Chain Rule

Example 1: Let $f(x) = (3x+5)^2$. Find $f'(x)$.

Using regular method:

$f(x) = (3x+5)^2 = 9x^2 + 30x + 25$

$f'(x) = 9 \cdot D_x\left(x^2\right) + 30 \cdot D_x\left(x\right) + D_x\left(25\right)$

$f'(x) = 9 \cdot \left(2x\right) + 30 \cdot \left(1\right) + \left(0\right) = 18x + 30$

Using Chain Rule method:

$f(x) = (3x+5)^2$

Let $u = 3x+5 \qquad \Rightarrow \qquad \dfrac{du}{dx} = D_x\left(3x\right) + \cdot D_x\left(5\right) = 3 + 0 = 3$

$f(x) = (3x+5)^2 = u^2$

Note: $y = f(x)$

$y = u^2$

$$\frac{dy}{du} = 2u = 2\left(3x+5\right) = 6x + 10$$

By Chain Rule: $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$f'(x) = \frac{dy}{dx} = \left[6x+10\right] \cdot \left[3\right]$$

$$f'(x) = \frac{dy}{dx} = 18x + 30$$

Example 2: Let $f(x)=\sqrt{3x+5}$. Find $f'(x)$.

Using Chain Rule method:

$$f(x)=\sqrt{3x+5}=(3x+5)^{1/2}$$

Let $u=3x+5$ $\quad\Rightarrow\quad$ $\dfrac{du}{dx}=D_x\left(3x\right)+\cdot D_x\left(5\right)=3+0=3$

$$f(x)=\sqrt{3x+5}=(3x+5)^{1/2}=u^{1/2}$$

Note: $y=f(x)$

$$y=u^{1/2}\quad\Rightarrow\quad\frac{dy}{du}=\frac{1}{2}u^{1/2-1}=\frac{1}{2}u^{-1/2}=\frac{1}{2}(3x+5)^{-1/2}$$

By Chain Rule: $f'(x)=\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$f'(x)=\frac{dy}{dx}=\left[\frac{1}{2}(3x+5)^{-1/2}\right]\left[3\right]$$

$$f'(x)=\frac{dy}{dx}=\frac{3}{2}(3x+5)^{-1/2}$$

$$f'(x)=\frac{dy}{dx}=\frac{3}{2}\cdot\frac{1}{(3x+5)^{1/2}}$$

Example 3: Let $f(x) = \sqrt[3]{3x^4 + 5}$. Find $f'(x)$.

Using Chain Rule method:

$$f(x) = \sqrt[3]{3x^4 + 5} = (3x^4 + 5)^{1/3}$$

Let $u = 3x^4 + 5 \quad \Rightarrow \quad \dfrac{du}{dx} = D_x\left(3x^4\right) + \cdot D_x\left(5\right) = 12x^3 + 0 = 12x^3$

$$f(x) = \sqrt[3]{3x^4 + 5} = (3x^4 + 5)^{1/3} = u^{1/3}$$

Note: $y = f(x)$

$$y = u^{1/3} \quad \Rightarrow \quad \frac{dy}{du} = \frac{1}{3}u^{1/3-1} = \frac{1}{3}u^{-2/3} = \frac{1}{3}(3x^4 + 5)^{-2/3}$$

By Chain Rule: $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$f'(x) = \frac{dy}{dx} = \left[\frac{1}{3}(3x^4 + 5)^{-2/3}\right]\left[12x^3\right]$$

$$f'(x) = \frac{dy}{dx} = 4x^3(3x^4 + 5)^{-2/3}$$

$$f'(x) = \frac{dy}{dx} = \frac{4x^3}{(3x^4 + 5)^{2/3}}$$

Example 4: Let $f(x) = \sqrt[4]{x^4 + 5x^2}$. Find $f'(x)$.

Using Chain Rule method:

$$f(x) = \sqrt[4]{x^4 + 5x^2} = (x^4 + 5x^2)^{1/4}$$

Let $u = x^4 + 5x^2 \qquad \Rightarrow \qquad \dfrac{du}{dx} = D_x\left(x^4\right) + D_x\left(5x^2\right) = 4x^3 + 10x$

$$f(x) = \sqrt[4]{x^4 + 5x^2} = (x^4 + 5x^2)^{1/4} = u^{1/4}$$

Note: $y = f(x)$

$y = u^{1/4} \qquad \Rightarrow \qquad \dfrac{dy}{du} = \dfrac{1}{4}u^{1/4-1} = \dfrac{1}{4}u^{-3/4} = \dfrac{1}{4}(x^4 + 5x^2)^{-3/4}$

By Chain Rule: $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$f'(x) = \frac{dy}{dx} = \left[\frac{1}{4}(x^4 + 5x^2)^{-3/4}\right]\left[4x^3 + 10x\right]$$

$$f'(x) = \frac{dy}{dx} = \left[\frac{1}{4} \cdot \frac{1}{(x^4 + 5x^2)^{3/4}}\right]\left[4x^3 + 10x\right]$$

Example 5: Let $f(x) = \dfrac{4}{(x-5)^3}$. Find $f'(x)$.

Using Chain Rule method:

$$f(x) = \frac{4}{(x-5)^3} = 4 \cdot (x-5)^{-3}$$

Let $u = x - 5$ $\quad \Rightarrow \quad \dfrac{du}{dx} = D_x(x) - D_x(5) = 1$

$$f(x) = \frac{4}{(x-5)^3} = 4 \cdot (x-5)^{-3} = 4 \cdot u^{-3}$$

Note: $y = f(x)$

$$y = 4 \cdot u^{-3} \quad \Rightarrow \quad \frac{dy}{du} = 4 \cdot \left[-3u^{-3-1}\right] = -12u^{-4} = -12(x-5)^{-4}$$

By Chain Rule: $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$f'(x) = \frac{dy}{dx} = \left[-12(x-5)^{-4}\right][1]$$

$$f'(x) = \frac{dy}{dx} = -12(x-5)^{-4}$$

$$f'(x) = \frac{dy}{dx} = -12 \cdot \frac{1}{(x-5)^4} = \frac{-12}{(x-5)^4}$$

Example 6: Let $f(x)=\left(\dfrac{4}{x-1}\right)^3$. Find $f'(x)$.

Using Chain Rule method:

$$f(x)=\left(\frac{4}{x-1}\right)^3=\frac{(4)^3}{(x-1)^3}=\frac{64}{(x-1)^3}=64\cdot(x-1)^{-3}$$

Let $u=x-1 \qquad \Rightarrow \qquad \dfrac{du}{dx}=D_x(x)-D_x(1)=1+0=1$

$$f(x)=\left(\frac{4}{x-1}\right)^3=\frac{(4)^3}{(x-1)^3}=\frac{64}{(x-1)^3}=64\cdot(x-1)^{-3}=64\cdot u^{-3}$$

Note: $y=f(x)$

$y=64\cdot u^{-3} \qquad \Rightarrow \qquad \dfrac{dy}{du}=64\cdot\left[-3u^{-3-1}\right]=-192u^{-4}=-192(x-1)^{-4}$

By Chain Rule: $f'(x)=\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$f'(x)=\frac{dy}{dx}=\left[-192(x-1)^{-4}\right][1]$$

$$f'(x)=\frac{dy}{dx}=-192(x-1)^{-4}$$

$$f'(x)=\frac{dy}{dx}=-192\cdot\frac{1}{(x-1)^4}=\frac{-192}{(x-1)^4}$$

Example 6: Let $f(x)=\cos^3 x$. Find $f'(x)$.

Using Chain Rule method:

$$f(x)=\cos^3 x=(\cos x)^3$$

Let $u=\cos x \qquad \Rightarrow \qquad \dfrac{du}{dx}=D_x(\cos x)=-\sin x$

$$f(x)=\cos^3 x=(\cos x)^3=u^3$$

Note: $y=f(x)$

$$y=u^3 \qquad \Rightarrow \qquad \frac{dy}{du}=3u^2=3(\cos x)^2=3\cos^2 x$$

By Chain Rule: $f'(x)=\dfrac{dy}{dx}=\dfrac{dy}{du}\cdot\dfrac{du}{dx}$

$$f'(x)=\frac{dy}{dx}=\left[3\cos^2 x\right]\left[-\sin x\right]$$

$$f'(x)=\frac{dy}{dx}=-3\cos^2 x\cdot\sin x$$

Example 7: Let $f(x) = \tan^4 x$. Find $f'(x)$.

Using Chain Rule method:

$$f(x) = \tan^4 x = (\tan x)^4$$

Let $u = \tan x$ $\quad\Rightarrow\quad \dfrac{du}{dx} = D_x(\tan x) = \sec^2 x$

$$f(x) = \tan^4 x = (\tan x)^4 = u^4$$

Note: $y = f(x)$

$$y = u^4 \quad\Rightarrow\quad \frac{dy}{du} = 4u^3 = 4(\tan x)^3 = 3\tan^3 x$$

By Chain Rule: $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$$f'(x) = \frac{dy}{dx} = \left[3\tan^3 x\right]\left[\sec^2 x\right]$$

$$f'(x) = \frac{dy}{dx} = 3\tan^3 x \cdot \sec^2 x$$

Example 7: Let $f(x) = (x+2)^2(x-1)^3$. Find $f'(x)$.

Using Product Rule:

$$f(x) = (x+2)^2(x-1)^3$$

$$f'(x) = (x+2)^2 \cdot D_x\left[(x-1)^3\right] + (x-1)^3 \cdot D_x\left[(x+2)^2\right]$$

Now use Chain Rule to find $D_x\left[(x-1)^3\right]$ and $D_x\left[(x+2)^2\right]$:

$$D_x\left[(x-1)^3\right] = 3(x-1)^2 \cdot D_x\left[x-1\right] = 3(x-1)^2 \cdot [1] = 3(x-1)^2$$

$$D_x\left[(x+2)^2\right] = 2(x+2) \cdot D_x\left[x+2\right] = 2(x+2) \cdot [1] = 2(x+2)$$

Hence,

$$f'(x) = (x+2)^2 \cdot D_x\left[(x-1)^3\right] + (x-1)^3 \cdot D_x\left[(x+2)^2\right]$$

$$f'(x) = (x+2)^2 \cdot \left[3(x-1)^2\right] + (x-1)^3 \cdot \left[2(x+2)\right]$$

Example 8: Let $f(x) = \dfrac{(x-1)^3}{(x+2)^2}$. Find $f'(x)$.

Using Quotient Rule:

$$f(x) = \frac{(x+2)^2 \cdot D_x\left[(x-1)^3\right] - (x-1)^3 \cdot D_x\left[(x+2)^2\right]}{\left[(x+2)^2\right]^2}$$

$$f'(x) = \frac{(x+2)^2 \cdot D_x\left[(x-1)^3\right] - (x-1)^3 \cdot D_x\left[(x+2)^2\right]}{(x+2)^4}$$

Now use Chain Rule to find $D_x\left[(x-1)^3\right]$ and $D_x\left[(x+2)^2\right]$:

$$D_x\left[(x-1)^3\right] = 3(x-1)^2 \cdot D_x\left[x-1\right] = 3(x-1)^2 \cdot \left[1\right] = 3(x-1)^2$$

$$D_x\left[(x+2)^2\right] = 2(x+2) \cdot D_x\left[x+2\right] = 2(x+2) \cdot \left[1\right] = 2(x+2)$$

Hence,

$$f'(x) = \frac{(x+2)^2 \cdot D_x\left[(x-1)^3\right] - (x-1)^3 \cdot D_x\left[(x+2)^2\right]}{(x+2)^4}$$

$$f'(x) = \frac{(x+2)^2 \cdot \left[3(x-1)^2\right] - (x-1)^3 \cdot D_x\left[2(x+2)\right]}{(x+2)^4}$$

Example 9: Let $f(x) = \dfrac{x^2}{\sqrt{x+4}}$. Find $f'(x)$.

Using Quotient Rule:

$$f'(x) = \frac{\sqrt{x+4} \cdot D_x\left[x^2\right] - \sqrt{x+4} \cdot D_x\left[\sqrt{x+4}\right]}{\left[\sqrt{x+4}\right]^2}$$

$$f'(x) = \frac{\sqrt{x+4} \cdot D_x\left[x^2\right] - \sqrt{x+4} \cdot D_x\left[\sqrt{x+4}\right]}{x+4}$$

Now use Chain Rule to find $D_x\left[x^2\right]$ and $D_x\left[\sqrt{x+4}\right]$:

$$D_x\left[x^2\right] = 2x$$

$$D_x\left[\sqrt{x+4}\right] = D_x\left[(x+4)^{1/2}\right] = \frac{1}{2}(x+4)^{-1/2} \cdot D_x[x+4]$$

$$= \frac{1}{2}(x+4)^{-1/2} \cdot [1] = \frac{1}{2}(x+4)^{-1/2}$$

Hence,

$$f'(x) = \frac{\sqrt{x+4} \cdot D_x\left[x^2\right] - \sqrt{x+4} \cdot D_x\left[\sqrt{x+4}\right]}{x+4}$$

$$f'(x) = \frac{\sqrt{x+4} \cdot [2x] - \sqrt{x+4} \cdot \left[\frac{1}{2}(x+4)^{-1/2}\right]}{x+4}$$

Example 10: Let $f(x)=\left(\frac{4-5x}{5+x}\right)^2$. Find $f'(x)$.

$$Note: f(x)=\left(\frac{4-5x}{5+x}\right)^2=\frac{(4-5x)^2}{(5+x)^2}$$

Using Quotient Rule:

$$f'(x)=\frac{(5+x)^2\cdot D_x\left[(4-5x)^2\right]-(4-5x)^2\cdot D_x\left[(5+x)^2\right]}{\left[(5+x)^2\right]^2}$$

$$f'(x)=\frac{(5+x)^2\cdot D_x\left[(4-5x)^2\right]-(4-5x)^2\cdot D_x\left[(5+x)^2\right]}{[5+x]^4}$$

Now use Chain Rule to find $D_x\left[(4-5x)^2\right]$ and $D_x\left[(5+x)^2\right]$:

$$D_x\left[(4-5x)^2\right]=2(4-5x)\cdot D_x[4-5x]=2(4-5x)\cdot[-5]=-10(4-5x)$$

$$D_x\left[(5+x)^2\right]=2(5+x)\cdot D_x[(5+x)]=2(5+x)\cdot[1]=2(5+x)$$

Hence,

$$f'(x)=\frac{(5+x)^2\cdot D_x\left[(4-5x)^2\right]-(4-5x)^2\cdot D_x\left[(5+x)^2\right]}{[5+x]^4}$$

$$f'(x)=\frac{(5+x)^2\cdot[-10(4-5x)]-(4-5x)^2\cdot[2(5+x)]}{[5+x]^4}$$

Example 11: Let $f(x)=\cos \pi x$. Find $f'(x)$.

Using Chain Rule:

Let $u=\pi x \qquad \Rightarrow \dfrac{du}{dx}=\pi$

$y=f(x)=\cos \pi x=\cos u \qquad \Rightarrow \qquad \dfrac{dy}{du}=-\sin u=-\sin \pi x$

$$f'(x)=\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\left[-\sin \pi x\right]\left[\pi\right]=-\pi \sin \pi x$$

Example 12: Let $f(x) = \cos(5x)^2$. Find $f'(x)$.

$Note: \cos(5x)^2$ is not the same as $\cos^2 5x$

Using Chain Rule:

Let $u = (5x)^2 = 25x^2 \quad \Rightarrow \dfrac{du}{dx} = 50x$

$y = f(x) = \cos(5x)^2 = \cos u \quad \Rightarrow \quad \dfrac{dy}{du} = -\sin u = -\sin\left(25x^2\right)$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[-\sin\left(25x^2\right)\right][50x] = -50x\sin\left(25x^2\right)$$

Example 12: Let $f(x) = 10\sec^3 x$. Find $f'(x)$.

Using Chain Rule:

$$f(x) = 10\sec^3 x = 10(\sec x)^3$$

Let $u = \sec x \quad \Rightarrow \dfrac{du}{dx} = \sec x \cdot \tan x$

$$y = f(x) = 10(\sec x)^3 = 10u^3 \quad \Rightarrow \quad \frac{dy}{du} = 30u^2 = 30(\sec x)^2$$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \left[30(\sec x)^2\right]\left[\sec x \cdot \tan x\right] = 30\sec^3 x \cdot \tan x$$

Example 13: Let $f(x) = 10\sin^2(5x-7)$. Find $f'(x)$.

Using Chain Rule:

$$f(x) = 10\sin^2(5x-7) = 10\left[\sin(5x-7)\right]^2$$

Let $u = \sin(5x-7)$ $\quad \Rightarrow \dfrac{du}{dx} = \cos(5x-7)\cdot D_x\left[5x-7\right]$

$$= \cos(5x-7)\cdot\left[5\right] = 5\cos(5x-7)$$

$y = f(x) = 10\left[\sin(5x-7)\right]^2 = 10u^2 \quad \Rightarrow \quad \dfrac{dy}{du} = 20u = 20\left(\sin(5x-7)\right)^2$

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \left[20\left(\sin(5x-7)\right)^2\right]\left[5\cos(5x-7)\right]$$

$$= 100\left(\sin(5x-7)\right)^2\cos(5x-7)$$