Continuity and One-Sided Limits

Definition of Continuity:

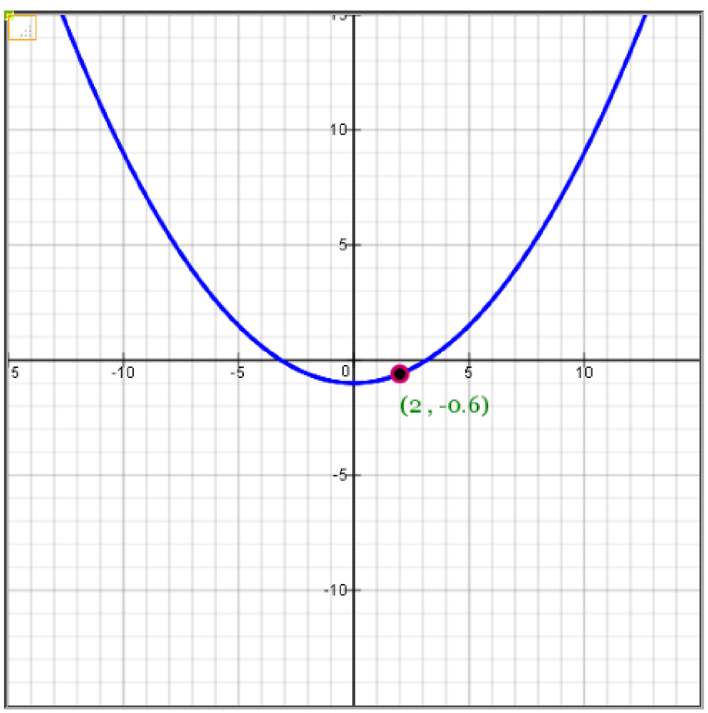
The function $f(x)$ is continuous at c if:

1) $f(c)$ is defined.

2) $\lim\limits_{x \to c} f(x) = f(c)$

Example 1: Polynomial Function

Let f(x) = 0.1x² - 1



Show that function $f(x)$ is continuous at $x = 2$.

a) Show that $f(x)$ is defined at $x = 2$: $f(2) = 0.1(2)^2 - 1 = -0.6$

b) Show $\lim_{x \to 2} f(x) = f(2)$

Limit from the left: $\lim_{x \to 2^-} f(x) = -0.6$

Limit from the right: $\lim_{x \to 2^+} f(x) = -0.6$

Hence, $\lim_{x \to 2} f(x) = -0.6$.

Hence, $f(2) = \lim_{x \to 2} f(x) = -0.6$.

Therefore, the function $f(x) = 0.1x^2 - 1$ is continuous at $x = 2$.

Show that function $f(x) = 0.1x^2 - 1$ is continuous at $x = 0$.

a) Show that $f(x)$ is defined at $x = 0$: $f(0) = 0.1(0)^2 - 1 = -1$ Note: $f(0) =$ finite number

b) Show that $f(0) = \lim_{x \to 2} f(x)$

Limit from the left: $\lim_{x \to 0^-} f(x) = -1$

Limit from the right: $\lim_{x \to 0^+} f(x) = -1$

Hence, $\lim_{x \to 2} f(x) = -1$

Hence, $f(0) = \lim_{x \to 2} f(x) = -1$.

Therefore, the function $f(x) = 0.1x^2 - 1$ is continuous at $x = 1$.

Show that function $f(x) = 0.1x^2 - 1$ is continuous at $x = -3$.

a) Show that $f(x)$ is defined at $x = -3$: $f(-3) = 0.1(-3)^2$ - 1 = -0.1 Note: $f(-3) =$ finite number

b) Show that $f(-3) = \lim_{x \to -3} f(x)$

Limit from the left: $\lim_{x \to -3^-} f(x) = -0.1$

Limit from the right: $\lim_{x \to -3^+} f(x) = -0.1$

Hence, $\lim_{x \to 2} f(x) = -0.1$

Hence, $f(0) = \lim_{x \to 2} f(x) = -0.1$.

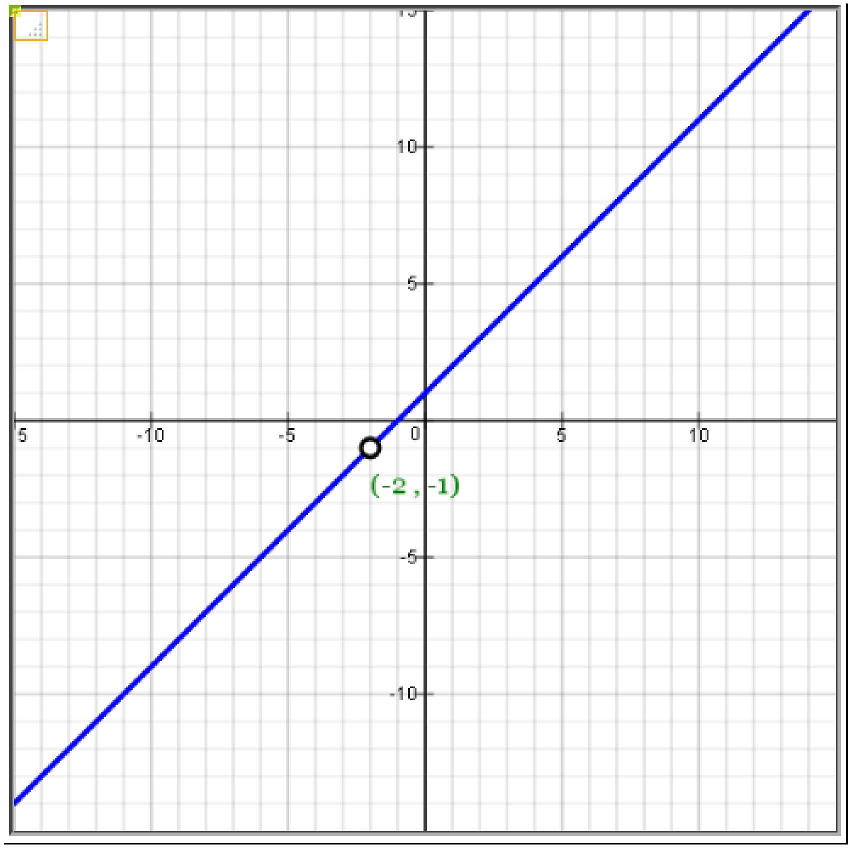
Therefore, the function $f(x) = 0.1x^2 - 1$ is continuous at $x = -3$.

Example 2:

Let $f(x) = \dfrac{x^2 + 3x + 2}{x + 2}$

Note: $\dfrac{x^2 + 3x + 2}{x + 2} = \dfrac{(x+2)(x+1)}{(x+2)} = x + 1$

Also, when $x = -2$, $f(x) = \dfrac{x^2 + 3x + 2}{x + 2} = \dfrac{(-2)^2 + 3(-2) + 2}{(-2) + 2} = \dfrac{0}{0} =$ undefined



$f(x)$ is discontinuous at $x = -2$.

$f(x)$ has a removable discontinuity at $x = -2$.

$f(x)$ is continuous on $(-\infty, -2) \cup (-2, \infty)$

Show that the function $f(x) = \dfrac{x^2 + 3x + 2}{x + 2}$ is not continuous at $x = -2$.

a) $f(x)$ is undefined at $x = 2$: $\quad f(2) = \dfrac{(-2)^2 + 3(-2) + 2}{(-2) + 2} = \dfrac{0}{0} = \text{undefined}$

b) Limit from the left: $\lim\limits_{x \to -2^-} f(x) = -1$

Limit from the right: $\lim\limits_{x \to -2^+} f(x) = -1$

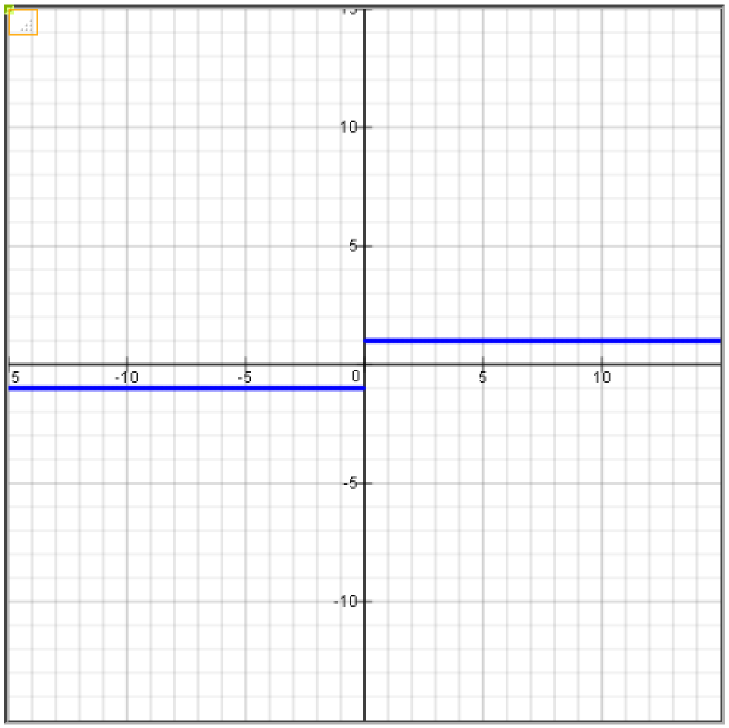
Hence, $\lim\limits_{x \to -2} f(x) = -1$

Hence, $f(2) \neq \lim\limits_{x \to -2^-} f(x)$.

Therefore, the function $f(x) = \dfrac{x^2 + 3x + 2}{x + 2}$ is not continuous at $x = -2$.

Example 3: Let $f(x) = \dfrac{|x|}{x}$

Note: When x = 0, $f(x) = \dfrac{|x|}{x} = \dfrac{0}{0} = undefined$



Show that $f(x) = \dfrac{|x|}{x}$ is not continuous at $x = 0$.

a) $f(0) = \dfrac{|0|}{0} = \dfrac{0}{0} = undefined$; $f(x)$ is undefined at $x = 0$.

b) Limit from the left: $\lim\limits_{x \to 0^-} f(x) = -1$

Limit from the right: $\lim\limits_{x \to 0^+} f(x) = 1$

Hence, $\lim\limits_{x \to 0} f(x) = 1$

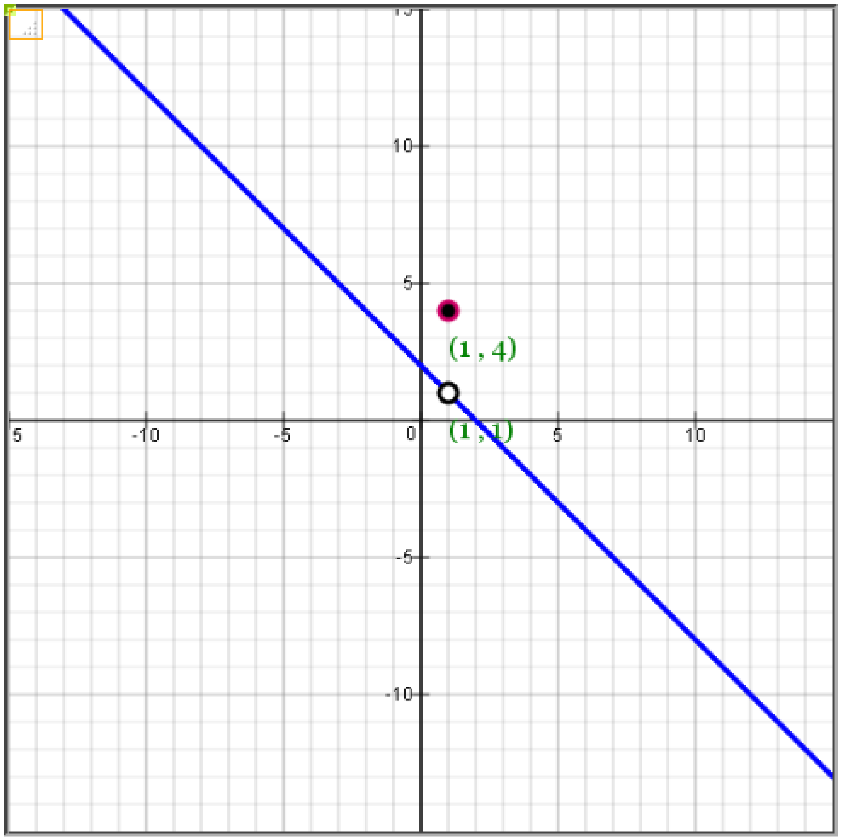
Hence, $f(0) \neq \lim\limits_{x \to 0} f(x)$

Therefore, $f(x) = \dfrac{|x|}{x}$ is not continuous at $x = 0$.

Also, $f(x) = \dfrac{|x|}{x}$ has nonremovable discontinuity at $x = 0$.

Example 4:

$$\text{Let } f(x) = \begin{cases} 2 - x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$$



$f(x)$ is discontinuous at $x = 1$.

$f(x)$ has a removable discontinuity at $x = 1$.

$f(x)$ is continuous on $(-\infty,1) \cup (1,\infty)$

Show that $f(x) = \begin{cases} 2 - x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$ is not continuous at $x = 1$.

a) $f(1) = 4$; $f(x)$ is defined at $x = 1$.

b) Limit from the left: $\lim_{x \to 1^-} f(x) = 1$

Limit from the right: $\lim_{x \to 1^+} f(x) = 1$

Hence, $\lim_{x \to 1} f(x) = 1$

Hence, $f(1) \neq \lim_{x \to 1} f(x)$

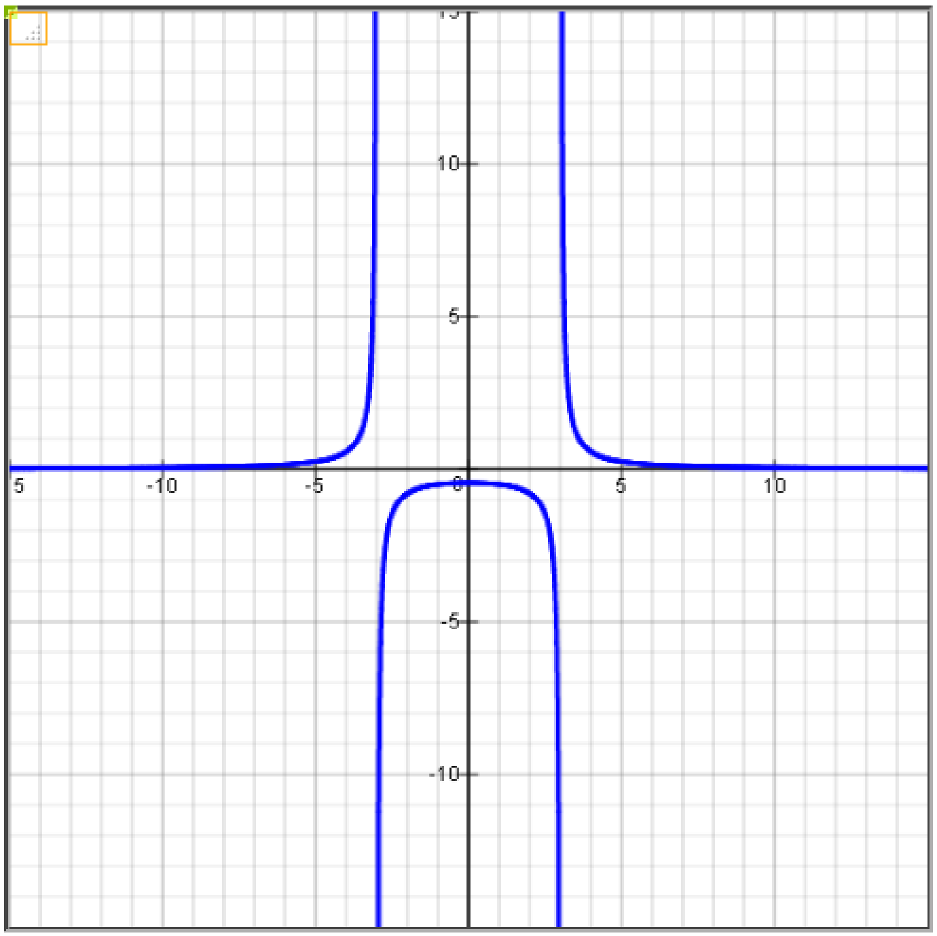
Therefore, $f(x) = \begin{cases} 2 - x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$ is not continuous at $x = 1$.

Also, $f(x) = \begin{cases} 2 - x & \text{if } x \neq 1 \\ 4 & \text{if } x = 1 \end{cases}$ has a removable discontinuity at $x = 1$.

Example 5:

Let $f(x) = \dfrac{4}{x^2 - 9}$

Discuss continuity of the function.



$f(x)$ is discontinuous at $x = -3$ and at $x = 3$.

$f(x)$ has nonremovable discontinuities at $x = -3$ and at $x = 3$.

$f(x)$ is continuous on $(-\infty,-3) \cup (-3,3) \cup (3,\infty)$

a) $f(-3) = undefined$. Therefore, $f(x)$ is discontuous at $x = -3$.

$f(x)$ has a nonremovable disconuity at $x = -3$.

b) $f(3) = undefined$. Therefore, $f(x)$ is discontuous at $x = 3$.

$f(x)$ has a nonremovable disconuity at $x = 3$.

c) $f(x)$ is continuous everywhere except at $x = -3$ and at $x = 3$.

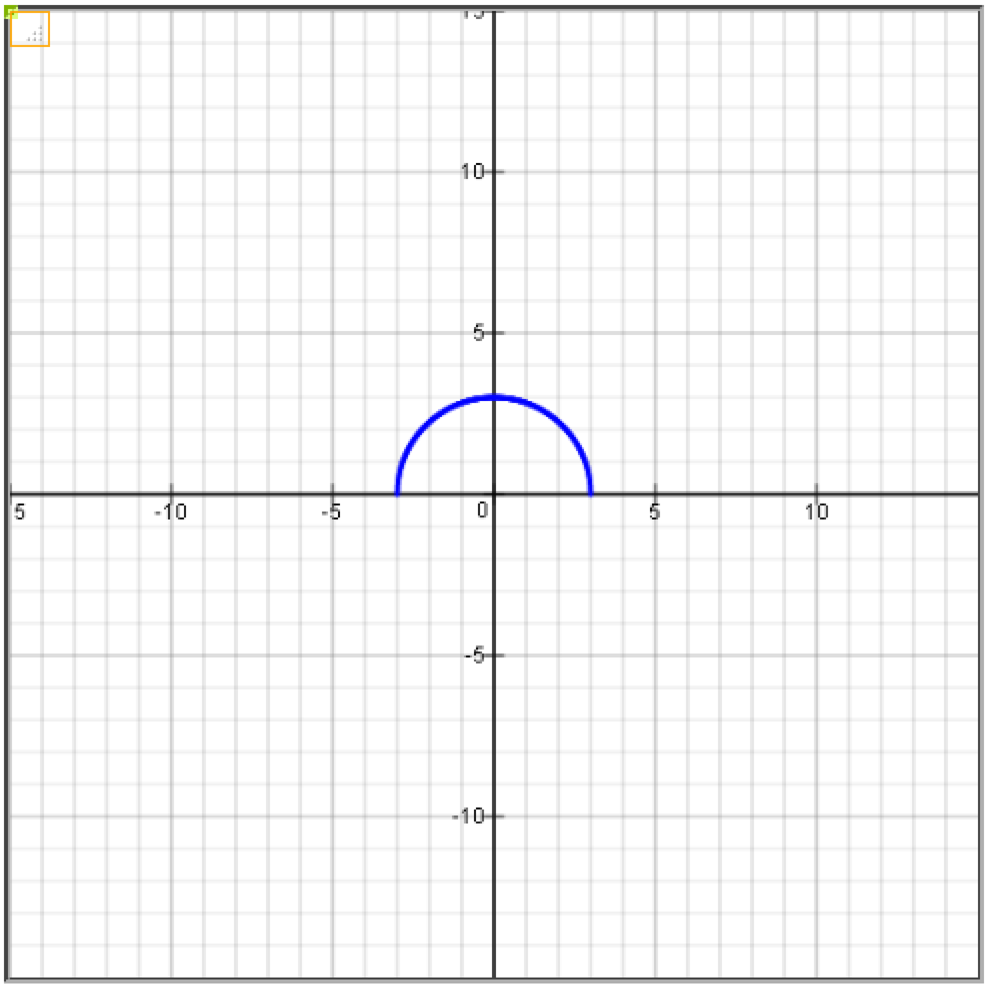
Summary:

$f(x)$ is continuous on $(-\infty, -3) \cup (-3,\ 3) \cup (3,\ \infty)$

Example 6:

Let $f(x) = \sqrt{9 - x^2}$.

Show that $f(x)$ is continuous on the interval [-3, 3].



We need to show:

a) f(x) is continuous on the open interval (-3, 3).

From graph, we can see that f(x) is continuous on (-3, 3).

b) $f(x) = \sqrt{9 - x^2}$.

Show $\lim_{x \to -3^+} f(x) = f(-3)$.

$$f(-3) = \sqrt{9 - (-3)^2} = 0$$

$$\lim_{x \to -3^+} f(x) = \lim_{x \to -3^+} \sqrt{9 - x^2} = \sqrt{\lim_{x \to -3^+} (9 - x^2)} = \sqrt{0} = 0$$

Therefore, $\lim_{x \to -3^+} f(x) = f(-3)$.

c) $f(x) = \sqrt{9 - x^2}$.

Show $\lim_{x \to 3^-} f(x) = f(3)$.

$$f(3) = \sqrt{9 - (3)^2} = 0$$

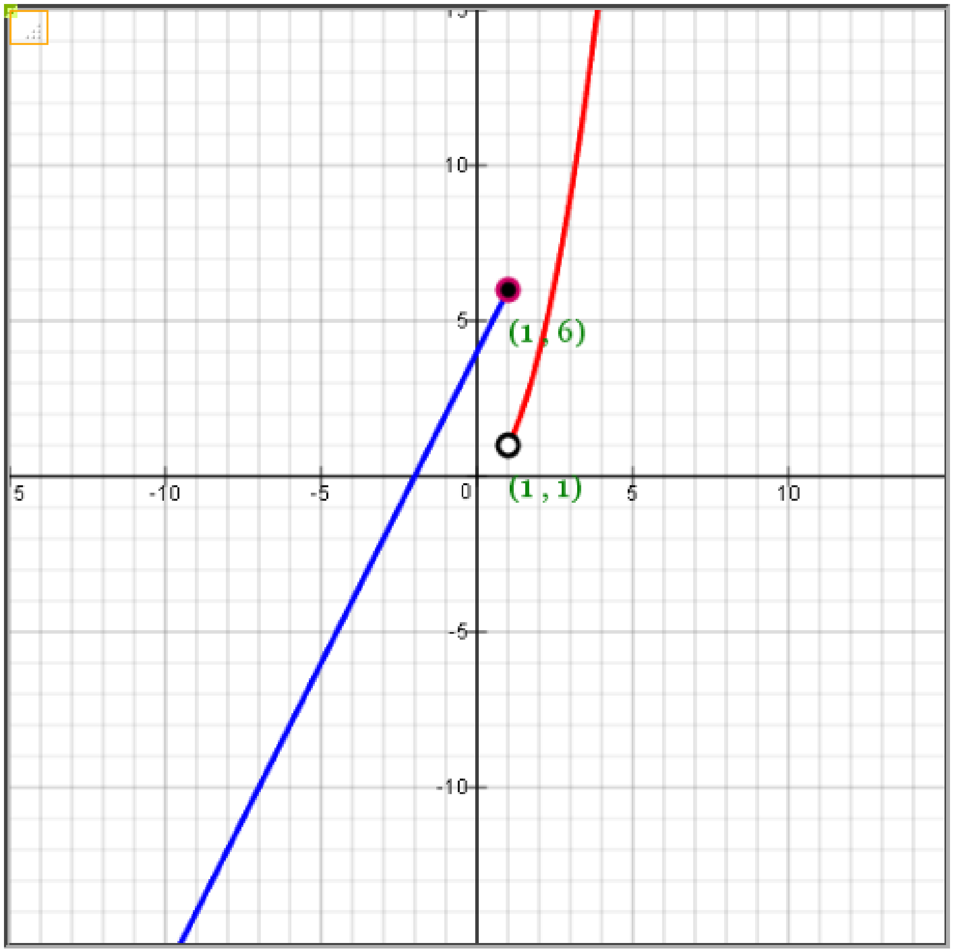
$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \sqrt{9 - x^2} = \sqrt{\lim_{x \to 3^-} (9 - x^2)} = \sqrt{0} = 0$$

Therefore, $\lim_{x \to 3^-} f(x) = f(3)$.

Example 7:

Let $f(x) = \begin{cases} 2x+4 & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$

Show that $f(x)$ is discontinuous at $x = 1$.



$f(x)$ is discontinuous at $x = 1$.

$f(x)$ has a nonremovable discontinuity at $x = 1$.

$f(x)$ is continuous on $(-\infty, 1) \cup (1, \infty)$

Let $f(x) = \begin{cases} 2x+4 & \text{if } x \le 1 \\ x^2 & \text{if } x > 1 \end{cases}$

a) $f(1) = 2x + 4 = 2(1) + 4 = 6$

b) $\lim_{x \to 1^-} f(x) = 6$

$\lim_{x \to 1^+} f(x) = 1$

Hence, $\lim_{x \to 1} f(x) = 1$

Hence, $f(1) \ne \lim_{x \to 1} f(x)$

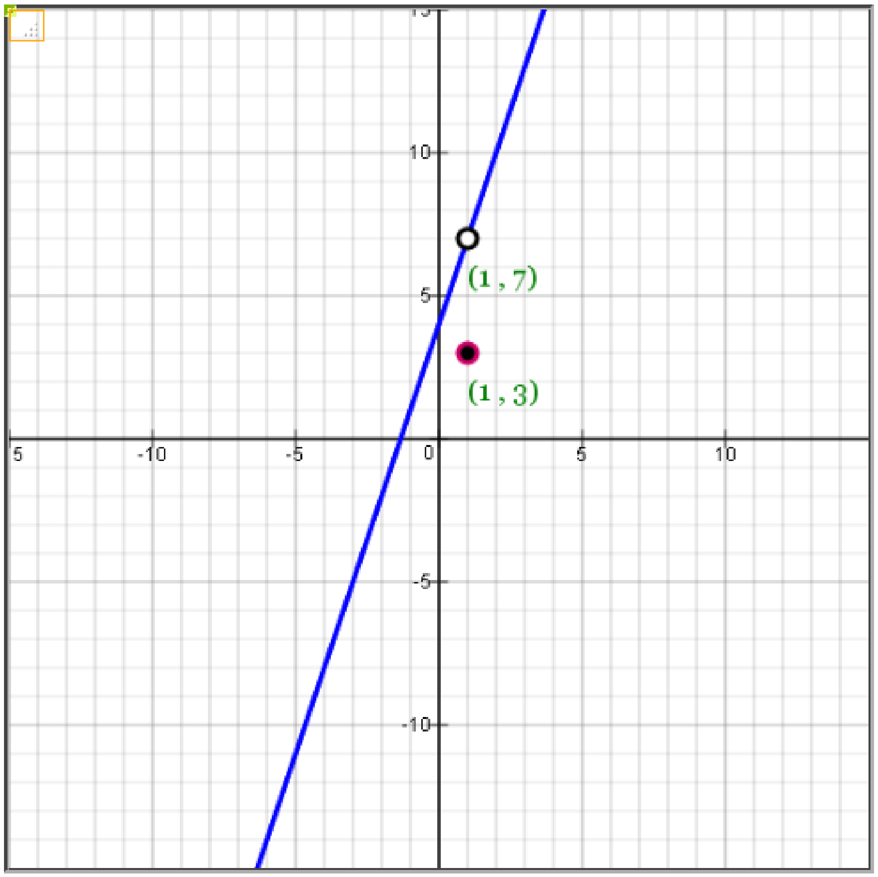
Therefore, $f(x)$ is discontinuous at $x = 1$.

Also, $f(x)$ has a nonremovable discontinuity at $x = 1$.

Example 8:

Let $f(x)=\begin{cases}3x+4 & \text{if } x\neq 1\\ 3 & \text{if } x=1\end{cases}$

Show that $f(x)$ is discontinuous at $x=1$.



$f(x)$ is discontinuous at $x=1$.

$f(x)$ has a removable discontinuity at $x=1$.

$f(x)$ is continuous on $(-\infty,1)\cup(1,\infty)$

Let $f(x) = \begin{cases} 3x+4 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$

a) $f(1) = 3$

b) $\lim_{x \to 1^-} f(x) = 7$

$\lim_{x \to 1^+} f(x) = 7$

Hence, $\lim_{x \to 1} f(x) = 7$

Hence, $f(1) \neq \lim_{x \to 1^+} f(x)$

Therefore, $f(x)$ is discontinuous at $x = 1$.

Also, $f(x)$ has a removable discontinuity at $x = 1$.