Chain Rule

Exmaple 1: Let $f(x)=(3x+5)^2$. Find $f'(x)$.

Solution:

Using regular method:

$$f(x)=(3x+5)^2 = 9x^2+30x+25$$

$$f'(x)=9\cdot D_x\left(x^2\right)+30\cdot D_x\left(x\right)+D_x\left(25\right)$$

$$f'(x)=9\cdot\left(2x\right)+30\cdot\left(1\right)+\left(0\right)=18x+30$$

Using Chain Rule method:

$$f(x)=(3x+5)^2$$

Let B = Base $=3x+5$ and E = Exponent = 2

Chain Rule: $f'(x)=\mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E-1}}\cdot D_x\left(\mathrm{B}\right)$

$$f'(x)=2\left(3x+5\right)\cdot 3=18x+30$$

Example 2: Let $f(x)=\sqrt{3x+5}=\left(3x+5\right)^{1/2}$. Find $f'(x)$.

Solution:

Let B = Base $=3x+5$ and E = Exponent = 1/2

Chain Rule: $f'(x)=\mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E-1}}\cdot D_x\left(\mathrm{B}\right)$

$$f'(x)=\frac{1}{2}\left(3x+5\right)^{\frac{1}{2}-1}\cdot D_x\left(3x+5\right)=\frac{1}{2}\left(3x+5\right)^{-\frac{1}{2}}\cdot\left(3\right)=\frac{3}{2\left(3x+5\right)^{1/2}}$$

Example 3: Let $f(x) = \sqrt[3]{3x^4 + 5} = \left(3x^4 + 5\right)^{1/3}$. Find $f'(x)$.

Solution:

Let B = Base $= 3x^4 + 5$ and E = Exponent = 1/3

Chain Rule: $f'(x) = \mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E-1}} \cdot D_x\left(\mathrm{B}\right)$

$$f'(x) = \frac{1}{3}\left(3x^4 + 5\right)^{\frac{1}{3}-1} \cdot D_x\left(3x^4 + 5\right)$$

$$f'(x) = \frac{1}{3}\left(3x^4 + 5\right)^{-\frac{2}{3}} \cdot \left(12x^3\right) = 4x^3 \cdot \left(3x^4 + 5\right)^{-\frac{2}{3}} = \frac{4x^3}{\left(3x^4 + 5\right)^{\frac{2}{3}}}$$

Example 4: Let $f(x)=\sqrt[4]{x^4+5x^2}=\left(x^4+5x^2\right)^{1/4}$. Find $f'(x)$.

Solution:

Let B = Base $= x^4+5x^2$ and E = Exponent = 1/4

Chain Rule: $f'(x)=\mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E}\text{-}1}\cdot D_x\left(\mathrm{B}\right)$

$$f'(x)=\frac{1}{4}\left(x^4+5x^2\right)^{\frac{1}{4}\text{-}1}\cdot D_x\left(x^4+5x^2\right)$$

$$f'(x)=\frac{1}{4}\left(x^4+5x^2\right)^{\frac{-3}{4}}\cdot\left(4x^3+10x\right)=\frac{\left(4x^3+10x\right)}{4\left(x^4+5x^2\right)^{\frac{3}{4}}}=\frac{\left(2x^3+5x\right)}{2\left(x^4+5x^2\right)^{\frac{3}{4}}}$$

Example 5: Let $f(x) = \dfrac{4}{(x-5)^3} = 4(x-5)^{-3}$. Find $f'(x)$.

Solution:

Let B = Base $= x-5$ and E = Exponent = -3

Chain Rule: $f'(x) = \mathrm{E}(\mathrm{B})^{\mathrm{E}\text{-}1} \cdot D_x(\mathrm{B})$

$$f'(x) = 4 \cdot \left[-3(x-5)^{-3-1} \cdot D_x(x-5)\right]$$

$$f'(x) = 4 \cdot \left[-3(x-5)^{-4} \cdot (1)\right] = -12(x-5)^{-4} = \frac{-12}{(x-5)^4}$$

Example 6: Let $f(x) = \left(\frac{4}{x-1}\right)^3$. Find $f'(x)$.

Solution:

Let B = Base = $\frac{4}{x-1}$ and E = Exponent = 3

Chain Rule: $f'(x) = \mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E-1}} \cdot D_x\left(\mathrm{B}\right)$

$$f'(x) = 3\left(\frac{4}{x-1}\right)^{3-1} \cdot D_x\left(\frac{4}{x-1}\right)$$

$$Note: D_x\left(\frac{4}{x-1}\right) = \frac{\left(x-1\right)D_x\left(4\right)-\left(4\right)D_x\left(x-1\right)}{\left(x-1\right)^2}$$ Using Quotient Rule

$$D_x\left(\frac{4}{x-1}\right) = \frac{\left(x-1\right)\left(0\right)-\left(4\right)\left(1\right)}{\left(x-1\right)^2} = \frac{-4}{\left(x-1\right)^2}$$

Hence,

$$f'(x) = 3\left(\frac{4}{x-1}\right)^{3-1} \cdot D_x\left(\frac{4}{x-1}\right) = 3\left(\frac{4}{x-1}\right)^2 \cdot \frac{-4}{\left(x-1\right)^2} = 3\frac{\left(4\right)^2}{\left(x-1\right)^2} \cdot \frac{-4}{\left(x-1\right)^2} = \frac{-192}{\left(x-1\right)^4}$$

Example 7: Let $f(x) = \cos^3 x = \left(\cos x\right)^3$. Find $f'(x)$.

Solution:

Let B = Base $= \cos x$ and E = Exponent = 3

Chain Rule: $f'(x) = \mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E}\text{-}1} \cdot D_x\left(\mathrm{B}\right)$

$$f'(x) = 3\left(\cos x\right)^{3\text{-}1} \cdot D_x\left(\cos x\right)$$

$$f'(x) = 3\left(\cos x\right)^2 \cdot \left(-\sin x\right)$$

$$f'(x) = -3\left(\cos x\right)^2 \cdot \left(\sin x\right)$$

$$f'(x) = -3\left(\cos^2 x\right) \cdot \left(\sin x\right)$$

Example 8: Let $f(x) = \tan^4 x = \left(\tan x\right)^4$. Find $f'(x)$.

Solution:

Let B = Base = $\tan x$ and E = Exponent = 4

Chain Rule: $f'(x) = \mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E\text{-}1}} \cdot D_x\left(\mathrm{B}\right)$

$$f'(x) = 4\left(\tan x\right)^{4\text{-}1} \cdot D_x\left(\tan x\right)$$

$$f'(x) = 4\left(\tan x\right)^3 \cdot \left(\sec^2 x\right)$$

$$f'(x) = 4\left(\tan^3 x\right) \cdot \left(\sec^2 x\right)$$

Example 9: Let $f(x)=(x+2)^2(x-1)^3$. Find $f'(x)$.

Solution:

Product Rule: $f'(x)=F\cdot D_x(S)+S\cdot D_x(F)$

$$f'(x)=(x+2)^2\cdot D_x\left((x-1)^3\right)+(x-1)^3\cdot D_x\left((x+2)^2\right)$$

For $D_x\left((x+2)^2\right)$ we can use Chain Rule: $D_x\left((x+2)^2\right)=2(x+2)^{2-1}\cdot D_x(x+2)=2(x+2)^1\cdot(1)=2(x+2)$

For $D_x\left((x-1)^3\right)$ we can use Chain Rule: $D_x\left((x-1)^3\right)=3(x-1)^{3-1}\cdot D_x(x-1)=3(x-1)^2\cdot(1)=3(x-1)^2$

Hence,

$$f'(x)=(x+2)^2\cdot D_x\left((x-1)^3\right)+(x-1)^3\cdot D_x\left((x+2)^2\right)$$

$$f'(x)=(x+2)^2\cdot\left[3(x-12)\right]+(x-1)^3\cdot\left[2(x+2)\right]$$

Example 10: Let $f(x) = \dfrac{(x-1)^3}{(x+2)^2}$. Find $f'(x)$.

Solution:

Using Quotient Rule: $f'(x) = \dfrac{(x+2)^2 \cdot D_x\left((x-1)^3\right) - (x-1)^3 \cdot D_x\left((x+2)^2\right)}{\left[(x+2)^2\right]^2}$

For $D_x\left((x+2)^2\right)$ we can use Chain Rule: $D_x\left((x+2)^2\right) = 2(x+2)^{2-1} \cdot D_x(x+2) = 2(x+2)^1 \cdot (1) = 2(x+2)$

For $D_x\left((x-1)^3\right)$ we can use Chain Rule: $D_x\left((x-1)^3\right) = 3(x-1)^{3-1} \cdot D_x(x-1) = 3(x-1)^2 \cdot (1) = 3(x-1)^2$

Hence,

$$f'(x) = \frac{(x+2)^2 \cdot D_x\left((x-1)^3\right) - (x-1)^3 \cdot D_x\left((x+2)^2\right)}{\left[(x+2)^2\right]^2}$$

$$f'(x) = \frac{(x+2)^2 \cdot \left[3(x-1)^2\right] - (x-1)^3 \cdot \left[2(x+2)\right]}{(x+2)^4}$$

Example 11: Let $f(x)=\dfrac{x^2}{\sqrt{x+4}}$. Find $f'(x)$.

Solution:

Using Quotient Rule: $f'(x)=\dfrac{\sqrt{x+4}\cdot D_x\left(x^2\right)-\left(x^2\right)\cdot D_x\left(\sqrt{x+4}\right)}{\left[\sqrt{x+4}\right]^2}$

For $D_x\left(\sqrt{x+4}\right)$ we can use Chain Rule: $D_x\left(\sqrt{x+4}\right)=D_x\left(\left(x+4\right)^{1/2}\right)=\dfrac{1}{2}\left(x+4\right)^{1/2-1}\cdot D_x\left(x+4\right)$

$$=\frac{1}{2}\left(x+4\right)^{-1/2}\cdot(1)=\frac{1}{2\left(x+4\right)^{1/2}}$$

Note: $\left[\sqrt{x+4}\right]^2=\left[\left(x+4\right)^{1/2}\right]^2=x+4$

Hence,

$$f'(x)=\frac{\sqrt{x+4}\cdot D_x\left(x^2\right)-\left(x^2\right)\cdot D_x\left(\sqrt{x+4}\right)}{\left[\sqrt{x+4}\right]^2}$$

$$f'(x)=\frac{\sqrt{x+4}\cdot[2x]-\left(x^2\right)\cdot\left[\dfrac{1}{2\left(x+4\right)^{1/2}}\right]}{x+4}$$

Example 12: Let $f(x)=\left(\dfrac{4-5x}{5+x}\right)^2$. Find $f'(x)$.

Solution:

Solution:

Let B = Base = $\dfrac{4-5x}{5+x}$ and E = Exponent = 2

Chain Rule: $f'(x)=\mathrm{E}\left(\mathrm{B}\right)^{\text{E-1}}\cdot D_x\left(\mathrm{B}\right)$

$$f'(x)=2\left(\frac{4-5x}{5+x}\right)^{2\text{-}1}\cdot D_x\left(\frac{4-5x}{5+x}\right)$$

For $D_x\left(\dfrac{4-5x}{5+x}\right)$ we can using Quotient Rule: $D_x\left(\dfrac{4-5x}{5+x}\right)=\dfrac{(5+x)\cdot D_x(4-5x)-(4-5x)\cdot D_x(5+x)}{[5+x]^2}$

$$D_x\left(\frac{4-5x}{5+x}\right)=\frac{(5+x)\cdot(-5)-(4-5x)\cdot(1)}{[5+x]^2}=\frac{-25-5x-4+5x}{[5+x]^2}=\frac{-29}{[5+x]^2}$$

Hence,

$$f'(x)=2\left(\frac{4-5x}{5+x}\right)^{2\text{-}1}\cdot D_x\left(\frac{4-5x}{5+x}\right)=2\left(\frac{4-5x}{5+x}\right)^{2\text{-}1}\cdot\left(\frac{-29}{[5+x]^2}\right)$$

Chain Rule for Trigonometric Functions:

1) If $f\left(x\right)=\cos\left(\text{expression}\right)$, then $f'(x)=-\sin(\text{expression})\cdot D_x\left(\text{expression}\right)$

2) If $f\left(x\right)=\sin\left(\text{expression}\right)$, then $f'(x)=\cos(\text{expression})\cdot D_x\left(\text{expression}\right)$

3) If $f\left(x\right)=\tan\left(\text{expression}\right)$, then $f'(x)=\sec^2(\text{expression})\cdot D_x\left(\text{expression}\right)$

4) If $f\left(x\right)=\sec\left(\text{expression}\right)$, then $f'(x)=\sec(\text{expression})\cdot\sec(\text{expression})\cdot D_x\left(\text{expression}\right)$

5) If $f\left(x\right)=\csc\left(\text{expression}\right)$, then $f'(x)=-\csc(\text{expression})\cdot\cot(\text{expression})\cdot D_x\left(\text{expression}\right)$

6) If $f\left(x\right)=\cot\left(\text{expression}\right)$, then $f'(x)=-\csc^2(\text{expression})\cdot D_x\left(\text{expression}\right)$

Example 12: Let $f(x) = \cos \pi x$. Find $f'(x)$.

Solution:

Chain Rule for Trigonometric Functions:

If $f\left(x\right) = \cos\left(\text{expression}\right)$, then $f'(x) = -\sin(\text{expression}) \cdot D_x\left(\text{expression}\right)$

$$f'(x) = -\sin\left(\pi x\right) \cdot D_x\left(\pi x\right) = -\sin\left(\pi x\right) \cdot \left(\pi\right) = -\pi \sin\left(\pi x\right)$$

Example 13: Let $f(x) = \cos\left(5x\right)^2 = \cos\left(25x^2\right)$. Find $f'(x)$.

Solution:

Chain Rule for Trigonometric Functions:

If $f\left(x\right) = \cos\left(\text{expression}\right)$, then $f'(x) = -\sin(\text{expression}) \cdot D_x\left(\text{expression}\right)$

$$f'(x) = -\sin\left(25x^2\right) \cdot D_x\left(25x^2\right) = -\sin\left(25x^2\right) \cdot \left(50x\right) = -50\sin\left(25x^2\right)$$

Example 14: Let $f(x) = 10\sec^3 x = 10(\sec x)^3$. Find $f'(x)$.

Solution:

Let B = Base = $\sec x$ and E = Exponent = 3:

Chain Rule: $f'(x) = \mathrm{E}(\mathrm{B})^{\mathrm{E-1}} \cdot D_x(\mathrm{B})$

$$f'(x) = 10\left[3(\sec x)^{3-1} \cdot D_x(\sec x)\right] = 10\left[3(\sec x)^2 \cdot (\sec x \tan x)\right] = 30\sec^3 x \cdot \tan x$$

Example 15: Let $f(x) = 10\left[\sin\left(5x-7\right)\right]^2$. Find $f'(x)$.

Solution:

Let B = Base $= \sin\left(5x-7\right)$ and E = Exponent = 2

Chain Rule: $f'(x) = \mathrm{E}\left(\mathrm{B}\right)^{\mathrm{E-1}} \cdot D_x\left(\mathrm{B}\right)$

$$f'(x) = 10 \cdot \left(2\left[\sin\left(5x-7\right)\right]^{2-1} \cdot D_x\left(\sin(5x-7)\right)\right)$$

$$f'(x) = 20\left[\sin\left(5x-7\right)\right] \cdot \left[\cos(5x-7) \cdot D_x\left(5x-7\right)\right]$$

$$f'(x) = 20\left[\sin\left(5x-7\right)\right] \cdot \left[\cos(5x-7) \cdot \left(5\right)\right]$$

$$f'(x) = 100\left[\sin\left(5x-7\right)\right] \cdot \left[\cos(5x-7)\right]$$