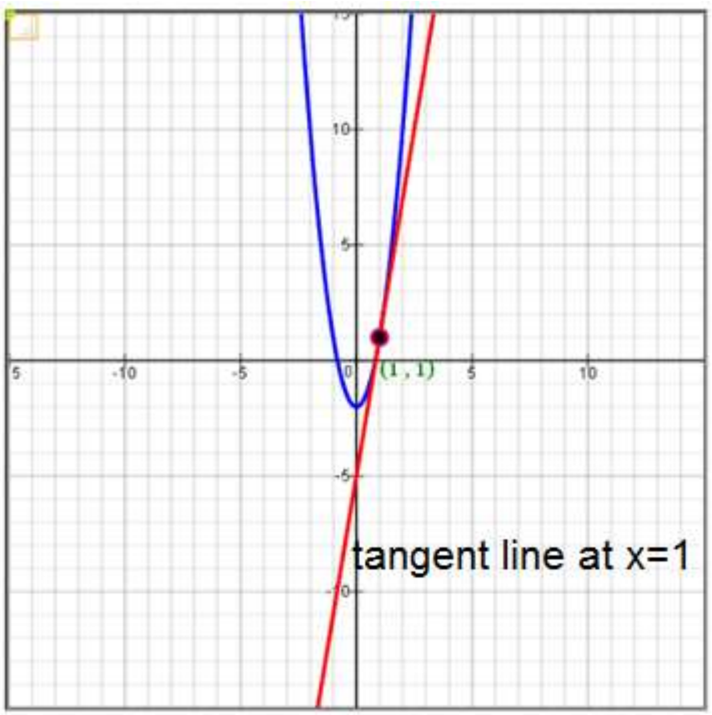
Derivative and Slope of Tangent Line

$$m = \text{Slope of Tangent Line at } c = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

Example 1: Let $f(x) = 3x^2 - 2$.

Find the slope of the tangent line at $x = 1$ or the tangent line passing through $(1,1)$.



$$f(x) = 3x^2 - 2$$

$$f(c) = f(1) = 3(1)^2 - 2 = 1$$ Note: When $x = 1, y = 1$

$$f(c + \Delta x) = f(1 + \Delta x) = 3(1 + \Delta x)^2 - 2$$

$$f(c + \Delta x) = 3\left[1^2 + 2(1)(\Delta x) + (\Delta x)^2\right] - 2$$

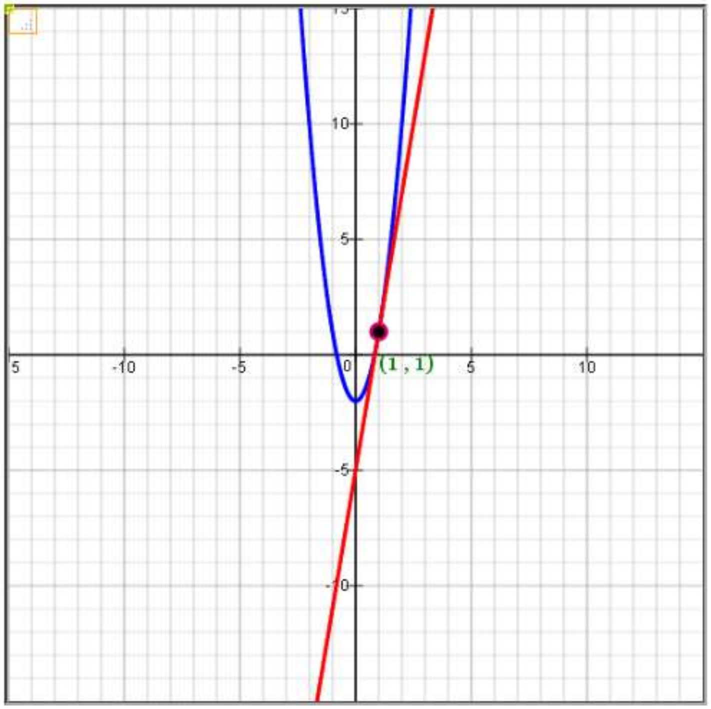
$$f(c + \Delta x) = 3 + 6\Delta x + 3(\Delta x)^2 - 2 = 1 + 6\Delta x + 3(\Delta x)^2$$

$$f(c + \Delta x) - f(c) = \left[1 + 6\Delta x + 3(\Delta x)^2\right] - [1] = 6\Delta x + 3(\Delta x)^2$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = \frac{6\Delta x + 3(\Delta x)^2}{\Delta x} = \frac{6\Delta x}{\Delta x} + \frac{3(\Delta x)^2}{\Delta x}$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = 6 + 3\Delta x$$

slope of tangent line = $\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} (6 + 3\Delta x) = 6 + 3(0) = 6$



Equation of Tangent Line: $y - y_1 = m(x - x_1)$

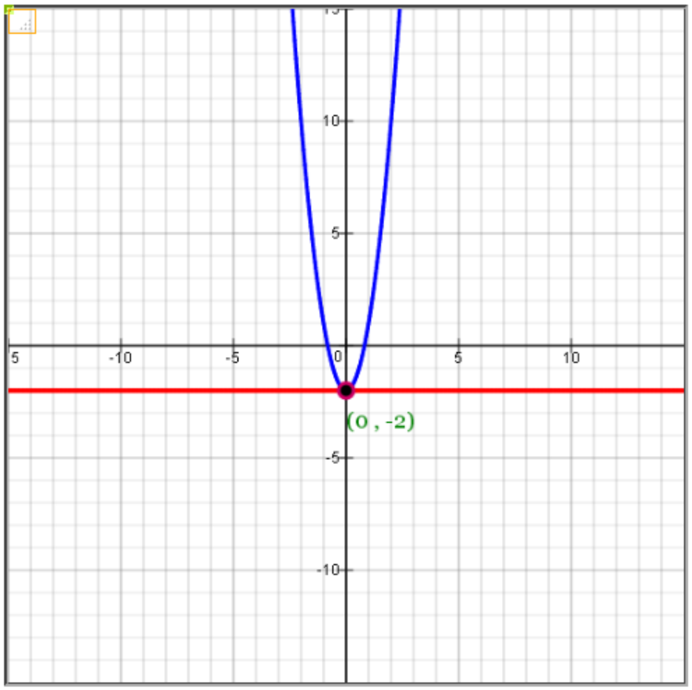
Line contains (1, 1) and has slope of 6.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 6(x - 1)$$

Example 2: Let $f(x) = 3x^2 - 2$.

Find the slope of the tangent line at $x = 0$ or the tangent line passing through $(0, \text{-}2)$.



$$f(x) = 3x^2 - 2$$

$$f(c) = f(0) = 3(0)^2 - 2 = -2$$

Note: When $x = 0, y = -2$

$$f(c + \Delta x) = f(0 + \Delta x) = 3(\Delta x)^2 - 2$$

$$f(c + \Delta x) = 3(\Delta x)^2 - 2$$

$$f(c + \Delta x) - f(c) = \left[3(\Delta x)^2 - 2\right] - [-2] = 3(\Delta x)^2$$

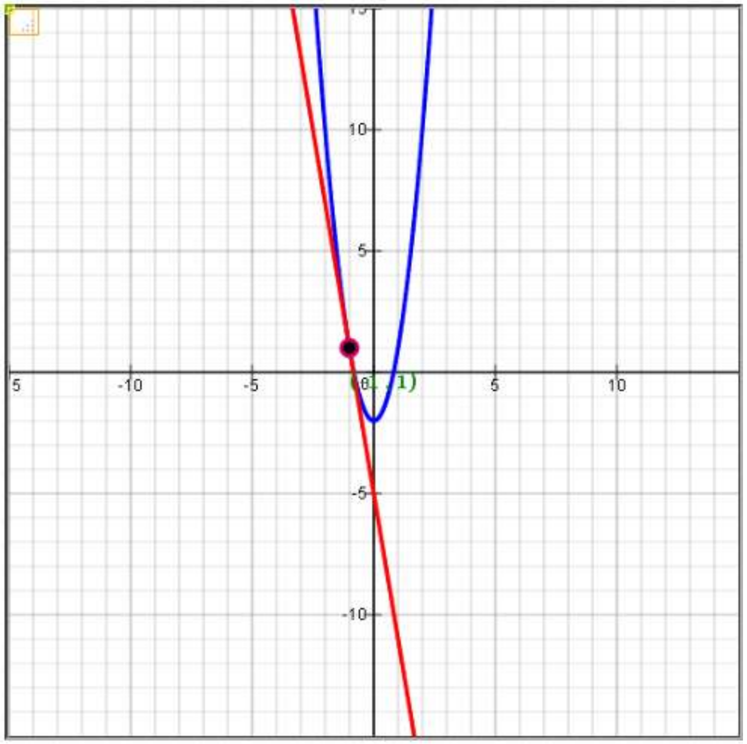
$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = \frac{3(\Delta x)^2}{\Delta x} = 3\Delta x$$

$$\text{slope of tangent line} = \lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} (3\Delta x) = 3(0) = 0$$

Equation of Tangent Line: $y = \text{-}2$

Example 3: Let $f(x) = 3x^2 - 2$.

Find the slope of the tangent line at $x = -1$ or the tangent line passing through $(-1, 1)$.



$$f(x) = 3x^2 - 2$$

$$f(c) = f(-1) = 3(-1)^2 - 2 = 1$$

Note: When $x = -1, y = 1$

$$f(c + \Delta x) = f(-1 + \Delta x) = 3\left(-1 + \Delta x\right)^2 - 2$$

$$f(c + \Delta x) = 3\left[\left(-1\right)^2 + 2(-1)(\Delta x) + \left(\Delta x\right)^2\right] - 2$$

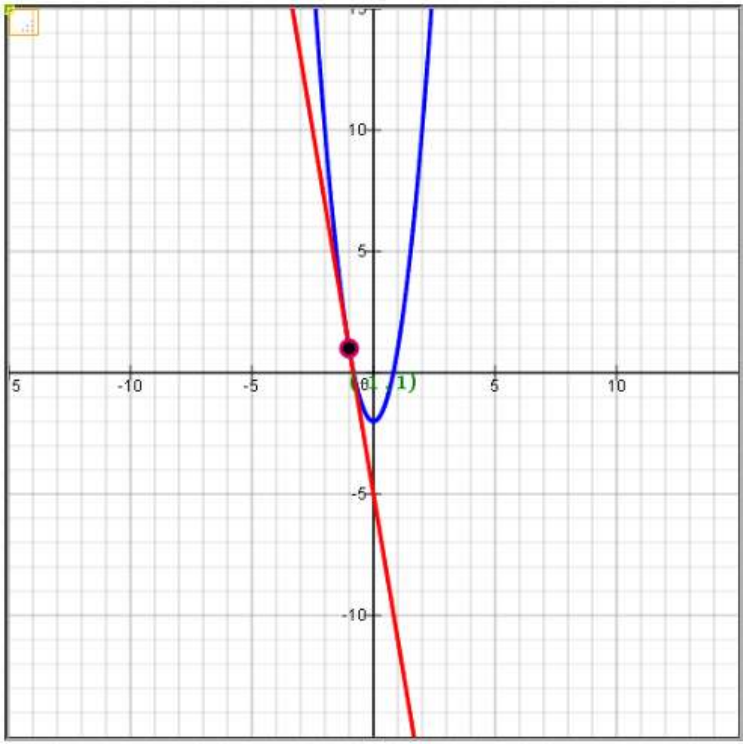
$$f(c + \Delta x) = 3 - 6\Delta x + 3\left(\Delta x\right)^2 - 2 = 1 - 6\Delta x + 3\left(\Delta x\right)^2$$

$$f(c + \Delta x) - f(c) = \left[1 - 6\Delta x + 3\left(\Delta x\right)^2\right] - [1] = -6\Delta x + 3\left(\Delta x\right)^2$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = \frac{-6\Delta x + 3\left(\Delta x\right)^2}{\Delta x} = \frac{-6\Delta x}{\Delta x} + \frac{3\left(\Delta x\right)^2}{\Delta x}$$

$$\frac{f(c + \Delta x) - f(c)}{\Delta x} = -6 + 3\Delta x$$

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = \lim_{\Delta x \to 0} \left(-6 + 3\Delta x\right) = -6 + 3(0) = -6$$



Equation of Tangent Line: $y - y_1 = m(x - x_1)$

Line contains (-1, 1) and has slope of -6.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -6(x - -1)$$

$$y - 1 = -6(x + 1)$$

Example 5: Let $f(x) = \dfrac{1}{x+4}$.

Find the derivative of the function $f(x) = f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f(x) = \frac{1}{x+4}$$

$$f(x+\Delta x) = \frac{1}{(x+\Delta x)+4}$$

$$f(x+\Delta x) - f(x) = \frac{1}{(x+\Delta x)+4} - \frac{1}{x+4}$$

Note: $\dfrac{a}{b} \pm \dfrac{c}{d} = \dfrac{ad \pm bc}{bd}$

$$f(x+\Delta x)-f(x)=\frac{1}{(x+\Delta x)+4}-\frac{1}{x+4}=\frac{(1)(x+4)-(1)\left[(x+\Delta x)+4\right]}{\left[(x+\Delta x)+4\right]\left[x+4\right]}$$

$$=\frac{x+4-x-\Delta x-4}{\left[(x+\Delta x)+4\right]\left[x+4\right]}=\frac{-\Delta x}{\left[(x+\Delta x)+4\right]\left[x+4\right]}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\dfrac{-\Delta x}{\left[(x+\Delta x)+4\right]\left[x+4\right]}}{\Delta x}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{-\Delta x}{\left[(x+\Delta x)+4\right]\left[x+4\right]}\cdot\frac{1}{\Delta x}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{-1}{\left[(x+\Delta x)+4\right]\left[x+4\right]}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{-1}{\left[(x+\Delta x)+4\right]\left[x+4\right]}$$

$$=\frac{-1}{\left[(x+0)+4\right]\left[x+4\right]}=\frac{-1}{\left[x+4\right]\left[x+4\right]}=\frac{-1}{(x+4)^2}$$

Example 6: Let $f(x) = \sqrt{x+4}$

Find the derivative of the function $f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f(x) = \sqrt{x+4}$$

$$f(x+\Delta x) = \sqrt{x+\Delta x+4}$$

$$f(x+\Delta x) - f(x) = \sqrt{x+\Delta x+4} - \sqrt{x+4}$$

Note: $\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{x}\sqrt{x} - \sqrt{y}\sqrt{y} = x - y$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\sqrt{x+\Delta x+4}-\sqrt{x+4}}{\Delta x}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\left(\sqrt{x+\Delta x+4}-\sqrt{x+4}\right)}{(\Delta x)}\cdot\frac{\left(\sqrt{x+\Delta x+4}+\sqrt{x+4}\right)}{\left(\sqrt{x+\Delta x+4}+\sqrt{x+4}\right)}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\sqrt{x+\Delta x+4}\sqrt{x+\Delta x+4}-\sqrt{x+4}\sqrt{x+4}}{(\Delta x)\left(\sqrt{x+\Delta x+4}+\sqrt{x+4}\right)}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\left(x+\Delta x+4\right)-\left(x+4\right)}{(\Delta x)\left(\sqrt{x+\Delta x+4}+\sqrt{x+4}\right)}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{\Delta x}{(\Delta x)\left(\sqrt{x+\Delta x+4}+\sqrt{x+4}\right)}$$

$$\frac{f(x+\Delta x)-f(x)}{\Delta x}=\frac{1}{\left(\sqrt{x+\Delta x+4}+\sqrt{x+4}\right)}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}=\lim_{\Delta x\to 0}\frac{1}{\left(\sqrt{x+\Delta x+4}+\sqrt{x+4}\right)}$$

$$=\frac{1}{\left(\sqrt{x+0+4}+\sqrt{x+4}\right)}=\frac{1}{\left(\sqrt{x+4}+\sqrt{x+4}\right)}=\frac{1}{2\sqrt{x+4}}$$

Example 7: Let $f(x) = 4$ Constant Function

Find the derivative of the function $f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f(x) = 4$$

$$f(x+\Delta x) = 4$$

$$f(x+\Delta x) - f(x) = [4] - [4]$$

$$f(x+\Delta x) - f(x) = 0$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = \lim_{\Delta x \to 0} (0) = \lim_{\Delta x \to 0} (0) = 0$$

For constant function, $f'(x) = 0$.

Example 8: Let $f(x) = x$.

Find the derivative of the function $f(x) = f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f(x) = x$$

$$f(x+\Delta x) = x + \Delta x$$

$$f(x+\Delta x) - f(x) = \left[x + \Delta x\right] - \left[x\right]$$

$$f(x+\Delta x) - f(x) = \Delta x$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(1\right)$$

$$= \lim_{\Delta x \to 0} \left(1\right) = 1$$

Example 9: Let $f(x) = x^2$.

Find the derivative of the function $f(x) = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f(x) = x^2$$

$$f(x+\Delta x) = (x+\Delta x)^2 = x^2 + 2x \cdot \Delta x + (\Delta x)^2$$

$$f(x+\Delta x) - f(x) = \left[x^2 + 2x \cdot \Delta x + (\Delta x)^2\right] - \left[x^2\right]$$

$$f(x+\Delta x) - f(x) = 2x \cdot \Delta x + (\Delta x)^2$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x \cdot \Delta x + (\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{2x \cdot \Delta x}{\Delta x} + \frac{(\Delta x)^2}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} (2x + \Delta x) = \lim_{\Delta x \to 0} (2x + 0) = 2x$$

Example 10: Let $f(x) = x^3$.

Find the derivative of the function $f(x) = f'(x) = \lim_{\Delta x \to 0} \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$.

$$f(x) = x^3$$

$$f(x+\Delta x) = (x+\Delta x)^3 = (x+\Delta x)^2 (x+\Delta x)$$

$$= \left[x^2 + 2x \cdot \Delta x + (\Delta x)^2\right](x+\Delta x)$$

$$= x^3 + 2x^2 \cdot \Delta x + x(\Delta x)^2 + x^2 \cdot \Delta x + 2x \cdot (\Delta x)^2 + (\Delta x)^3 = x^3 + 3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$f(x+\Delta x) - f(x) = \left[x^3 + 3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3\right] - \left[x^3\right]$$

$$f(x+\Delta x) - f(x) = 3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3x^2 \cdot \Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(\frac{3x^2 \cdot \Delta x}{\Delta x} + \frac{3x(\Delta x)^2}{\Delta x} + \frac{(\Delta x)^3}{\Delta x} \right)$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = \lim_{\Delta x \to 0} \left(3x^2 + 3x(0) + (0)^2\right) = 3x^2$$

Summary:

For $f(x) = c$ (constant function), $f'(x) = 0$

For $f(x) = x$ (constant function), $f'(x) = 1$

For $f(x) = x^2$ (constant function), $f'(x) = 2x$

For $f(x) = x^3$ (constant function), $f'(x) = 3x^2$

For $f(x) = x^4$ (constant function), $f'(x) = 4x^3$

For $f(x) = x^5$ (constant function), $f'(x) = 5x^4$

In general:

Let $f(x) = x^n$. Find the derivative of the function $f(x)$.

$$f'(x) = nx^{n-1}$$