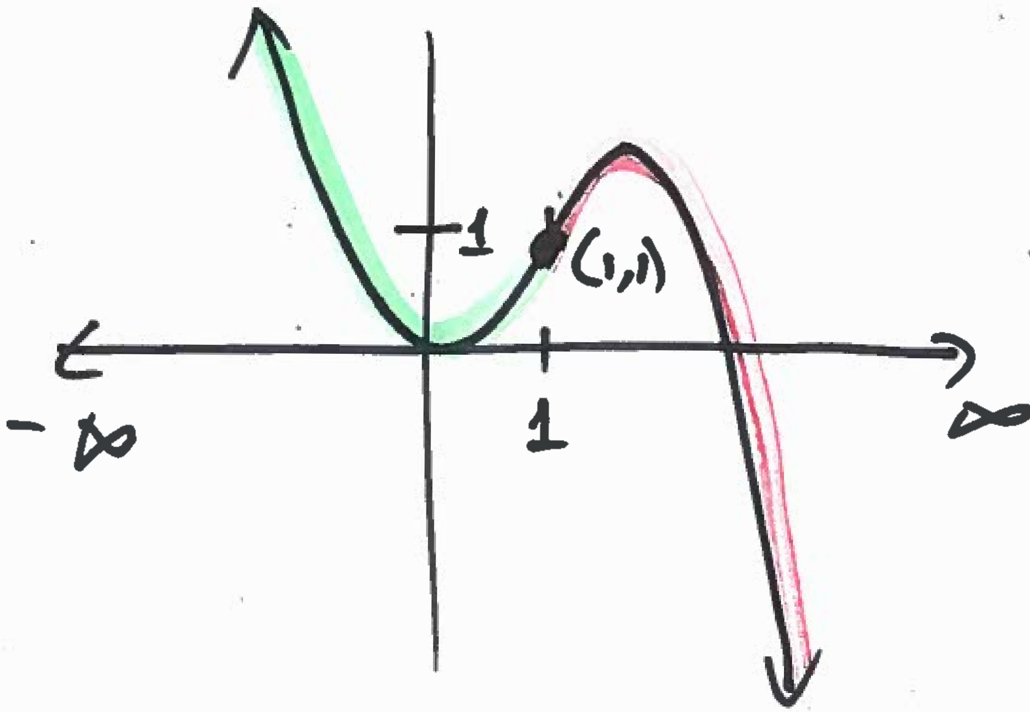
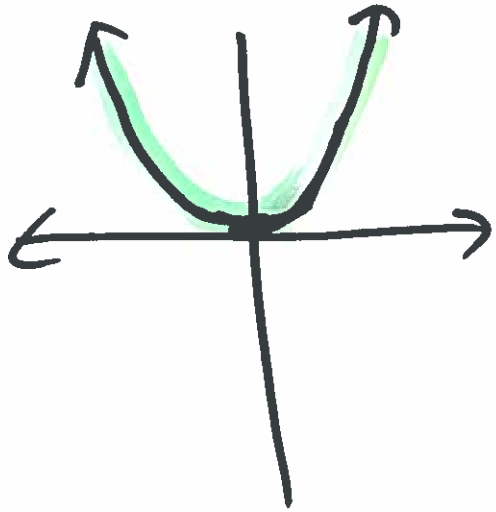
3.4 Concavity



(1, 1) is called point of inflection

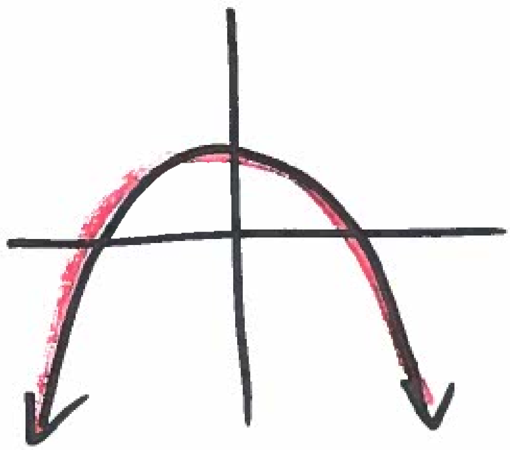
Graph concaves up on $(-\infty, 1)$

Graph concaves down on $(1, \infty)$



Graph concaves up on $(-\infty, \infty)$

No point of inflection;
No changes in concavity.



Graph concaves down
on $(-\infty, \infty)$

Note: No point of inflection;
No changes in concavity

Example 1

#1

$g(x) = 3x^2 - x^3$

Find point of inflection and intervals on which graph concaves up or down.

$g'(x) = 6x - 3x^2$

$g'(x) = 6 - 6x$

set $g''(x) = 0$

$6 - 6x = 0$

$6 = 6x$

$x = 1$

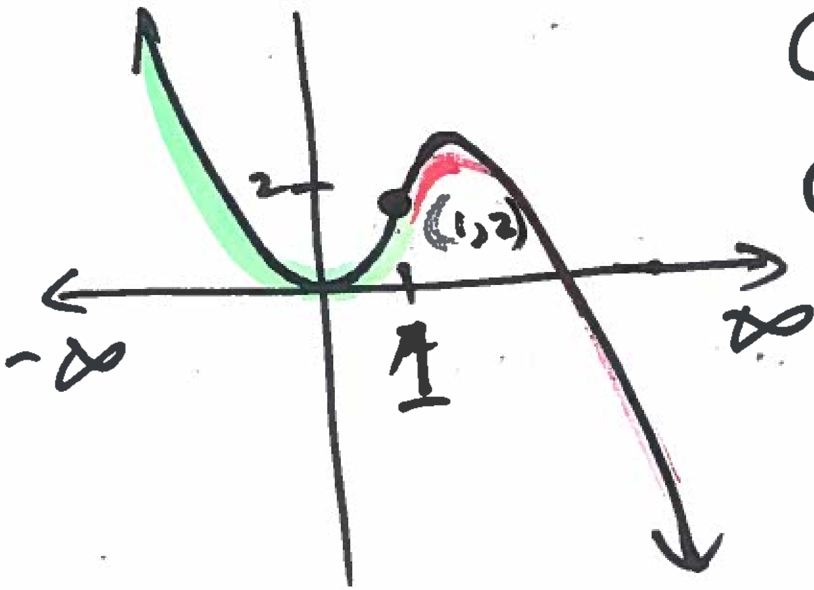
when $x = 1$, $y = 3x^2 - x^3$

$y = 3(1)^2 - (1)^3 = 2$

Point of inflection $= (1, 2)$

**1

#1



Graph concaves up on $(-\infty, 1)$

Graph concaves down on $(1, \infty)$

**1

Example 2

$f(x) = -x^3 + 6x^2 - 5$ #2

a) Find point of inflection.

b) Find where graph concaves up or down.?

$f'(x) = -3x^2 + 12x$

$f''(x) = -6x + 12$

set $f''(x) = 0$

$-6x + 12 = 0$

$-6x = -12$

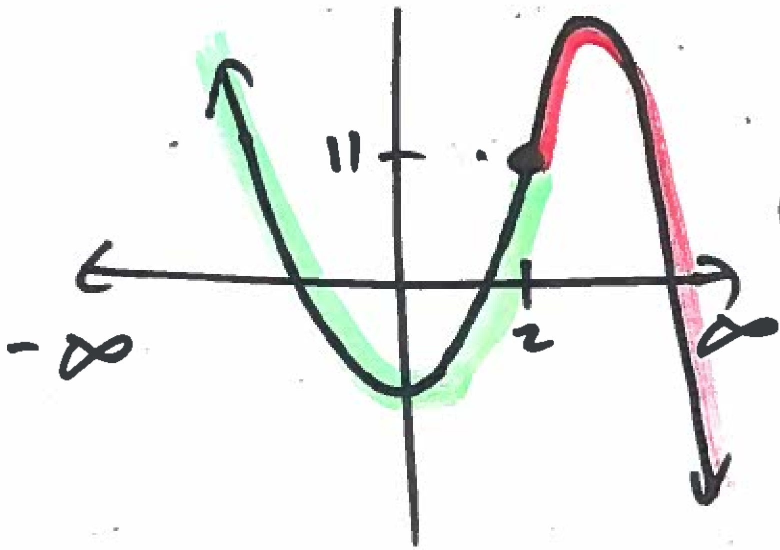
$x = 2$

$y = -x^3 + 6x^2 - 5 = -(2)^3 + 6(2)^2 - 5 = 11$

*2 Point of inflection $= (2, 11)$

#2

Point of inflection = (2, 11)



Graph concaves up on $(-\infty, 2)$

Graph concaves down On $(2, \infty)$

*2

Recall:

$$D_x\left[\text{base}^{\text{Exp}}\right]$$

$$= \text{Exp}\cdot(\text{base})^{\text{Exp}-1}\cdot D_x(\text{base})$$

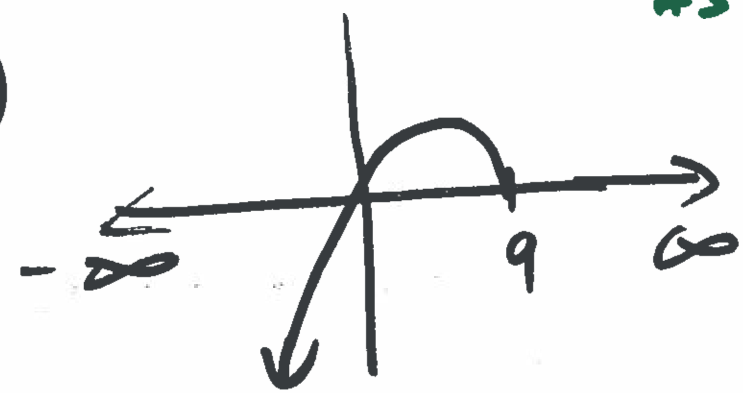
Chain Rule Shortcut

Example 3

#3

$$f(x) = x \cdot \sqrt{9-x} = x \cdot \sqrt{\ }(9-x)$$

$$f(x) = \underbrace{x}_{F} \cdot \underbrace{(9-x)^{+1/2}}_{S}$$



$$f'(x) = (x) \cdot D_x\left[(9-x)^{1/2}\right] + (9-x)^{1/2} \cdot D_x(x)$$

$$f'(x) = (x) \cdot \left[\tfrac{1}{2}(9-x)^{-1/2} \cdot (-1)\right] + (9-x)^{1/2}(1)$$

$$f'(x) = \frac{-x}{2(9-x)^{1/2}} + \frac{(9-x)^{\frac{1}{2}}}{1}$$

$$f'(x) = \frac{(-x)(1) + \left[2(9-x)^{1/2}\right](9-x)^{1/2}}{2(9-x)^{1/2}}$$

$\boxed{*\ 3}$

#3

$$f'(x) = \frac{-x + 2\cdot(9-x)}{2(9-x)^{1/2}} = \frac{-x+18-2x}{2(9-x)^{1/2}}$$

$$f'(x) = \frac{-3x+18}{2(9-x)^{1/2}}$$

$$f''(x) = \frac{\left[2(9-x)^{1/2}\right]\cdot D_x(-3x+18) - (-3x+18)\cdot D_x\left(2(9-x)^{1/2}\right)}{\left[2\cdot(9-x)^{1/2}\right]^2}$$

$$f''(x) = \frac{2(9-x)^{1/2}(-3) - (-3x+18)\left[2\cdot\frac{1}{2}(9-x)^{-1/2}\cdot(-1)\right]}{4\cdot(9-x)}$$

$\boxed{*\ 3}$

#3

$$f''(x) = \frac{-6(9-x)^{1/2} + (-3x+18)(9-x)^{-1/2}}{36-4x}$$

set $-6(9-x)^{1/2} + (-3x+18)(9-x)^{-1/2} = 0$

$$\frac{-6(9-x)^{1/2}(9-x)^{1/2}}{1\cdot(9-x)^{1/2}} + \frac{(-3x+18)}{(9-x)^{1/2}} = 0$$

$$\frac{-6(9-x) + -3x + 18}{(9-x)^{1/2}} = 0$$

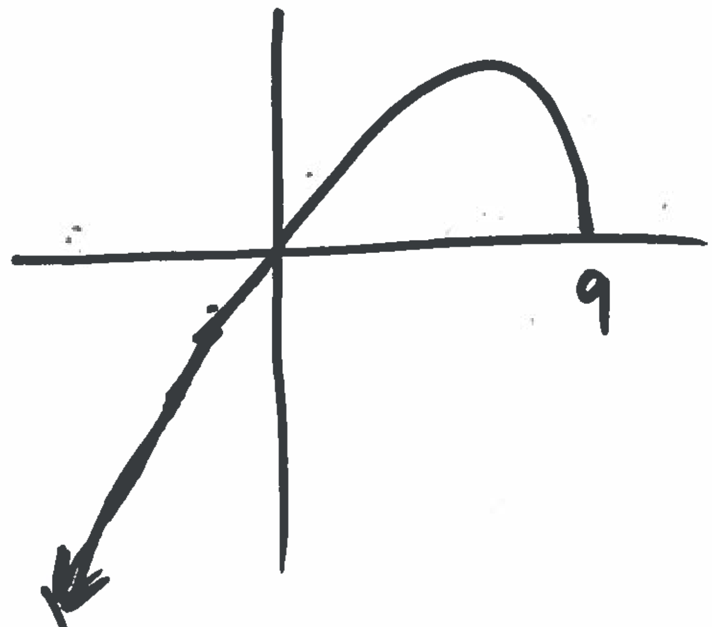
$$\frac{3x-36}{(9-x)^{1/2}} = 0$$

$3x - 36 = 0$

$x = 12$

set $36-4x=0$

$x = 9$



$\boxed{*3}$ Graph concaves down $(-\infty, 9)$

#3

$$\frac{-6(9-x)^{1/2}\cdot(9-x)^{1/2}}{1\cdot(9-x)^{1/2}}+\frac{(-3x+18)}{(9-x)^{1/2}}=0$$

$$\frac{-6(9-x)+(-3x+18)}{(9-x)^{1/2}}=0$$

$$-54+6x-3x+18=0$$

$$-36+3x=0$$

$$x=12$$

Example 4

#4

$$f(x) = \frac{2x^2}{3x^2+1}$$

Find point(s) of inflection.
Find where graph concaves up or down.

$$f'(x) = \frac{(3x^2+1)\,D_x(2x^2) - (2x^2)\cdot D_x(3x^2+1)}{(3x^2+1)^2}$$

$$= \frac{(3x^2+1)(4x) - (2x^2)(6x)}{(3x^2+1)^2}$$

$$f'(x) = \frac{\cancel{12x^3} + 4x - \cancel{12x^3}}{(3x^2+1)^2} = \frac{4x}{9x^4+6x^2+1}$$

#5

#4

$$f''(x) = \frac{(9x^4+6x^2+1)\cdot(4) - (4x)(36x^3+12x)}{(9x^4+6x^2+1)^2}$$

set

$$36x^4 + 24x^2 + 4 - 144x^4 - 48x^2 = 0$$

$$-108x^4 - 24x^2 + 4 = 0$$

Divide by -4 :

$$27x^4 + 6x^2 - 1 = 0$$

$$(3x-1)(3x+1)(3x^2+1) = 0$$

set

$$\underbrace{(9x^4+6x^2+1)}_{\text{Can never be 0}}{}^2 = 0$$

Set $3x-1=0$

$x = 1/3$

$3x+1=0$

$x = -1/3$

$$\underbrace{3x^2+1}_{\text{Can never be zero}} = 0$$

#5

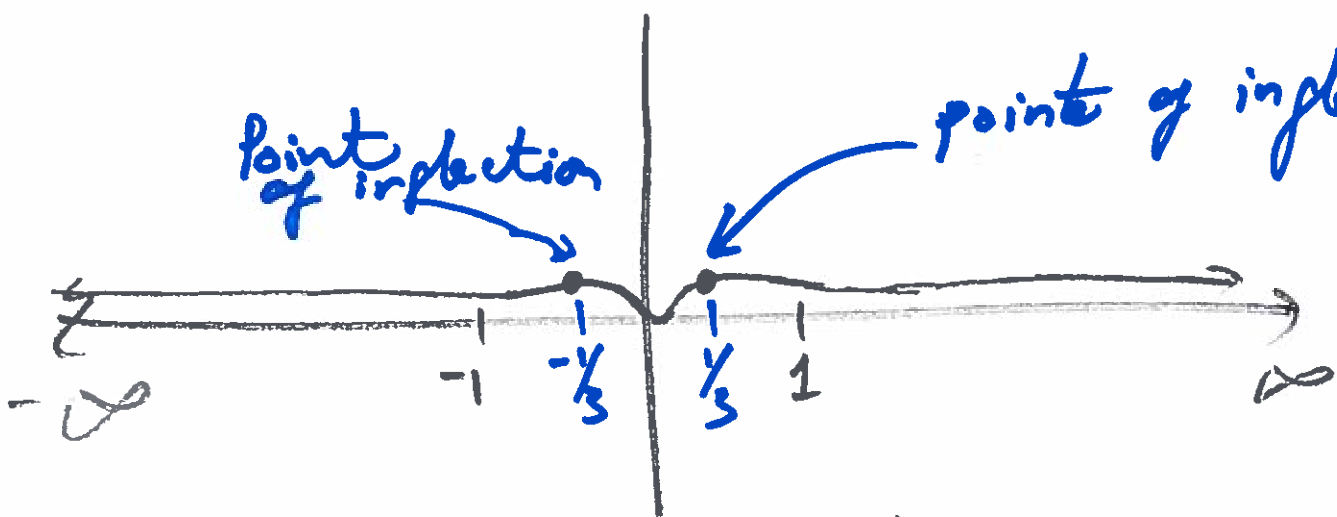
#4

When $x = 1/3$

$$y = \frac{2x^2}{3x^2+1} = \frac{2(1/3)^2}{3(1/3)^2+1} = 1/6 \qquad \left(\frac{1}{3}, \frac{1}{6}\right)$$

When $x = -1/3$, $y = 1/6$ $\qquad (-1/3, 1/6)$

Points of inflection : $(1/3, 1/6)$; $(-1/3, 1/6)$



Graph concaves up on $(-1/3, 1/3)$

Graph concaves down on $(-\infty, -1/3) \cup (1/3, \infty)$

#5

Example 5 #4

$$f(x) = x + 2\cdot\cos(2x) \qquad x \in [0, 2\pi]$$

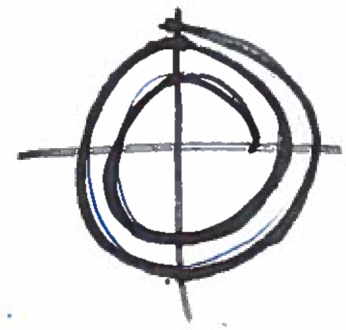
$$2x \in [0, 4\pi]$$

$$f'(x) = 1 + 2\cdot\left[-\sin(2x)\cdot 2\right]$$

$$f'(x) = 1 - 4\cdot\sin(2x)$$

$$f''(x) = 0 - 4\cdot\left[\cos(2x)\cdot 2\right]$$

$$f''(x) = -8\cdot\cos 2x$$



Set $f''(x) = 0$

$$-8\cdot\cos 2x = 0$$

$$\cos(2x) = 0$$

$$2x = \frac{1\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2},$$

$$x = \pi/4, \ 3\pi/4, \ 5\pi/4, \ 7\pi/4$$

#1,

#4

Graph concaves up: $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right) \cup \left(\frac{5\pi}{4}, \frac{7\pi}{4}\right)$

Graph concaves down: $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \frac{5\pi}{4}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right)$

*6