Formula A: $\int \frac{1}{\sqrt{a^2 - u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$

Formula B: $\int \frac{1}{a^2 + u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$

Formula C: $\int \frac{1}{u\sqrt{u^2 - a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right) + C$

Example 1: Find $\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{1}{\sqrt{4-x^2}}\,dx$

Recall: $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$

Let $a^2 = 4 \quad \Rightarrow \quad a = 2$

Let $u^2 = x^2 \quad \Rightarrow \; u = x \quad \Rightarrow \; \frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$

$$\int \frac{dx}{\sqrt{4-x^2}} = \int \frac{1}{\sqrt{4-x^2}}\,dx = \int \frac{1}{\sqrt{a^2-u^2}}\,du$$

$$= \arcsin\left(\frac{u}{a}\right) + C = \arcsin\left(\frac{x}{2}\right) + C$$

Example 2: Find $\int \frac{1}{x\sqrt{25x^2-1}}\,dx$

Recall: $\int \frac{1}{u\sqrt{u^2-a^2}}\,du = \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C$

Let $a^2=1 \quad \Rightarrow \quad a=1$

Let $u^2=25x^2 \quad \Rightarrow \quad u=5x \quad \Rightarrow \quad x=\frac{1}{5}u$

Also, $\frac{du}{dx}=5 \Rightarrow \quad du=5dx \Rightarrow \quad \frac{1}{5}du=dx$

$$\int \frac{1}{x\sqrt{25x^2-1}}\,dx = \int \frac{1}{\frac{1}{5}u\sqrt{u^2-a^2}}\,\frac{1}{5}\,du = \int \frac{1}{u\sqrt{u^2-a^2}}\,du$$

$$= \frac{1}{a}\operatorname{arcsec}\left(\frac{|u|}{a}\right)+C = \operatorname{arcsec}\left(\frac{|5x|}{1}\right)+C$$

Example 3: Find $\int \frac{1}{\sqrt{16-(x+8)^2}}\,dx$

Recall: $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right)+C$

Let $a^2 = 16 \quad \Rightarrow \quad a = 4$

Let $u^2 = (x+8)^2 \quad \Rightarrow \quad u = x+8 \quad \Rightarrow \; x = u-8$

Also $\frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$

$$\int \frac{1}{\sqrt{16-(x+8)^2}}\,dx = \int \frac{1}{\sqrt{a^2-u^2}}\,dx = \arcsin\left(\frac{u}{a}\right)+C$$

$$= \arcsin\left(\frac{x+8}{4}\right)+C$$

Example 4: Find $\int \frac{\cos x}{\sqrt{25-\sin^2 x}}dx = \int \frac{1}{\sqrt{25-\sin^2 x}}\cos x dx$

Recall: $\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\left(\frac{u}{a}\right)+C$

Let $a^2 = 25 \quad \Rightarrow \quad a = 5$

Let $u^2 = \sin^2 x \quad \Rightarrow \quad u = \sin x$

Also, $\frac{du}{dx} = \cos x \quad \Rightarrow \quad du = \cos x dx$

$$\int \frac{\cos x}{\sqrt{25-\sin^2 x}}dx = \int \frac{1}{\sqrt{25-\sin^2 x}}\cos x dx = \int \frac{1}{\sqrt{a^2-u^2}}du$$

$$= \arcsin\left(\frac{u}{a}\right)+C = \arcsin\left(\frac{\sin x}{5}\right)+C$$

Example 5: Find $\int \frac{2x+3}{x^2+1}dx = \int \frac{2x}{x^2+1}dx + \int \frac{3}{x^2+1}dx$

For $\int \frac{2x}{x^2+1}dx$:

Let $u = x^2+1$; $\quad \frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2xdx$

$$\int \frac{2x}{x^2+1}dx = \int \frac{1}{x^2+1}2xdx = \int \frac{1}{u}du = \ln|u| + C = \ln|x^2+1| + C$$

Example 5 (con't):

For $\int \frac{3}{x^2+1}\,dx = 3\int \frac{1}{x^2+1}\,dx$:

Recall: $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

Let $a^2=1 \;\Rightarrow\; a=1;\quad u^2=x^2 \;\Rightarrow\; u=x$; Also, $\frac{du}{dx}=1 \;\Rightarrow\; du=dx$

$$\int \frac{3}{x^2+1}\,dx = 3\int \frac{1}{x^2+1}\,dx = 3\int \frac{1}{a^2+u^2}\,du = 3\cdot\frac{1}{a}\left[\arctan\left(\frac{u}{a}\right)+C\right]$$

$$= 3\left[\arctan\left(\frac{x}{1}\right)+C\right]$$

So $\int \frac{2x+3}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx + \int \frac{3}{x^2+1}\,dx = \ln\left|x^2+1\right| + 3\left[\arctan\left(\frac{x}{1}\right)\right]+C$

Example 6: Find $\int \frac{8dx}{\sqrt{1-12x^2}} = 8\int \frac{1}{\sqrt{1-12x^2}}dx$

Recall: $\int \frac{1}{\sqrt{a^2-u^2}}du = \arcsin\left(\frac{u}{a}\right)+C$

Let $a^2 = 1 \quad \Rightarrow \quad a = 1$

Let $u^2 = 12x^2 \quad \Rightarrow \quad u = \sqrt{12}x \quad \Rightarrow \quad x = \frac{1}{\sqrt{12}}u$

Also $\frac{du}{dx} = \sqrt{12} \Rightarrow \quad du = \sqrt{12}dx \quad \Rightarrow \quad \frac{1}{\sqrt{12}}du = dx$

$$\int \frac{8dx}{\sqrt{1-12x^2}} = 8\int \frac{1}{\sqrt{1-12x^2}}dx = 8\left[\int \frac{1}{\sqrt{a^2-u^2}}\frac{1}{\sqrt{12}}du\right]$$

$$= \frac{8}{\sqrt{12}}\left[\int \frac{1}{\sqrt{a^2-u^2}}du\right] = \frac{8}{\sqrt{12}}\left[\arcsin\left(\frac{u}{a}\right)\right]$$

$$= \frac{8}{\sqrt{12}}\left[\arcsin\left(\frac{\sqrt{12}x}{1}\right)\right]+C$$

Example 7: Find $\int_3^5 \frac{1}{8+(x-7)^2}\,dx$

Recall: $\int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

Let $a^2 = 8 \quad \Rightarrow \quad a = \sqrt{8}$

Let $u^2 = (x-7)^2 \quad \Rightarrow \quad u = x-7 \quad \Rightarrow \quad \frac{du}{dx} = 1 \quad \Rightarrow \quad du = dx$

$$\int \frac{1}{8+(x-7)^2}\,dx = \int \frac{1}{a^2+u^2}\,du = \frac{1}{a}\arctan\left(\frac{u}{a}\right) = \frac{1}{\sqrt{8}}\arctan\left(\frac{x-7}{\sqrt{8}}\right)$$

$$\int_3^5 \frac{1}{8+(x-7)^2}\,dx = \left[\frac{1}{\sqrt{8}}\arctan\left(\frac{x-7}{\sqrt{8}}\right)\right]_3^5$$

$$= \left[\frac{1}{\sqrt{8}}\arctan\left(\frac{5-7}{\sqrt{8}}\right)\right] - \left[\frac{1}{\sqrt{8}}\arctan\left(\frac{3-7}{\sqrt{8}}\right)\right]$$

Example 8: Find $\int \frac{x}{x^4+25}dx = \int \frac{x}{(x^2)^2+25}dx$

Recall: $\int \frac{1}{a^2+u^2}du = \frac{1}{a}\arctan\left(\frac{u}{a}\right)+C$

Let $a^2 = 25 \quad \Rightarrow \quad a = 5; \quad$ Let $u^2 = x^4 = (x^2)^2 \quad \Rightarrow \quad u = (x^2)$

Also, $\frac{du}{dx} = 2x \quad \Rightarrow \quad du = 2xdx \quad \Rightarrow \quad \frac{1}{2}du = xdx$

$$\int \frac{x}{x^4+25}dx = \int \frac{1}{x^4+25}xdx = \int \frac{1}{u^2+a^2}\cdot\frac{1}{2}du = \frac{1}{2}\int \frac{1}{u^2+a^2}du$$

$$= \frac{1}{2}\left[\arctan\left(\frac{u}{a}\right)+C\right] = \frac{1}{2}\left[\arctan\left(\frac{x}{5}\right)+C\right]$$