Product and Quotient Rules for Derivative

Product Rule: $D_x\left(f(x)\cdot g(x)\right)= f(x)\cdot g'(x)+g(x)\cdot f'(x)$

Quotient Rule: $D_x\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left[g(x)\right]^2}$

Example 1:

Let $f(x)=(x^2+4)(x^2-5x)$.

Find $f'(x)$.

Product Rule: $D_x\left(f(x)\cdot g(x)\right)= f(x)\cdot g'(x)+g(x)\cdot f'(x)$

$f'(x)=(x^2+4)\cdot D_x\left[x^2-5x\right]+(x^2-5x)\cdot D_x\left[x^2+4\right]$

$f'(x)=(x^2+4)\cdot\left[2x-5\right]+(x^2-5x)\cdot\left[2x\right]$

$f'(x)=2x^3-5x^2+8x-20+2x^3-10x^2$

$f'(x)=4x^3-15x^2+8x-20$

Example 2:

Let $f(x) = \sqrt{x}(x^2 + 4)$.

Find $f'(x)$.

Product Rule: $D_x\left(f(x)\cdot g(x)\right) = f(x)\cdot g'(x) + g(x)\cdot f'(x)$

$$f'(x) = \sqrt{x}\cdot D_x\left[x^2+4\right] + (x^2+4)\cdot D_x\left[\sqrt{x}\right]$$

Note: $D_x\left[\sqrt{x}\right] = D_x\left[x^{1/2}\right] = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2}x^{-1/2}$

$$f'(x) = \sqrt{x}\cdot\left[2x+0\right] + (x^2+4)\cdot\left[\frac{1}{2}x^{-1/2}\right]$$

$$f'(x) = 2x\sqrt{x} + (x^2+4)\cdot\left[\frac{1}{2}x^{-1/2}\right]$$

$$f'(x) = 2x\sqrt{x} + (x^2+4)\cdot\left[\frac{1}{2x^{1/2}}\right]$$

Example 3:

Let $f(x) = \sqrt[3]{x^2} \sin x$.

Find $f'(x)$.

Product Rule: $D_x\left(f(x) \cdot g(x)\right) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

$$f'(x) = \sqrt[3]{x^2} \cdot D_x\left[\sin x\right] + \sin x \cdot D_x\left[\sqrt[3]{x^2}\right]$$

Note: $D_x\left[\sqrt[3]{x^2}\right] = D_x\left[x^{2/3}\right] = \frac{2}{3}x^{2/3-1} = \frac{2}{3}x^{-1/3}$

$$f'(x) = \sqrt[3]{x^2} \cdot \left[\cos x\right] + \sin x \cdot \left[\frac{2}{3}x^{-1/3}\right]$$

$$f'(x) = \sqrt[3]{x^2} \cdot \left[\cos x\right] + \sin x \cdot \left[\frac{2}{3x^{1/3}}\right]$$

Example 4:

Let $f(x) = \dfrac{3x^2 - 1}{2x + 5}$.

Find $f'(x)$.

Quotient Rule: $D_x\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{\left[g(x)\right]^2}$

$$f'(x) = \frac{\left(2x+5\right) D_x\left(3x^2 - 1\right) - \left(3x^2 - 1\right) D_x\left(2x+5\right)}{\left[2x+5\right]^2}$$

$$f'(x) = \frac{\left(2x+5\right)\left(6x\right) - \left(3x^2 - 1\right)\left(2\right)}{\left[2x+5\right]^2}$$

Example 5:

Let $f(x) = \dfrac{x^3}{5\sqrt{x} + 3}$.

Find $f'(x)$.

Quotient Rule: $D_x\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2}$

$$f'(x) = \frac{\left(5\sqrt{x}+3\right)D_x\left(x^3\right) - \left(x^3\right)D_x\left(5\sqrt{x}+3\right)}{\left[5\sqrt{x}+3\right]^2}$$

$$Note: D_x\left(5\sqrt{x}+3\right) = D_x\left(5x^{1/2}+3\right) = 5\left(\frac{1}{2}x^{1/2-1}\right) + 0 = \frac{5}{2}x^{-1/2}$$

$$f'(x) = \frac{\left(5\sqrt{x}+3\right)\left(3x^2\right) - \left(x^3\right)\left(\frac{5}{2}x^{-1/2}\right)}{\left[5\sqrt{x}+3\right]^2}$$

Example 6:

Let $f(x) = \dfrac{\cos x}{x^2}$.

Find $f'(x)$.

Quotient Rule: $D_x\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2}$

$$f'(x) = \frac{\left(x^2\right)D_x\left(\cos x\right) - \left(\cos x\right)D_x\left(x^2\right)}{\left[x^2\right]^2}$$

$$f'(x) = \frac{\left(x^2\right)\left(-\sin x\right) - \left(\cos x\right)\left(2x\right)}{\left[x^2\right]^2}$$

Example 7:

Let $f(x) = \dfrac{3x^2 + 4x - 5}{x^2 - 4}$.

Find $f'(x)$.

Quotient Rule: $D_x\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{\left[g(x)\right]^2}$

$$f'(x) = \frac{\left(x^2 - 4\right)D_x\left(3x^2 + 4x - 5\right) - \left(3x^2 + 4x - 5\right)D_x\left(x^2 - 4\right)}{\left[x^2 - 4\right]^2}$$

$$f'(x) = \frac{\left(x^2 - 4\right)\left(6x + 4\right) - \left(3x^2 + 4x - 5\right)\left(2x\right)}{\left[x^2 - 4\right]^2}$$

Example 8:

Let $f(x)=\dfrac{5-\frac{4}{x}}{x^2-4}$.

Find $f'(x)$.

$$Note:\ f(x)=\frac{5-\frac{4}{x}}{x^2-4}=\ f(x)=\frac{5-4x^{-1}}{x^2-4}.$$

Quotient Rule: $D_x\left(\dfrac{f(x)}{g(x)}\right)=\dfrac{g(x)\cdot f'(x)-f(x)\cdot g'(x)}{\left[g(x)\right]^2}$

$$f'(x)=\frac{\left(x^2-4\right)D_x\left(5-4x^{-1}\right)-\left(5-4x^{-1}\right)D_x\left(x^2-4\right)}{\left[x^2-4\right]^2}$$

$$f'(x)=\frac{\left(x^2-4\right)\left(-4\left(-1x^{-2}\right)\right)-\left(5-4x^{-1}\right)\left(2x\right)}{\left[x^2-4\right]^2}$$

$$f'(x)=\frac{\left(x^2-4\right)\left(4x^{-2}\right)-\left(5-4x^{-1}\right)\left(2x\right)}{\left[x^2-4\right]^2}$$

Example 9:

Let $f(x) = \tan x \cdot \sin x$

Find $f'(x)$.

Product Rule: $D_x\left(f(x) \cdot g(x)\right) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

$$f'(x) = \tan x \cdot D_x\left(\sin x\right) + \sin x \cdot D_x(\tan x)$$

$$f'(x) = \tan x \cdot \left(\cos x\right) + \sin x \cdot \left[(\sec x)^2\right]$$

Note: $(\sec x)^2 = \sec^2 x$

Example 10:

Let $f(x) = \cos x \cdot \sin x$

Find $f'(x)$ and find tangent line at $\left(\frac{\pi}{2}, 0\right)$.

Product Rule: $D_x\left(f(x) \cdot g(x)\right) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$

$$f'(x) = \cos x \cdot D_x\left(\sin x\right) + \sin x \cdot D_x(\cos x)$$

$$f'(x) = \cos x \cdot \left(\cos x\right) + \sin x \cdot \left[-\sin x\right]$$

$$f'(x) = \left(\cos x\right)^2 - \left(\sin x\right)^2$$

Slope of tangent line at $\left(\frac{\pi}{2}, 0\right) = f'\left(\frac{\pi}{2}\right)$

$$f'\left(\frac{\pi}{2}\right) = \left(\cos\frac{\pi}{2}\right)^2 - \left(\sin\frac{\pi}{2}\right)^2 = (0)^2 - (1)^2 = -1$$

Equation of tangent line: $y - y_1 = m(x - x_1)$

Equation of tangent line: $y - 0 = -1\left(x - \frac{\pi}{2}\right)$

Example 11:

Let $f(x) = \dfrac{x+2}{x-2}$

Find $f'(x)$ and find tangent line at $(0,-1)$.

Quotient Rule: $D_x\left(\dfrac{f(x)}{g(x)}\right) = \dfrac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{[g(x)]^2}$

$$f'(x) = \frac{(x-2)\cdot D_x(x+2) - (x+2)\cdot D_x(x-2)}{[x-2]^2}$$

$$f'(x) = \frac{(x-2)\cdot(1) - (x+2)\cdot(1)}{[x-2]^2}$$

Slope of tangent line at $(0,-1) = f'(0)$

$$f'(0) = \frac{(x-2)\cdot(1) - (x+2)\cdot(1)}{[x-2]^2} = \frac{(0-2)\cdot(1) - (0+2)\cdot(1)}{[0-2]^2} = \frac{-4}{4} = -1$$

Equation of tangent line: $y - y_1 = m(x - x_1)$

Equation of tangent line: $y - -1 = -1(x-0)$