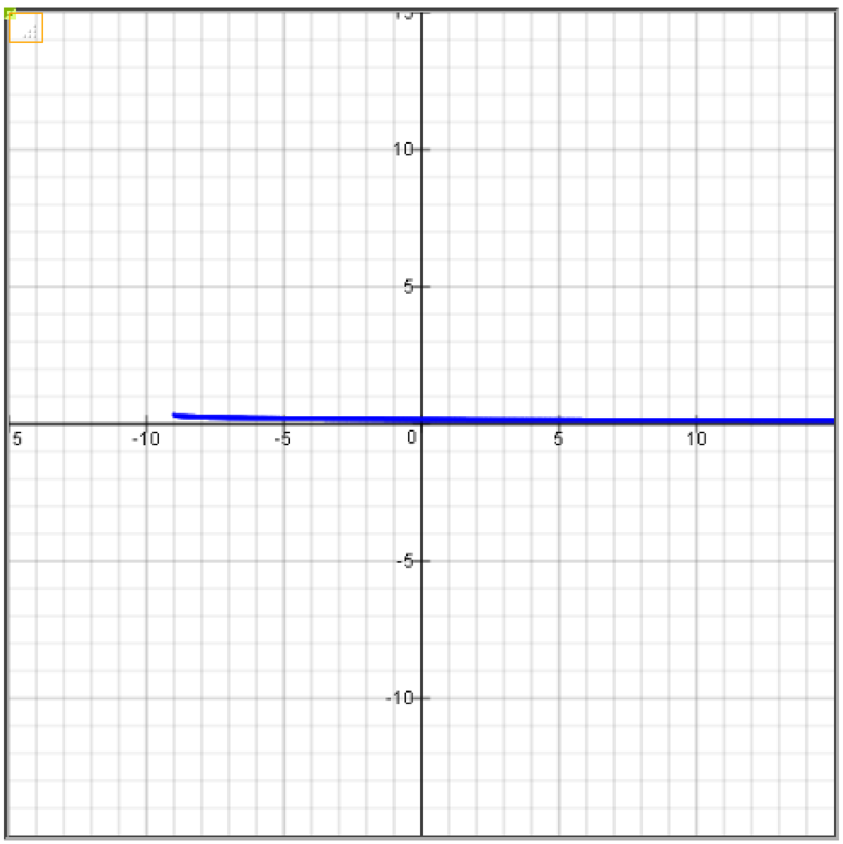
$$\frac{x}{\sqrt{x+4}-2}=\frac{x}{\sqrt{x+4}-2}\cdot\frac{\sqrt{x+4}+2}{\sqrt{x+4}+2}=\frac{(x)\left(\sqrt{x+4}+2\right)}{(x)}=\sqrt{x+4}+2$$

$$\lim_{x\to 0}\frac{x}{\sqrt{x+4}-2}=\lim_{x\to 0}\left(\sqrt{x+4}+2\right)=\sqrt{0+4}+2=2+2=4$$

Example 14: Function with radical expression in numerator or denominator

Let $f(x) = \dfrac{\sqrt{x+9}-3}{x}$

Note: when $x = 0$, $f(x) = \dfrac{\sqrt{x+9}-3}{x} = \dfrac{0}{0} = \text{undefined}$



Note:

Multiply expression by conjugate of denominator and simplify:

$$\frac{\sqrt{x+9}-3}{x} = \frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$$

Recall:

$$\left(\sqrt{2}\right)^2 = \left(\sqrt{2}\right)\left(\sqrt{2}\right) = \sqrt{4} = 2; \qquad \left(\sqrt{5}\right)^2 = \left(\sqrt{5}\right)\left(\sqrt{5}\right) = \sqrt{25} = 5$$

$$\left(\sqrt{x}\right)^2 = x; \qquad \left(\sqrt{x+5}\right)^2 = x+5$$

$$\left(\sqrt{a+b}\right)^2 = a+b$$

$$(a-b)(a+b) = a^2 - b^2$$

$$(\sqrt{a}-b)(\sqrt{a}+b) = \left(\sqrt{a}\right)^2 - b^2 = a - b^2$$

$$\left(\sqrt{x+9}-3\right)\left(\sqrt{x+9}+3\right) = \left(\sqrt{x+9}\right)^2 - 3^2 = x+9-9 = x$$

Hence,

$$\frac{\sqrt{x+9}-3}{x} = \frac{\sqrt{x+9}-3}{x} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3} = \frac{(x)}{(x)\left(\sqrt{x+9}+3\right)} = \frac{1}{\sqrt{x+9}+3}$$

$$\lim_{x\to 0} \frac{\sqrt{x+9}-3}{x} = \lim_{x\to 0}\left(\frac{1}{\sqrt{x+9}+3}\right) = \frac{1}{\sqrt{0+9}+3} = \frac{1}{6}$$

Example 15: Function with degree of numerator larger or equal to degree of denominator

$$\text{Let } f(x) = \frac{x^2 + 5x + 4}{x + 4}$$

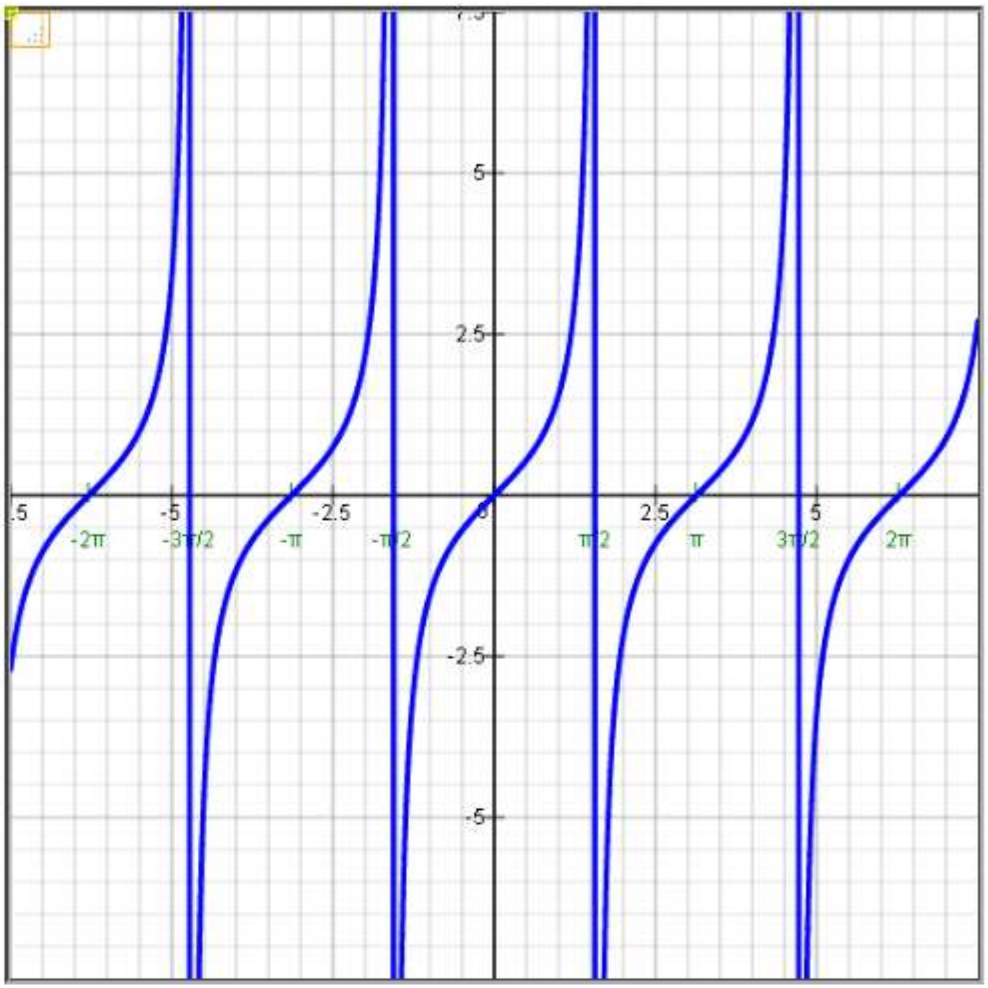
$$\text{Note: } \frac{x^2 + 5x + 4}{x + 4} = \frac{(x+4)(x+1)}{(x+4)} = x + 1$$

$$\lim_{x \to -4} f(x) = \lim_{x \to -4} \frac{x^2 + 5x + 4}{x + 4} = \lim_{x \to -4} (x + 1) = -3$$

Example 16: Trigonometric Function

Let $f(x) = \tan x$

Find $\lim_{x \to 2\pi} f(x)$.



Let $f(x) = \tan x$

Note from graph, $\lim_{x \to 2\pi^-} f(x) = 0$ and $\lim_{x \to 2\pi^+} f(x) = 0$.

Therefore, $\lim_{x \to 2\pi} f(x) = 0$

Example 17: Trigonometric Function

Let $f(x) = \sin \dfrac{\pi x}{4}$

Find $\lim\limits_{x \to 4} f(x)$.

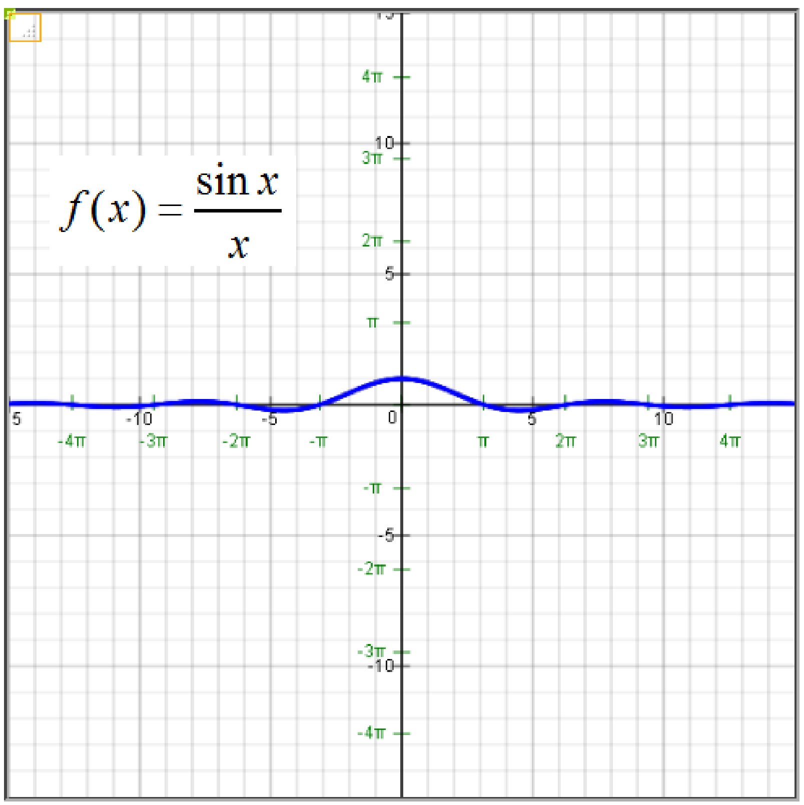
Note: As x approaches 4, $f(x) = \sin \dfrac{\pi x}{4}$ approaches $\sin \dfrac{\pi(4)}{4} = \sin \pi = 0$.

Therefore,

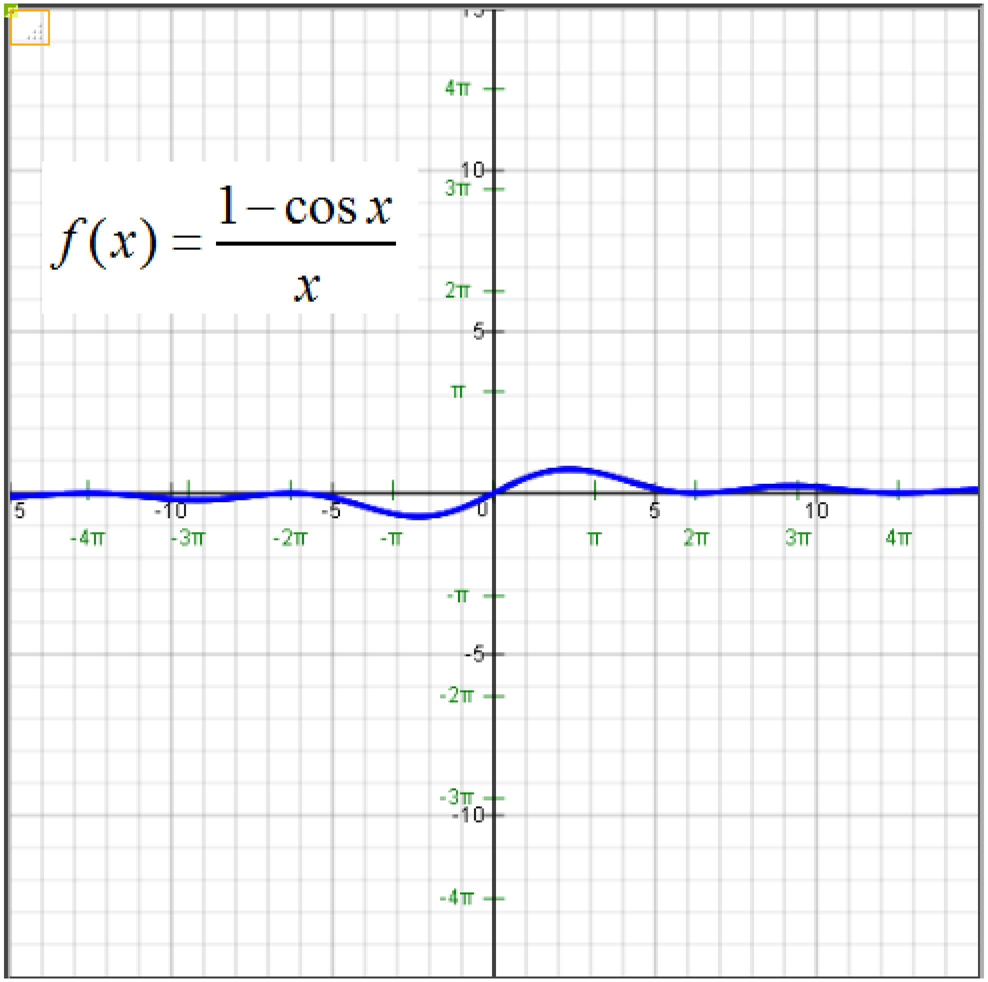
Find $\lim\limits_{x \to 4} f(x) = \lim\limits_{x \to 4} \sin \dfrac{\pi x}{4} = \sin \pi = 0$.

Example 18: Special Trigonometric Limits

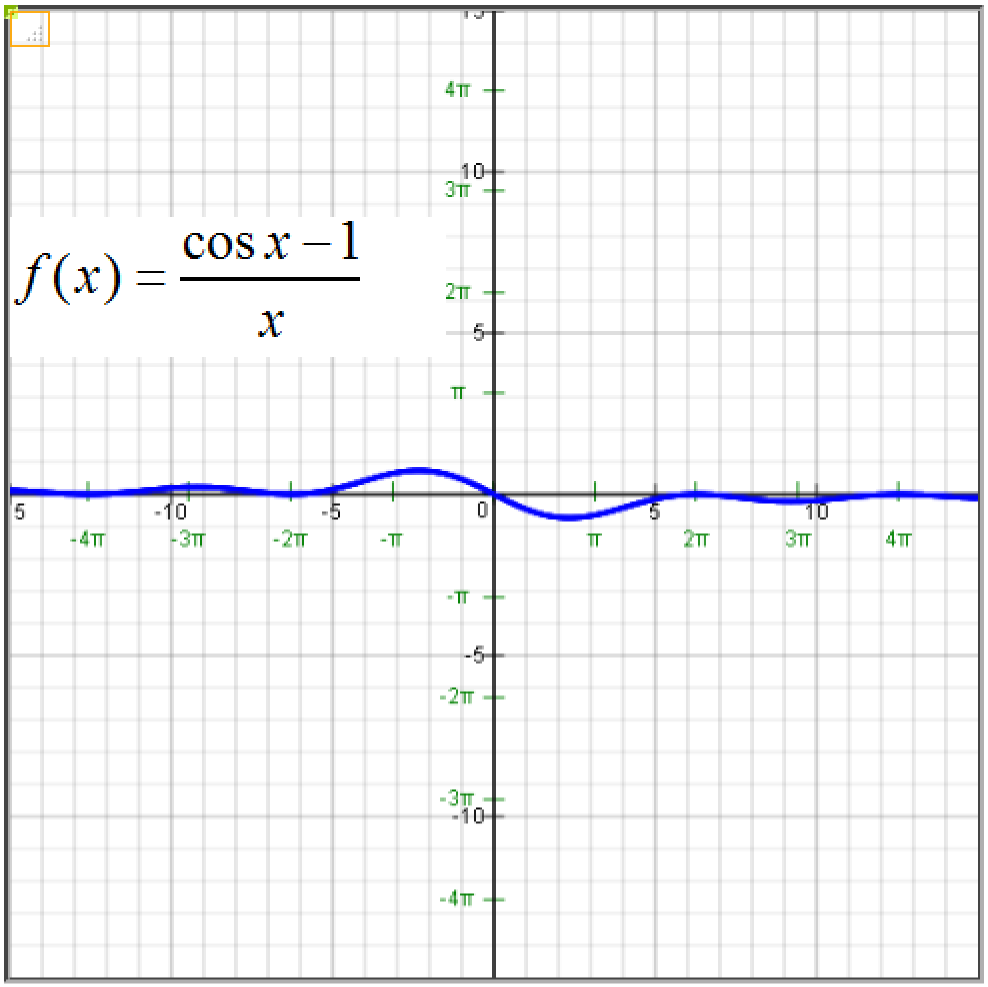
Theorem: $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 0;\qquad \lim\limits_{x\to 0}\dfrac{1-\cos x}{x} = 0;\qquad \lim\limits_{x\to 0}\dfrac{\cos x - 1}{x} = 0.$



Theorem: $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 0$



Theorem: $\lim_{x \to 0} \frac{1-\cos x}{x} = 0.$



Theorem: $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0.$

Find $\lim_{x \to 0} \frac{\sin x}{2x}$.

Note: $\frac{\sin x}{2x} = \frac{1}{2} \cdot \frac{\sin x}{x}$

$$\lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \left(\frac{1}{2} \cdot \frac{\sin x}{x} \right) = \left(\lim_{x \to 0} \frac{1}{2} \right) \cdot \left(\lim_{x \to 0} \frac{\sin x}{x} \right)$$

$$= (\text{limit of constant function}) \left(\lim_{x \to 0} \frac{\sin x}{x} \right)$$

$$= \left(\frac{1}{2} \right) \cdot (0) = 0$$

Find $\lim\limits_{x \to 0} \dfrac{3(1-\cos x)}{x}$.

Note: $\dfrac{3(1-\cos x)}{x} = 3 \cdot \dfrac{(1-\cos x)}{x}$

$$\lim_{x \to 0} \frac{3(1-\cos x)}{x} = \lim_{x \to 0}\left(3 \cdot \frac{(1-\cos x)}{x}\right) = \left(\lim_{x \to 0} 3\right) \cdot \left(\lim_{x \to 0} \frac{(1-\cos x)}{x}\right)$$

$$= \left(\text{limit of constant function}\right)\left(\lim_{x \to 0} \frac{\sin x}{x}\right)$$

$$= (3) \cdot (1) = 3$$

Example 19: Difference Quotient

Let $f(x) = 4x + 3.$

Find Difference Quotient = $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$ Think of Δx as a small number like 0.000001

Note:

$$f(x+\Delta x) = 4\left(x+\Delta x\right) + 3 = 4x + 4 \cdot \Delta x + 3$$

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{\left(4x + 4 \cdot \Delta x + 3\right) - (4x+3)}{\Delta x}$$

$$= \frac{4 \cdot \Delta x}{\Delta x} = 4$$

Note: x and Δx are two different variables.

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \left[\frac{\left(4x + 4 \cdot \Delta x + 3\right) - (4x+3)}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} \left[\frac{4 \cdot \Delta x}{\Delta x} \right]$$

$$= \lim_{\Delta x \to 0} 4 = \lim_{\Delta x \to 0} (\text{constant function}) = 4$$

Example 20: Difference Quotient

Let $f(x) = 4x^2 + 3$.

Find Difference Quotient = $\dfrac{f(x+\Delta x) - f(x)}{\Delta x}$. Think of Δx as a small number like 0.000001

Note:

$$f(x+\Delta x) = 4\left(x+\Delta x\right)^2 + 3 = 4\left(x+\Delta x\right)\left(x+\Delta x\right) + 3$$

$$= 4\left(x^2 + x\cdot\Delta x + x\cdot\Delta x + \left(\Delta x\right)^2\right) + 3$$

$$= 4\left(x^2 + 2x\cdot\Delta x + \left(\Delta x\right)^2\right) + 3$$

$$= 4x^2 + 8x\cdot\Delta x + 4\left(\Delta x\right)^2 + 3$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\left[4x^2+8x\cdot\Delta x+4\left(\Delta x\right)^2+3\right]-(4x^2+3)}{\Delta x}$$

$$=\frac{8x\cdot\Delta x+4\left(\Delta x\right)^2}{\Delta x}$$

$$=\frac{8x\cdot\Delta x}{\Delta x}+\frac{4\left(\Delta x\right)^2}{\Delta x}$$

$$=8x+4\Delta x$$

Note: $\dfrac{\left(\Delta x\right)^2}{\Delta x}=\dfrac{\left(\Delta x\right)\left(\Delta x\right)}{\Delta x}=\Delta x$

Note: x and Δx are two different variables.

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\left[8x+4\Delta x\right]$$

$$=\lim_{\Delta x\to 0}\left[8x\right]+\lim_{\Delta x\to 0}\left[4\cdot\Delta x\right]=8x+4(0)=8x$$

Note: Δx approaches 0; and x is not approaching 0.