

## Derivative and Slope of Tangent Line

$$m = \text{Slope of Tangent Line at } c = \lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x}$$

Example 1:

Let  $f(x) = 3x^2 - 2$ . Find the slope of the tangent line at  $c = 1$ .

$$f(x) = 3x^2 - 2$$

$$f(c) = f(1) = 3(1)^2 - 2 = 1 \quad \text{Note: When } x = 1, y = 1$$

$$f(c + \Delta x) = f(1 + \Delta x) = 3(1 + \Delta x)^2 - 2$$

$$f(c + \Delta x) = 3[1^2 + 2(1)(\Delta x) + (\Delta x)^2] - 2$$

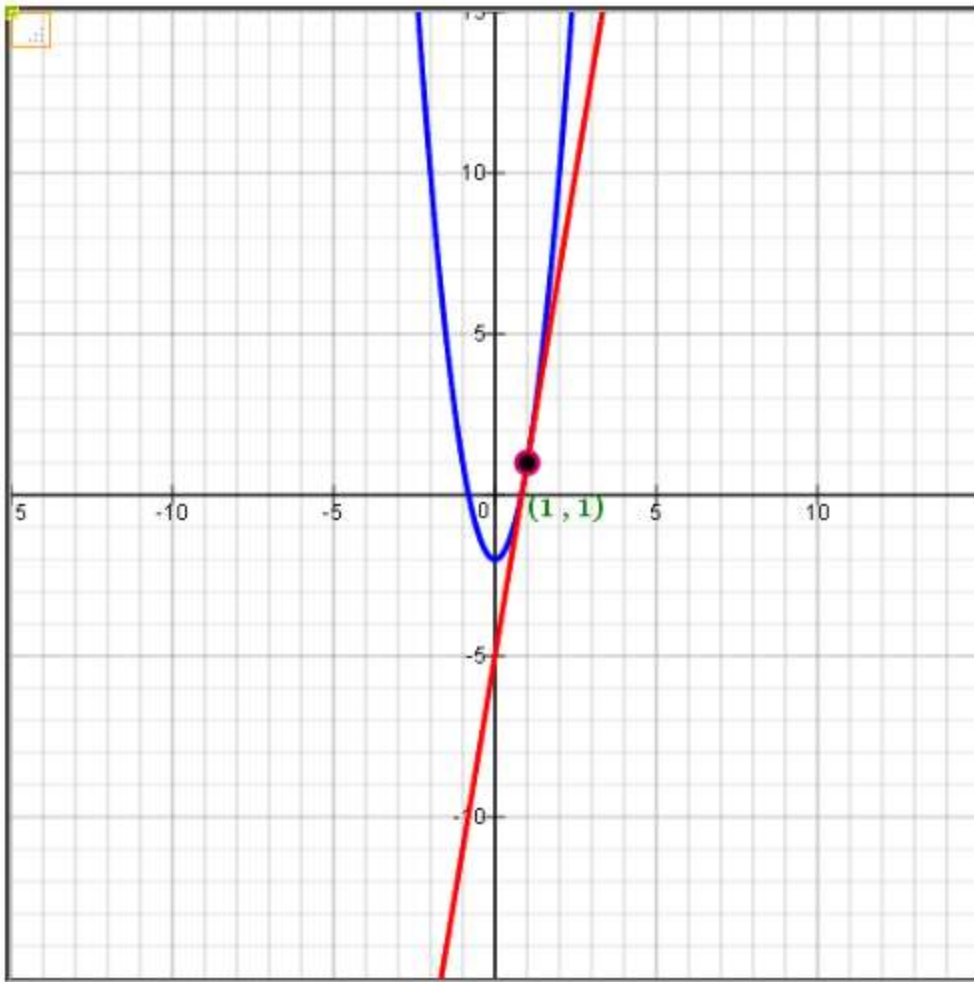
$$f(c + \Delta x) = 3 + 6\Delta x + 3(\Delta x)^2 - 2 = 1 + 6\Delta x + 3(\Delta x)^2$$

$$f(c + \Delta x) - f(c) = [1 + 6\Delta x + 3(\Delta x)^2] - [1] = 6\Delta x + 3(\Delta x)^2$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = \frac{6\Delta x + 3(\Delta x)^2}{\Delta x} = \frac{6\Delta x}{\Delta x} + \frac{3(\Delta x)^2}{\Delta x}$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = 6 + 3\Delta x$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (6 + 3\Delta x) = 6 + 3(0) = 6$$



Equation of Tangent Line:  $y - y_1 = m(x - x_1)$

Line contains (1, 1) and has slope of 6.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 6(x - 1)$$

Example 2:

Let  $f(x) = 3x^2 - 2$ . Find the slope of the tangent line at  $c = 0$ .

$$f(x) = 3x^2 - 2$$

$$f(c) = f(0) = 3(0)^2 - 2 = -2 \quad \text{Note: When } x = 0, y = -2$$

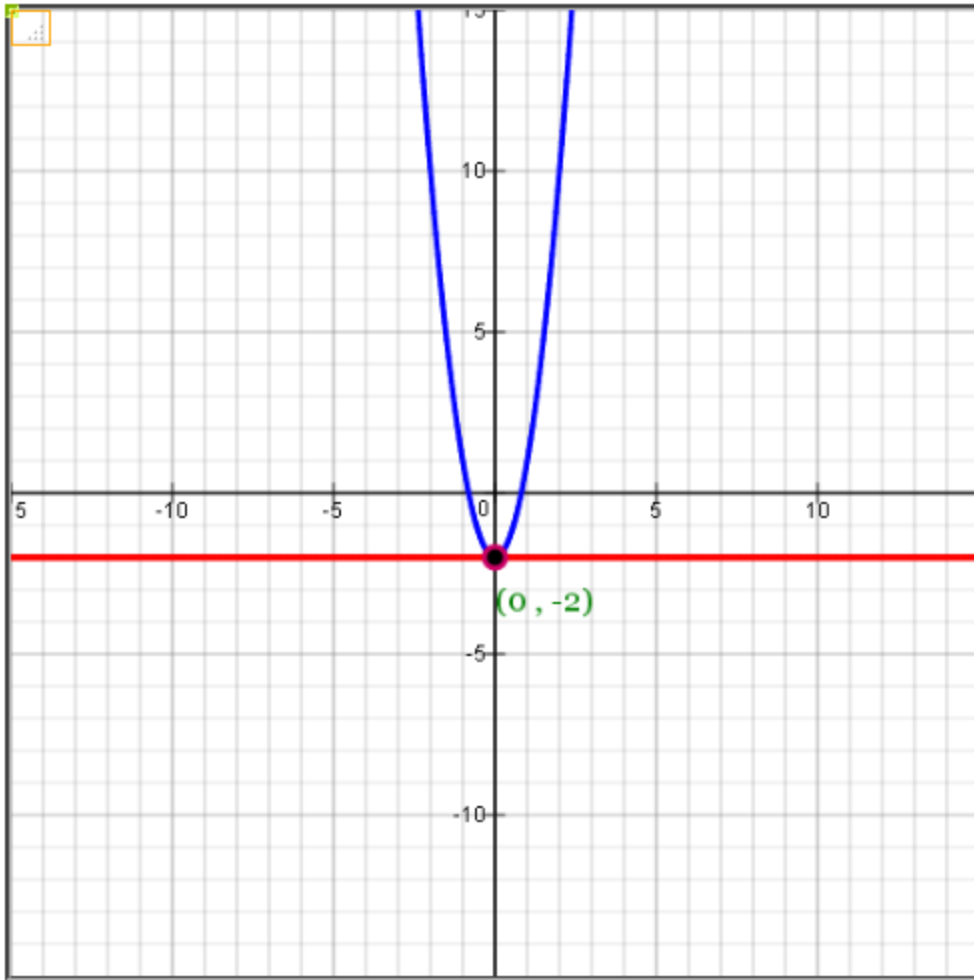
$$f(c + \Delta x) = f(0 + \Delta x) = 3(\Delta x)^2 - 2$$

$$f(c + \Delta x) = 3(\Delta x)^2 - 2$$

$$f(c + \Delta x) - f(c) = [3(\Delta x)^2 - 2] - [-2] = 3(\Delta x)^2$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = \frac{3(\Delta x)^2}{\Delta x} = 3\Delta x$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (3\Delta x) = 3(0) = 0$$



Equation of Tangent Line:  $y = -2$

### Example 3:

Let  $f(x) = 3x^2 - 2$ . Find the slope of the tangent line at  $c = -1$ .

$$f(x) = 3x^2 - 2$$

$$f(c) = f(-1) = 3(-1)^2 - 2 = 1$$

Note: When  $x = -1$ ,  $y = 1$

$$f(c + \Delta x) = f(-1 + \Delta x) = 3(-1 + \Delta x)^2 - 2$$

$$f(c + \Delta x) = 3\left[(-1)^2 + 2(-1)(\Delta x) + (\Delta x)^2\right] - 2$$

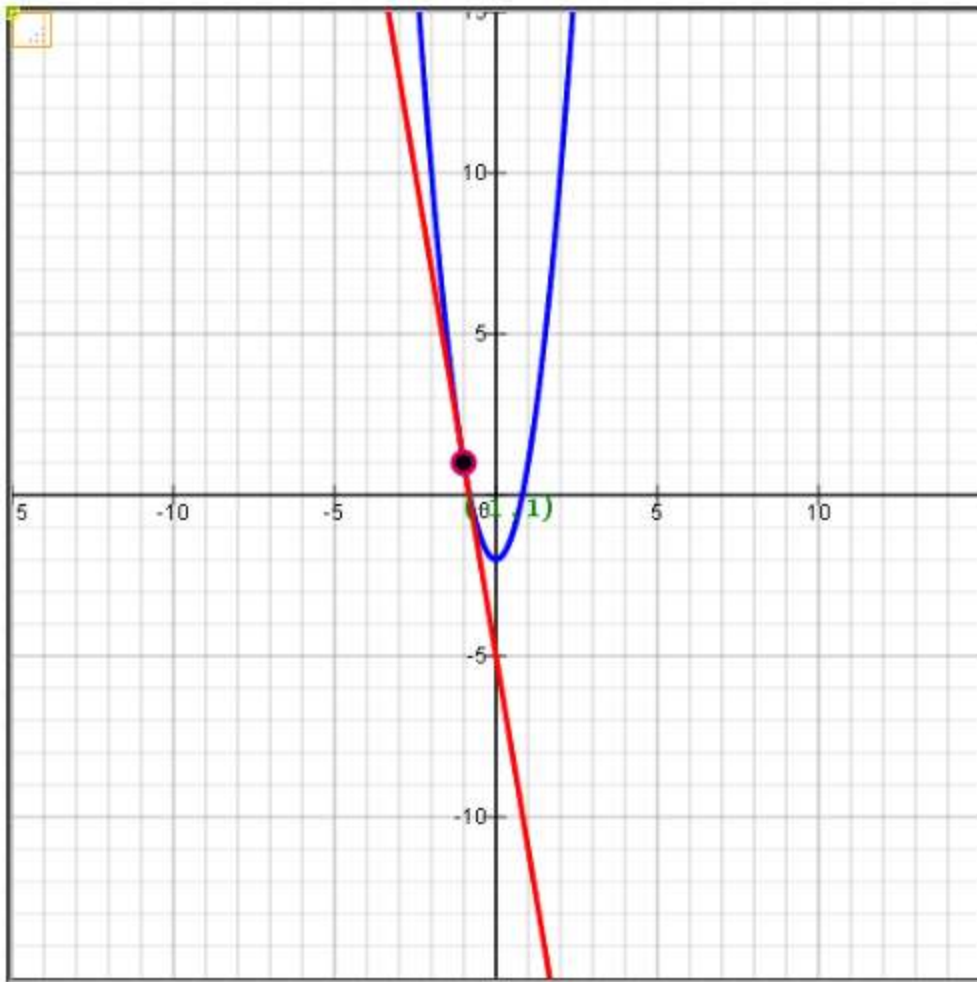
$$f(c + \Delta x) = 3 - 6\Delta x + 3(\Delta x)^2 - 2 = 1 - 6\Delta x + 3(\Delta x)^2$$

$$f(c + \Delta x) - f(c) = \left[1 - 6\Delta x + 3(\Delta x)^2\right] - [1] = -6\Delta x + 3(\Delta x)^2$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = \frac{-6\Delta x + 3(\Delta x)^2}{\Delta x} = \frac{-6\Delta x}{\Delta x} + \frac{3(\Delta x)^2}{\Delta x}$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = -6 + 3\Delta x$$

$$\lim_{\Delta x \rightarrow 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \rightarrow 0} (-6 + 3\Delta x) = -6 + 3(0) = -6$$



Equation of Tangent Line:  $y - y_1 = m(x - x_1)$

Line contains  $(-1, 1)$  and has slope of  $-6$ .

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -6(x - -1)$$

$$y - 1 = -6(x + 1)$$

Example 5:

Let  $f(x) = \frac{1}{x+4}$ . Find the derivative of the function  $f(x)$ .

$$f(x) = \frac{1}{x+4}$$

$$f(x + \Delta x) = \frac{1}{(x + \Delta x) + 4}$$

$$f(x + \Delta x) - f(x) = \frac{1}{(x + \Delta x) + 4} - \frac{1}{x + 4}$$

Note:  $\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd}$

$$f(x + \Delta x) - f(x) = \frac{1}{(x + \Delta x) + 4} - \frac{1}{x + 4}$$

$$= \frac{(1)(x + 4) - (1)[(x + \Delta x) + 4]}{[(x + \Delta x) + 4][x + 4]}$$

$$= \frac{x + 4 - x - \Delta x - 4}{[(x + \Delta x) + 4][x + 4]}$$

$$= \frac{-\Delta x}{[(x + \Delta x) + 4][x + 4]}$$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-\Delta x}{[(x + \Delta x) + 4][x + 4] \Delta x}$$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-\Delta x}{[(x + \Delta x) + 4][x + 4]} \cdot \frac{1}{\Delta x}$$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{-1}{[(x + \Delta x) + 4][x + 4]}$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{-1}{[(x + \Delta x) + 4][x + 4]}$$

$$= \frac{-1}{[(x + 0) + 4][x + 4]} = \frac{-1}{[x + 4][x + 4]} = \frac{-1}{(x + 4)^2}$$



Example 6:

Let  $f(x) = \sqrt{x+4}$ . Find the derivative of the function  $f(x)$ .

$$f(x) = \sqrt{x+4}$$

$$f(x + \Delta x) = \sqrt{x + \Delta x + 4}$$

$$f(x + \Delta x) - f(x) = \sqrt{x + \Delta x + 4} - \sqrt{x + 4}$$

Note:  $(\sqrt{x} - \sqrt{y})(\sqrt{x} + \sqrt{y}) = (\sqrt{x})^2 - (\sqrt{y})^2 = x - y$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\sqrt{x + \Delta x + 4} - \sqrt{x + 4}}{\Delta x}$$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(\sqrt{x + \Delta x + 4} - \sqrt{x + 4})}{(\Delta x)} \cdot \frac{(\sqrt{x + \Delta x + 4} + \sqrt{x + 4})}{(\sqrt{x + \Delta x + 4} + \sqrt{x + 4})}$$

Note:  $(\sqrt{a} - \sqrt{b})(\sqrt{a} + \sqrt{b}) = (\sqrt{a})^2 - (\sqrt{b})^2 = a - b$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{(x + \Delta x + 4) - (x + 4)}{(\Delta x)(\sqrt{x + \Delta x + 4} + \sqrt{x + 4})}$$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{\Delta x}{(\Delta x)(\sqrt{x + \Delta x + 4} + \sqrt{x + 4})}$$

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{1}{(\sqrt{x + \Delta x + 4} + \sqrt{x + 4})}$$

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{1}{(\sqrt{x + \Delta x + 4} + \sqrt{x + 4})} \\ &= \frac{1}{(\sqrt{x + 0 + 4} + \sqrt{x + 4})} = \frac{1}{(\sqrt{x + 4} + \sqrt{x + 4})} = \frac{1}{2\sqrt{x + 4}} \end{aligned}$$

## Example 7: Constant Function

Let  $f(x) = 4$ . Find the derivative of the function  $f(x)$ .

$$f(x) = 4$$

$$f(x + \Delta x) = 4$$

$$f(x + \Delta x) - f(x) = [4] - [4]$$

$$f(x + \Delta x) - f(x) = 0$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{0}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (0)$$

$$= \lim_{\Delta x \rightarrow 0} (0) = 0$$

For constant function,  $f'(x) = 0$ .

Example 8:

Let  $f(x) = x$ . Find the derivative of the function  $f(x)$ .

$$f(x) = x$$

$$f(x + \Delta x) = x + \Delta x$$

$$f(x + \Delta x) - f(x) = [x + \Delta x] - [x]$$

$$f(x + \Delta x) - f(x) = \Delta x$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} (1)$$

$$= \lim_{\Delta x \rightarrow 0} (1) = 1$$

Example 9:

Let  $f(x) = x^2$ . Find the derivative of the function  $f(x)$ .

$$f(x) = x^2$$

$$f(x + \Delta x) = (x + \Delta x)^2 = x^2 + 2x \cdot \Delta x + (\Delta x)^2$$

$$f(x + \Delta x) - f(x) = \left[ x^2 + 2x \cdot \Delta x + (\Delta x)^2 \right] - \left[ x^2 \right]$$

$$f(x + \Delta x) - f(x) = 2x \cdot \Delta x + (\Delta x)^2$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{2x \cdot \Delta x + (\Delta x)^2}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left( \frac{2x \cdot \Delta x}{\Delta x} + \frac{(\Delta x)^2}{\Delta x} \right)$$

$$= \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = \lim_{\Delta x \rightarrow 0} (2x + 0) = 2x$$

Example 10:

Let  $f(x) = x^3$ . Find the derivative of the function  $f(x)$ .

$$f(x) = x^3$$

$$f(x + \Delta x) = (x + \Delta x)^3 = (x + \Delta x)^2(x + \Delta x)$$

$$= \left[ x^2 + 2x \cdot \Delta x + (\Delta x)^2 \right] (x + \Delta x)$$

$$= x^3 + 2x^2 \cdot \Delta x + x(\Delta x)^2 + x^2 \cdot \Delta x + 2x \cdot (\Delta x)^2 + (\Delta x)^3$$

$$= x^3 + 3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$f(x + \Delta x) - f(x) = \left[ x^3 + 3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3 \right] - \left[ x^3 \right]$$

$$f(x + \Delta x) - f(x) = 3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \left( \frac{3x^2 \cdot \Delta x}{\Delta x} + \frac{3x(\Delta x)^2}{\Delta x} + \frac{(\Delta x)^3}{\Delta x} \right)$$

$$= \lim_{\Delta x \rightarrow 0} \left( 3x^2 + 3x\Delta x + (\Delta x)^2 \right) = \lim_{\Delta x \rightarrow 0} \left( 3x^2 + 3x(0) + (0)^2 \right) = 3x^2$$

## Summary:

For  $f(x) = c$  (constant function),  $f'(x) = 0$

For  $f(x) = x$  (constant function),  $f'(x) = 1$

For  $f(x) = x^2$  (constant function),  $f'(x) = 2x$

For  $f(x) = x^3$  (constant function),  $f'(x) = 3x^2$

For  $f(x) = x^4$  (constant function),  $f'(x) = 4x^3$

For  $f(x) = x^5$  (constant function),  $f'(x) = 5x^4$

## In general:

Let  $f(x) = x^n$ . Find the derivative of the function  $f(x)$ .

$$f'(x) = nx^{n-1}$$