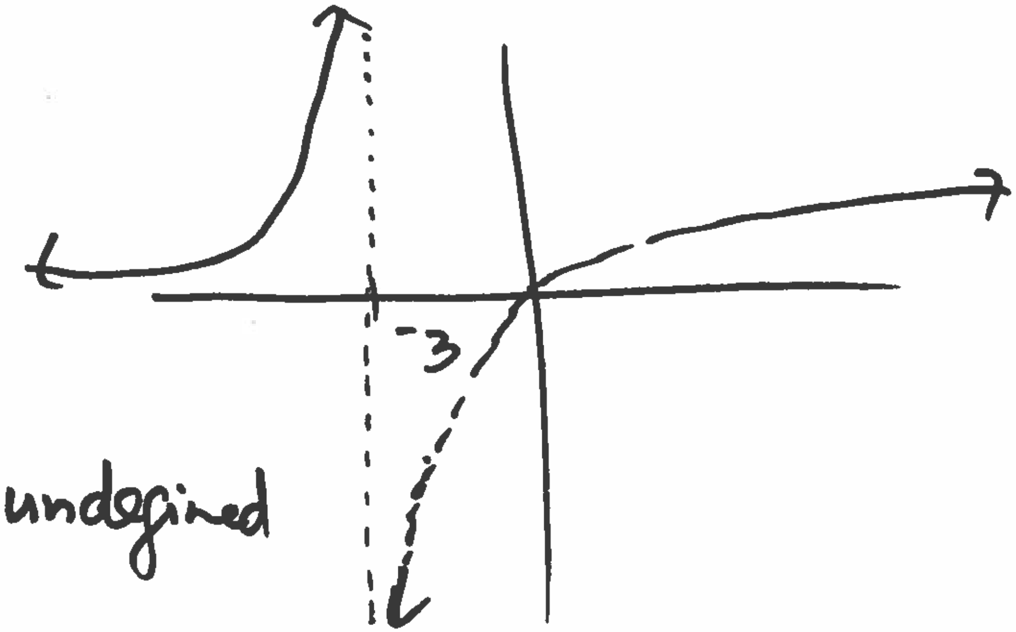
3.2 Rolle's Thm.. and Mean Value Thm.

$$f(x) = \frac{x+1}{x+3} = \frac{N}{D}$$

$$y = (x+1)/(x+3)$$



If $x = -3$, $y = \frac{-3+1}{-3+3} = \text{undefined}$

$f(x)$ is discontinuous at $x = -3$

$$f'(x) = \frac{(x+3) \cdot D_x(x+1) - (x+1) \cdot D_x(x+3)}{(x+3)^2}$$

$$f'(x) = \frac{(x+3)(1) - (x+1)(1)}{(x+3)^2} = \frac{2}{(x+3)^2}$$

$$f'(0) = \frac{2}{(0+3)^2} = 2/9$$

$f'(x)$ is defined at $x = 0$

$$f'(1) = \frac{2}{(1+3)^2} = 1/16$$

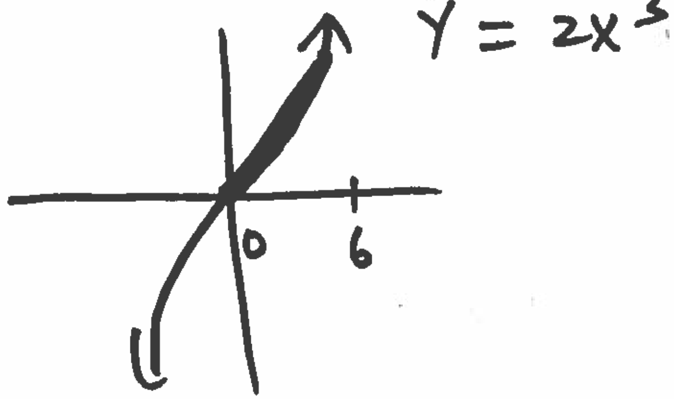
$f'(x)$ is defined at $x = 1$

$$f'(-3) = \frac{2}{(-3+3)^2} = \frac{2}{0} = \text{undefined}$$

$f'(x)$ is undefined at $x = -3$

$f'(x) = \dfrac{2}{(x+3)^2}$	$f(x) = \dfrac{x+1}{x+3}$
$f'(x)$ is defined at $x = 0$	$f(x)$ is differentiable at $x = 0$
$f'(x)$ is defined at $x = 1$	$f(x)$ is differentiable at $x = 1$
$f'(x)$ is undefined at $x = -3$	$f(x)$ is not differentiable at $x = -3$

$f(x) = 2x^3$ $[a, b] = [0, 6]$



$f(x)$ is continuous on $(0, 6)$

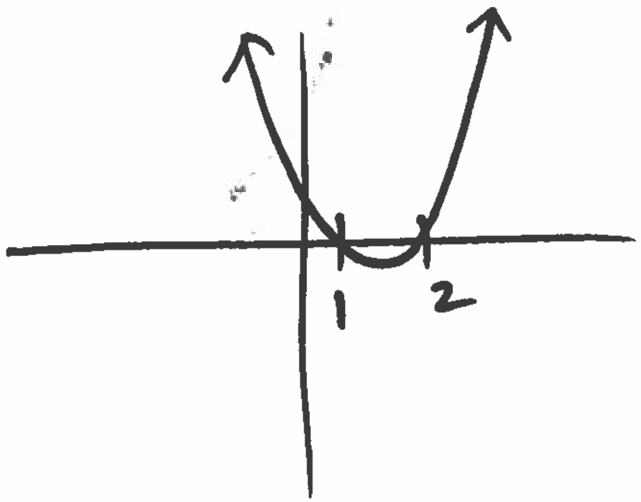
$f'(x) = 2 \cdot (3x^2) = 6x^2$

$f'(x)$ is defined (or exists) on $(0, 6)$

$f(a) = f(0) = 2(0)^3 = 0$

$f(b) = f(6) = 2 \cdot (6)^3 = 72$

$f(x) = x^2 - 3x + 2$ $\qquad [a, b] = [1, 2]$



$f(x)$ is continuous on $(1, 2)$

$f'(x) = 2x - 3$

$f'(x)$ is defined (or exists) on $(1, 2)$

$f(a) = f(1) = (1)^2 - 3(1) + 2 = 0$

$f(b) = f(2) = (2)^2 - 3(2) + 2 = 0$

Note: $f(a) = f(b)$

set $f'(x) = 0$

$2x - 3 = 0$

$x = 1.5$

Let $c = 1.5$

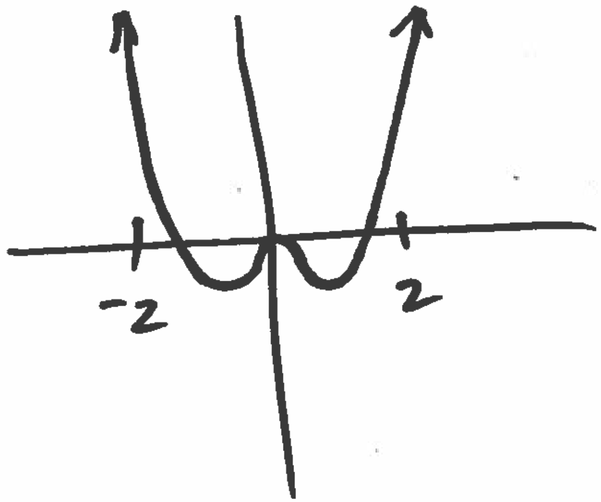
$$f'(c) = f'(1.5) = 2.(1.5)-(3) = 0$$

Rolle's Thm: If $f(x)$ is con't and $f'(x)$ exists on the interval (a, b) and $f(a) = f(b)$,

then there exists a numbe c between a and b such that $f'(c) = 0$.

$f(x) = x^4 - 2x^2 \qquad [a, b] = [-2, 2]$

Question: Can Rolle's Thm. be applied for $f(x)$? Yes



$f(x)$ is con't on $(-2, +2)$

$f'(x) = 4x^3 - 4x$.

$f'(x)$ is defined (or exists) on $(-2, 2)$

$f(a) = f(-2) = (-2)^4 - 2(-2)^2 = 8$

$f(b) = f(2) = (2)^4 - 2(2)^2 = 8$

$f(a) = f(b)$

So we can apply Rolle's Thm.

That means we can find a number $c \in (a, b)$ such that $f'(c) = 0$.

Finding c: set $f'(x) = 0$

$$4x^3 - 4x = 0$$

$$(4x)(x^2 - 1) = 0$$

$4x = 0$	$x^2 - 1 = 0$
$x = 0$	$x^2 = 1$
	$\sqrt{x^2} = \pm\sqrt{1}$
	$x = \pm 1$

Let $c = 0, -1, 1$

Note: $f'(0) = 0$
$f'(1) = 0$
$f'(-1) = 0$

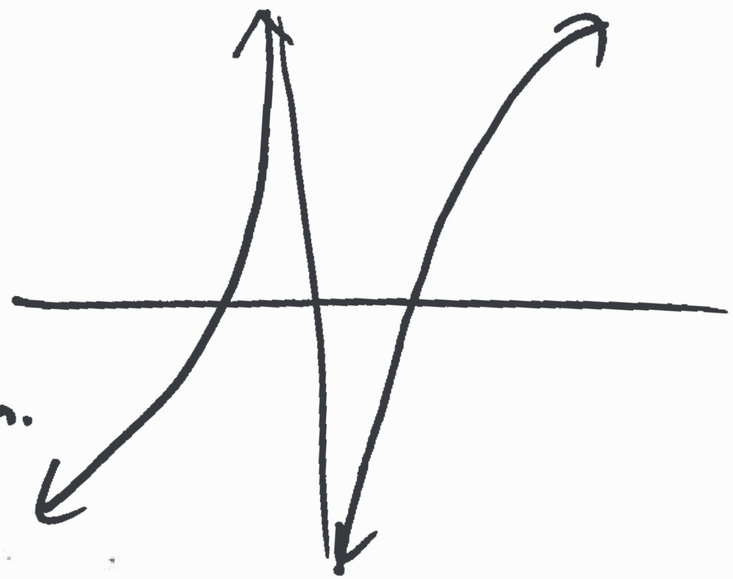
$$f(x) = \frac{x^2 - 1}{x} \qquad [a, b] = [-1, 1]$$

Can we apply Rolle's Thm.? No

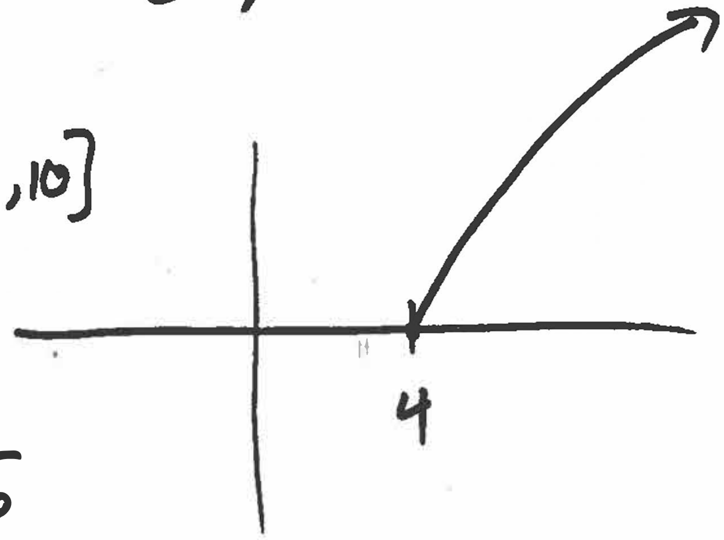
$f(x)$ is discon't at $x = 0$.

$f(x)$ is not con't on $(-1, 1)$

So we cannot apply Rolle's Thm.



$f(x) = x \cdot \sqrt{x-4}$ $\qquad [a, b] = [4, 10]$



$f(x)$ is con't on $[a,b] = [4,10]$

$f(a) = f(4) = 4 \cdot \sqrt{4-4} = 0$

$f(b) = f(b) = 10 \cdot \sqrt{10-4} = 10\sqrt{6}$

so $f(a) \neq f(b)$

Therefore, Rolle's Thm. cannot be applied.

Let $f(x) = |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$

Show that $f'(x)$ does not exist at $x = 0$.

Alternative Definition of Derivative: $f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$.

Hence, $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$.

Limit from the left: $\lim\limits_{x \to 0^-} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{-x - 0}{x - 0} = \lim\limits_{x \to 0^-} \dfrac{-x}{x} = \lim\limits_{x \to 0^-} (-1) = -1$

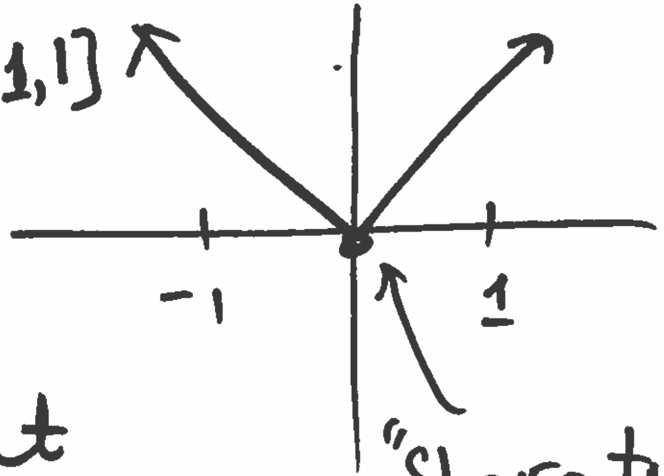
Note: For $x \to 0^-$, $f(x) = -x$; and $f(0) = |0| = 0$

Limit from the right: $\lim\limits_{x \to 0^+} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{x - 0}{x - 0} = \lim\limits_{x \to 0^+} \dfrac{x}{x} = \lim\limits_{x \to 0^+} (1) = 1$

Note: For $x \to 0^+$, $f(x) = x$; and $f(0) = |0| = 0$

Therefore, $f'(0) = \lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} =$ DNE

$f(x) = |x| \quad [a, b] = [-1, 1]$



"Sharp turn" point $(0, 0)$

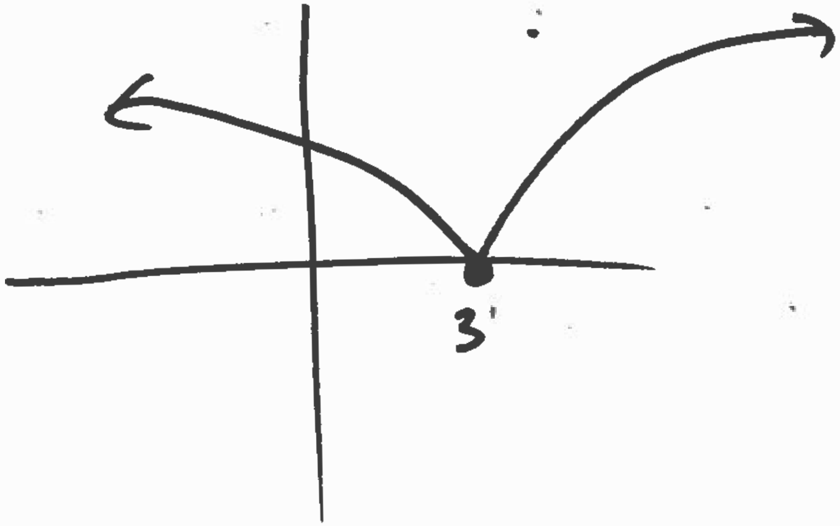
$f(x)$ is con't on $(-1, 1)$

$f'(x)$ does not exist at "sharp turn" point.

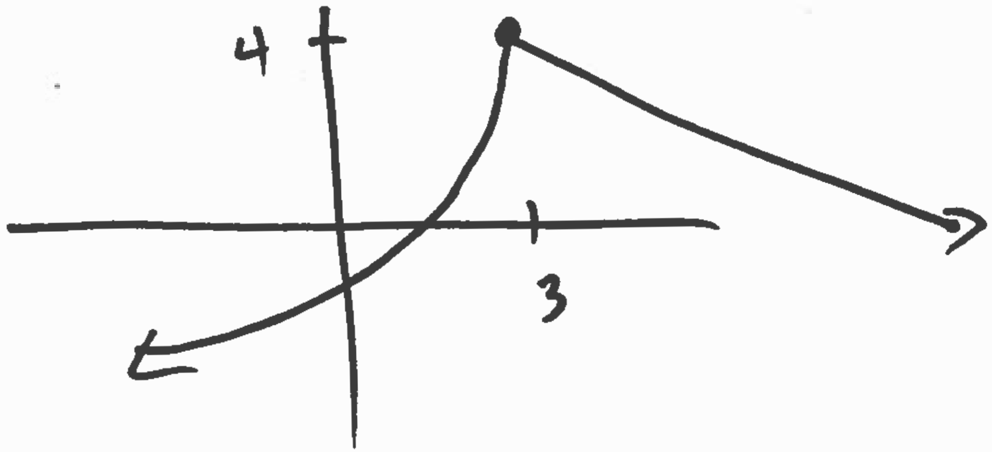
$f'(x)$ does not exist at $x = 0$.

So we cannot apply Rolle's Thm.

Examples of "sharp turn" point:



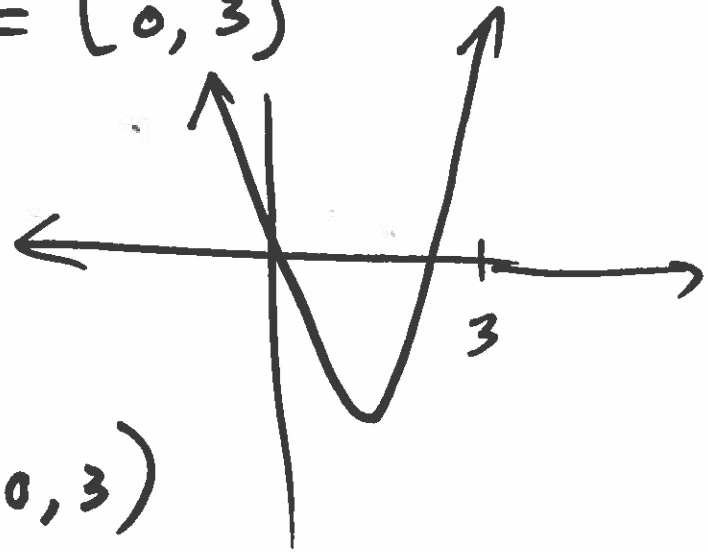
$f'(x)$ does not exist at $x=3$.



$f'(x)$ does not exist at $x=3$.

Mean Value Thm.

$f(x) = x^4 - 8x$ $\quad [a, b] = [0, 3]$



$f(x)$ is con't on $(0, 3)$.

$f'(x) = 4x^3 - 8$

$f'(x)$ is defined (or exists) on $(0, 3)$

$f(a) = f(0) = 0^4 - 8(0) = 0$

$f(b) = f(3) = 3^4 - 8 \cdot 3 = 57$.

$$\frac{f(b) - f(a)}{b - a} = \frac{57 - 0}{3 - 0} = 57/3 = 19$$

set $f'(x) = \dfrac{f(b) - f(a)}{b - a}$

$$4x^3 - 8 = 57/3 = 19$$

$$4x^3 = 27$$

$$x^3 = 27/4$$

$$\sqrt[3]{x^3} = \sqrt[3]{27/4} = 1.889$$

Let $c = \sqrt[3]{27/4} = 1.889$

Note : $f'(c) = \cancel{4}4(1.889)^3 - 8 = 19 = \dfrac{f(b) - f(a)}{b - a}$

Mean Value Thm:

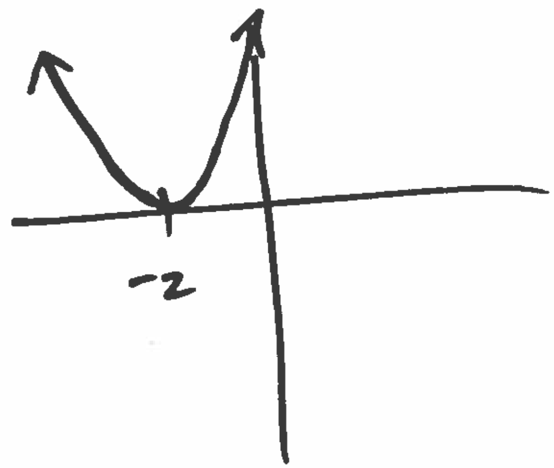
If $f(x)$ is con't and $f'(x)$ exists on (a, b).

Then, we can find a number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Mean Value Thm:

$f(x) = (x+2)^2$ $\quad [a, b] = [0, 2]$



$f(x)$ is con't on $(0, 2)$

$$f'(x) = 2 \cdot (x+2)^1 \cdot D_x(x+2)$$

$$= 2(x+2)(1)$$

$$f'(x) = 2x+4$$

$f'(x)$ is defined (or exists) on $(0, 2)$

So we can apply Mean Value Thm.

That means we can find a number $c \in (a, b)$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

Finding c:

$$f(a) = f(0) = (0+2)^2 = 4$$

$$f(b) = f(2) = (2+2)^2 = 16$$

$$\frac{f(b)-f(a)}{b-a} = \frac{16-4}{2-0} = \frac{12}{2} = 6$$

set $f'(x) = 6$

$$2x+4 = 6$$

$$2x = 2$$

$$x = 1$$

Let $c = 1$ Answer

Note: $f'(c) = f'(1) = 6 = \dfrac{f(b)-f(a)}{b-a}$