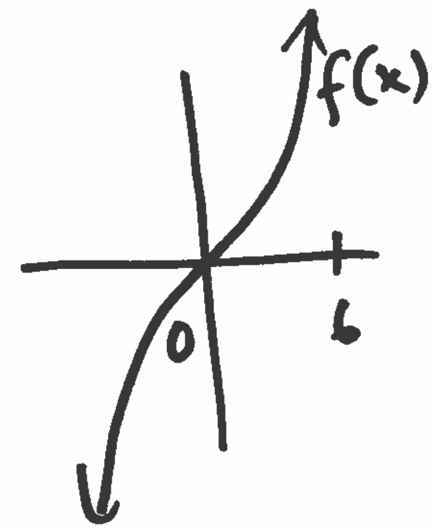
3.2 Rolle's Theorem and Mean Value Theorem

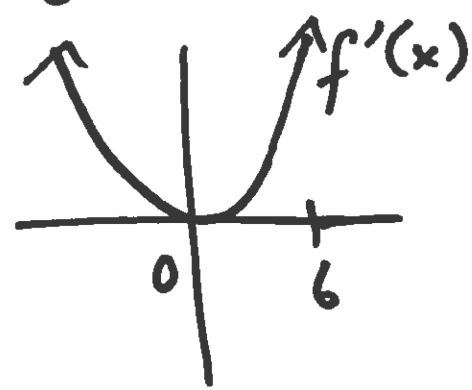
1) $f(x) = 2x^3$ $\quad [a, b] = [0, 6]$



a) Graph or $f(x)$ is continuous on $(0, 6)$

b) $f'(x) = 2 \cdot (3x^2) = 6x^2$

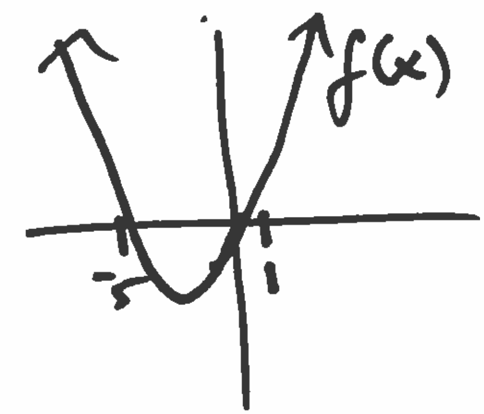
$f'(x)$ is defined on $(0, 6)$



2) $f(x) = x^2 + 4x - 5$ $\qquad [a,b] = [-5, 1]$

a) $f(a) = f(-5) = (-5)^2 + 4(-5) - 5 = 0$

$f(b) = f(1) = (1)^2 + 4(1) - 5 = 0$



b) $f(x)$ is continuous on $[-5, 1]$

c) $f'(x) = 2x + 4$

$f'(x)$ is defined (or exists) on $(-5, 1)$

set $f'(x) = 0$

$2x + 4 = 0$

$2x = -4$

$x = -2$

Note: $f'(x) = 2x + 4$

$f'(-2) = 2(-2) + 4 = 0$

Rolle's Theorem: If conditions ⓐ, ⓑ, ⓒ are met, then there a number "c" between (a, b) such that

$f'(c) = 0$.

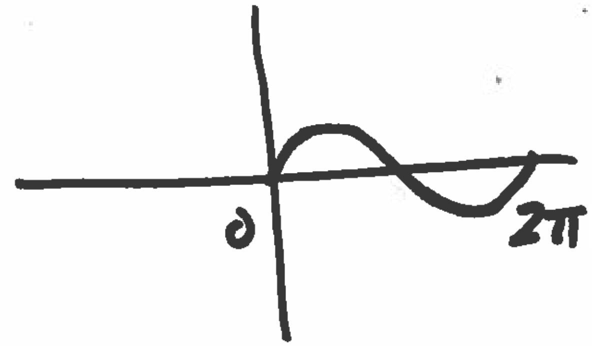
In this example, $c = -2$ and -2 is between $(-5, 1)$.

(2) $f(x) = \sin x$ $\qquad [a, b] = [0, 2\pi]$

a) $f(a) = f(0) = \sin 0 = 0$

$f(b) = f(2\pi) = \sin 2\pi = 0$

So $f(a) = f(b)$



b) $f(x)$ is continuous on $[0, 2\pi]$

c) $f'(x) = \cos x$

$f'(x)$ is defined (or exists) on $(0, 2\pi)$

Since conditions (a), (b), (c) are met, Rolle's Thm. says that we can find a number "c" between $(0, 2\pi)$ such that $f'(c) = 0$. Find c.

Finding C: set $f'(x) = 0$

$\cos x = 0$

Note: $\cos \frac{\pi}{2} = 0$; $\cos \frac{3\pi}{2} = 0$

So $x = \frac{\pi}{2}$; $x = \frac{3\pi}{2}$

So C can be $\pi/2$ or $3\pi/2$.

Note: $\frac{\pi}{2}$ is between $(0, 2\pi)$

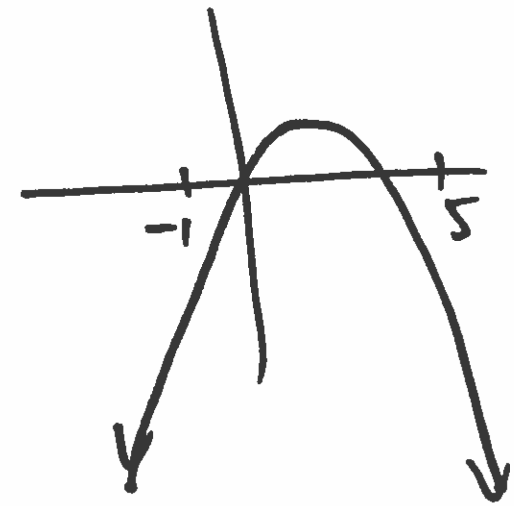
$\frac{3\pi}{2}$ is between $(0, 2\pi)$

(3) $f(x) = -x^2 + 4x \qquad [a, b] = [-1, 5]$

a) $f(a) = f(-1) = -(-1)^2 + 4(-1) = -5$

$f(b) = f(5) = -(5)^2 + 4(5) = -5$

So $f(a) = f(b)$



b) $f(x)$ is continuous on $[-1, 5]$

c) $f'(x) = -2x + 4$

$f'(x)$ is defined (or exists) on $(-1, 5)$

Since conditions (a), (b), (c) are met, Rollé's Thm. says that we can find a number "c" between $(-1, 5)$ such that $f'(c) = 0$. Find c.

Finding c: set $f'(x) = 0$

$$-2x + 4 = 0$$

$$-2x = -4$$

$$x = 2$$

So $c = 2$

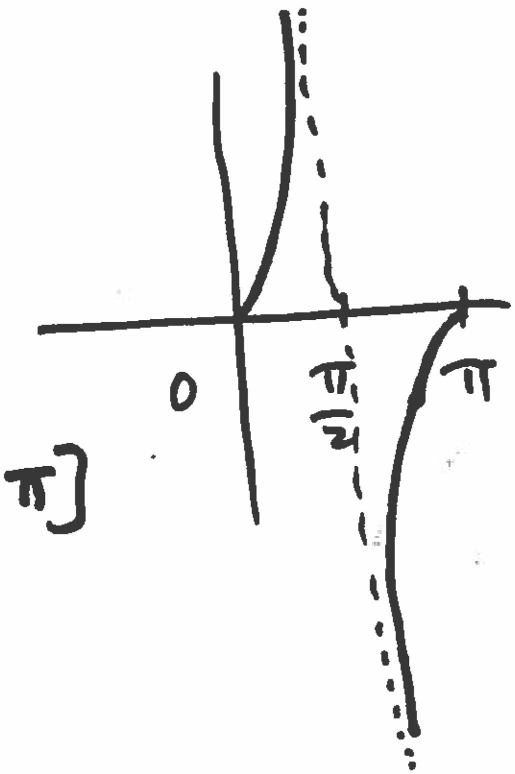
Note: c is between $(-1, 5)$.

(9) $f(x) = \tan x \qquad [a, b] = [0, \pi]$

a) $f(a) = f(0) = \tan 0 = 0$

$f(b) = f(\pi) = \tan \pi = 0$

So $f(a) = f(b)$



(b) $f(x)$ is discontinuous at $x = \pi/2$.

So $f(x)$ is not continuous on $[0, \pi]$

c) $f'(x) = \sec^2 x = (\sec x)^2$

$f'(\pi/2) = (\sec \pi/2)^2 = \text{undefined}$

Since $f'(\pi/2)$ is undefined, we say that the function $f(x) = \tan x$ is not differentiable at $x = \pi/2$.

$f'(0) = (\sec 0)^2 = 0$

Since $f'(0)$ is defined, we say that the function $f(x) = \tan x$ is differentiable at $x = 0$.

$f'(3\pi/2) = \left(\sec \frac{3\pi}{2}\right)^2 =$ undefined

Since $f'(3\pi/2)$ is undefined, the function $f(x) = \tan x$ is not differentiable at $x = 3\pi/2$

For the function $f(x) = \tan x$, conditions b, c are not met. We cannot apply Rolle's Thm.

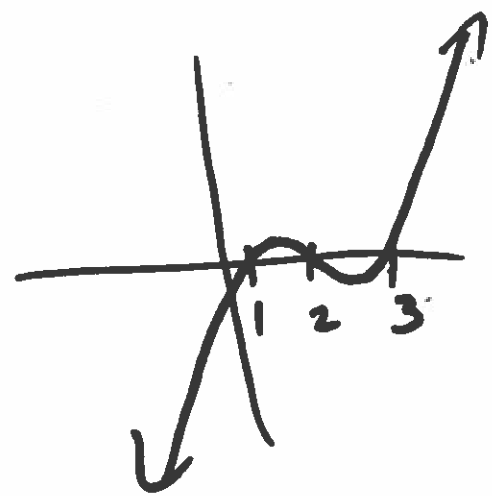
5 $f(x) = (x-1)(x-2)(x-3)$ $\qquad [a,b] = [1,3]$

Can we apply Rolle's Thm.?

a) $f(a) = f(1) = (1-1)(1-2)(1-3) = 0$

$f(b) = f(3) = (3-1)(3-2)(3-3) = 0$

So $f(a) = f(b)$



b) $f(x)$ is continuous on $[1,3]$

c) $f(x) = (x-1)(x-2)(x-3)$

$f(x) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 6\cdot(2x) + 11(1) - 0$

$f'(x) = 3x^2 - 12x + 11$

$f'(x)$ is defined (or exists) on $(1, 3)$

So we can apply Rolle's Thm. because conditions

a, b, c are met.

Finding c: set $f'(x) = 0$

$$3x^2 - 12x + 11 = 0$$

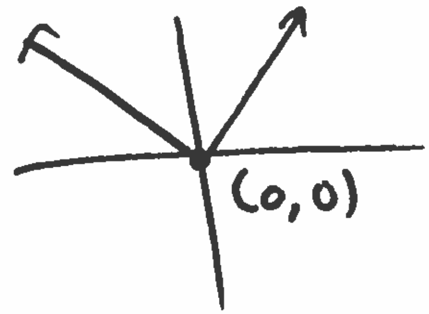
Use Quadratic Formula:

$$x = 2.57735026919$$

$$x = 1.42264973081$$

V-shaped Graph like $f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

Show that $f'(0)$ does not exist.



$$f'(c) = \lim_{\Delta x \to 0} \frac{f(x+c) - f(c)}{\Delta x}$$

Also,

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

Hence,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - c} = ??$$

(0,0) is called a "sharp turn" point and $f'(x)$ does not exist at (0,0).

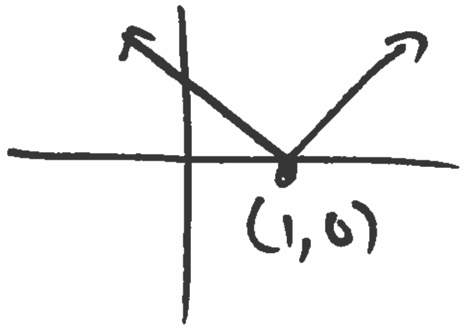
$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{-x - 0}{x - 0} = \lim_{x \to 0^-} \frac{-x}{x} = -1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x - 0}{x - 0} = \lim_{x \to 0^+} 1 = 1$$

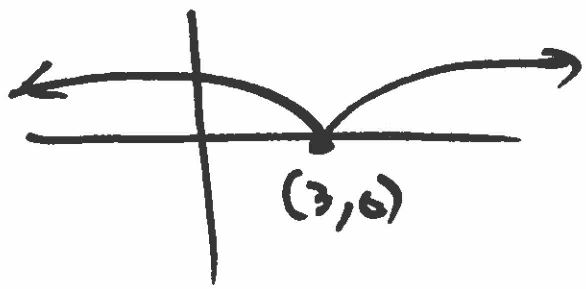
Therefore,

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \text{DNE}$$

In general, $f'(x)$ does not exist at "sharp turn" point.



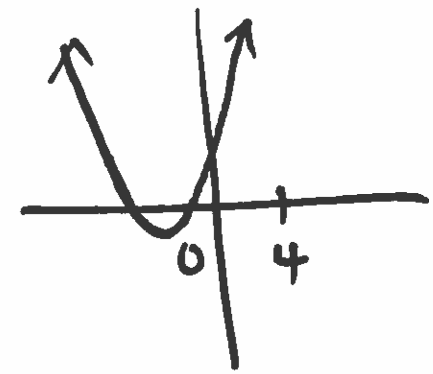
$f'(x)$ does not exist at $(1, 0)$



$f'(x)$ does not exist at $(3, 0)$

Mean Value Theorem (MVT)

6) $f(x) = x^2 + 3x + 2 \qquad [0, 4]$



$f(a) = f(0) = 0^2 + 3 \cdot 0 + 2 = 2$

$f(b) = f(4) = 4^2 + 3 \cdot 4 + 2 = 30$

$$S = \frac{f(b) - f(a)}{b - a} = \frac{30 - 2}{4 - 0} = \frac{28}{4} = 7$$

a) $f(x)$ is continuous on $[0, 4]$

b) $f'(x) = 2x + 3$

$f'(x)$ is defined on $(0, 4)$

If conditions ⓐ, ⓑ are met, then MVT says there a number "c" between $(0, 4)$ such that

$f'(c) = S = 7$

Find c: set $f'(x) = 7$

$$2x + 3 = 7$$
$$2x = 4$$
$$x = 2$$

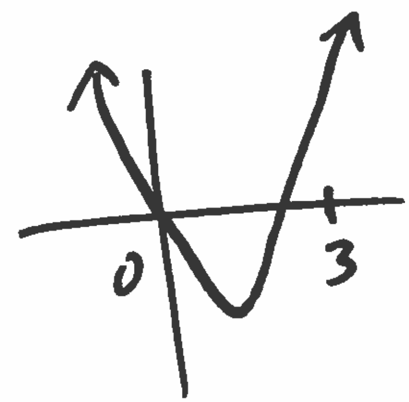
So $c = 2$

⑦ $f(x) = x^4 - 8x$ $\qquad [a, b] = [0, 3]$

$f(a) = f(0) = 0^4 - 8(0) = 0$

$f(b) = f(3) = 3^4 - 8(3) = 57$

$$S = \frac{f(b) - f(a)}{b - a} = \frac{57 - 0}{3 - 0} = 19$$



a) $f(x)$ is continuous on $[0, 3]$.

b) $f'(x) = 4x^3 - 8$

$f'(x)$ exists on $(0, 3)$

Since conditions ⓐ, ⓑ are met, MVT says there is a number "c" in $(0, 3)$ such that

$$f'(c) = S = 19$$

Find c: $\quad$ Set $f'(x) = S = 19$

$$4x^3 - 8 = 19$$

$$4x^3 = 27$$

$$x^3 = 27/4$$

$$\sqrt[3]{x^3} = \sqrt[3]{27/4}$$

$$x = \sqrt[3]{27/4} \simeq 1.889$$

So $C = \sqrt[3]{27/4} = 1.889$

(8)

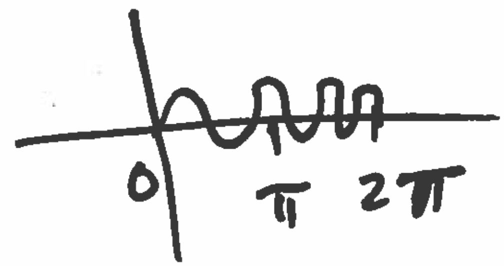
$f(x) = \sin 4x \qquad [a, b] = [0, 2\pi]$

$f(a) = \sin(4 \cdot 0) = \sin 0 = 0$

$f(b) = \sin(4 \cdot 2\pi) = \sin(8\pi) = 0$

$$S = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{2\pi - 0} = \frac{0}{2\pi} = 0$$

a) $f(x)$ is continuous on $[0, 2\pi]$



b) $f'(x) = \cos 4x \cdot D_x(4x) = \cos 4x \cdot 4 = 4 \cdot \cos 4x$

$f'(x)$ is defined (or exists) on $(0, 2\pi)$

Since conditions a and b are met, MVT says there is a number "c" in $(0, 2\pi)$ such that

$f'(c) = S = 0$

Find C: $\quad$ Set $f'(x) = S = 0$

$$4 \cdot \cos 4x = 0$$

$$\cos 4x = 0$$

Note: $\cos \pi/2 = 0$, $\cos \frac{3\pi}{2} = 0$, $\cos \frac{5\pi}{2} = 0$, ...

So $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} = 4x$

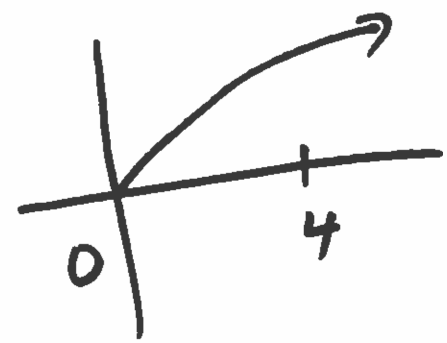
$\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8} = x$

So $C = \pi/8,\ 3\pi/8,\ 5\pi/8,\ 7\pi/8,\ \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8}$

(9) $f(x) = x^{1/2}$ $\qquad [a, b] = [0, 4]$

$$f(a) = f(0) = 0^{1/2} = 0$$
$$f(b) = f(4) = 4^{1/2} = 2$$

$$S = \frac{f(b) - f(a)}{b - a} = \frac{2-0}{4-0} = \frac{1}{2}$$



a) $f(x)$ is continuous on $[0, 4]$.

b) $f'(x) = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$

$f'(x)$ exists on $(0, 4)$

MVT says there is a number "c" between $(0,4)$ such that $f'(c) = S = \frac{1}{2}$

Finding C: set $f'(x) = S = \frac{1}{2}$

$$\frac{1}{2\sqrt{x}} = \frac{1}{2}$$

$$\frac{1}{2\sqrt{x}} - \frac{1}{2} = 0$$

$$\frac{2 - 2\sqrt{x}}{2\sqrt{x} \cdot 2} = 0$$

set $2 - 2\sqrt{x} = 0$

$$-2\sqrt{x} = -2$$

$$\sqrt{x} = 1$$

$$(\sqrt{x})^2 = 1^2$$

$$x = 1$$

set $2\sqrt{x} \cdot 2 = 0$

$$\sqrt{x} = 0$$

$$(\sqrt{x})^2 = 0$$

$$x = 0$$

Note: 0 is not in $(0,4)$

So $c = 1$