# Hyperbolic Functions

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2} \qquad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

Recall:

$$D_x\left(e^x\right) = e^x$$

$$D_x\left(e^{-x}\right) = -e^{-x}$$

$$D_x\left(e^{\text{Expr}}\right) = e^{\text{Expr}} \cdot D_x\left(\text{Expr}\right)$$

# Derivative of Hyperbolic Functions

Recall: $D_x\left(e^{\text{Expr}}\right)=e^{\text{Expr}}\cdot D_x\left(\text{Expr}\right)$

$$D_x\left(\sinh x\right)=D_x\left(\frac{e^x-e^{-x}}{2}\right)=D_x\left(\frac{1}{2}e^x-\frac{1}{2}e^{-x}\right)$$

$$=\frac{1}{2}\left(e^x\right)-\frac{1}{2}\left(e^{-x}\cdot(-1)\right)=\frac{1}{2}\left(e^x\right)+\frac{1}{2}\left(e^{-x}\right)=\cosh x$$

Simularly:

$$D_x\left(\cosh x\right)=\sinh x$$

$$D_x\left(\tanh x\right)=\left(\operatorname{sech}^2 x\right)$$

Chain Rule: $D_x\left[\sinh\left(\text{Expr}\right)\right] = \cosh\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right)$

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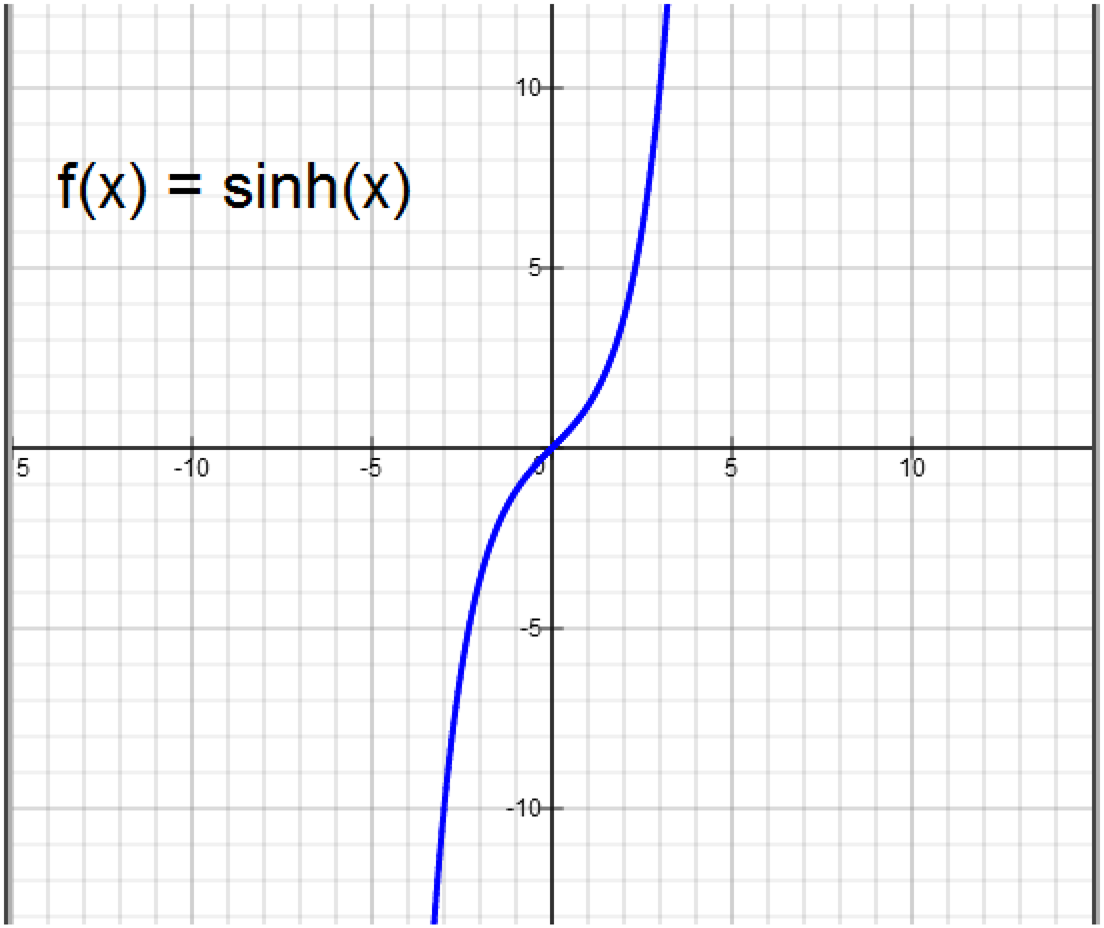
Chain Rule: $D_x\left[\tanh\left(\text{Expr}\right)\right] = \operatorname{sech}^2\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right)$

Chain Rule: $D_x\left[\operatorname{csch}\left(\text{Expr}\right)\right] = -\operatorname{csch}\left(\text{Expr}\right) \cdot \coth\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right)$

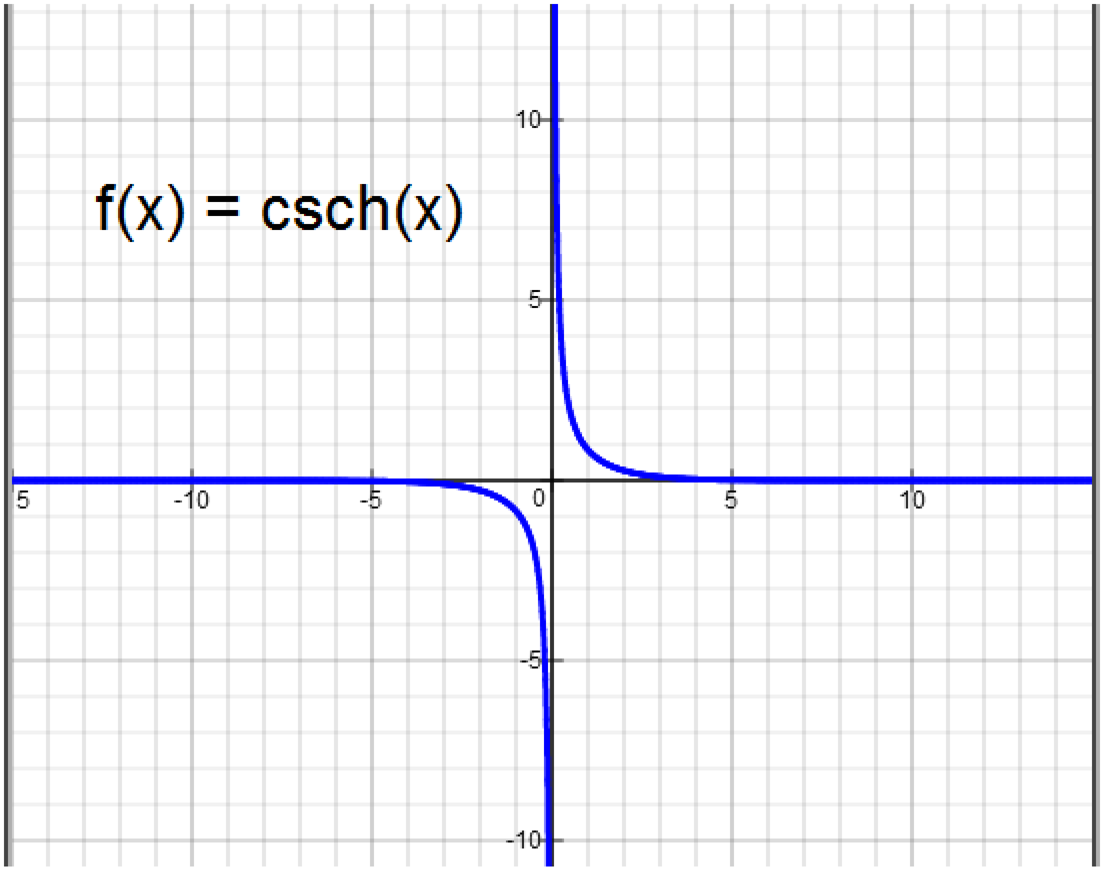
Chain Rule: $D_x\left[\operatorname{sech}\left(\text{Expr}\right)\right] = -\operatorname{sech}\left(\text{Expr}\right) \cdot \tanh\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right)$

Chain Rule: $D_x\left[\coth\left(\text{Expr}\right)\right] = -\operatorname{csch}^2\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right)$

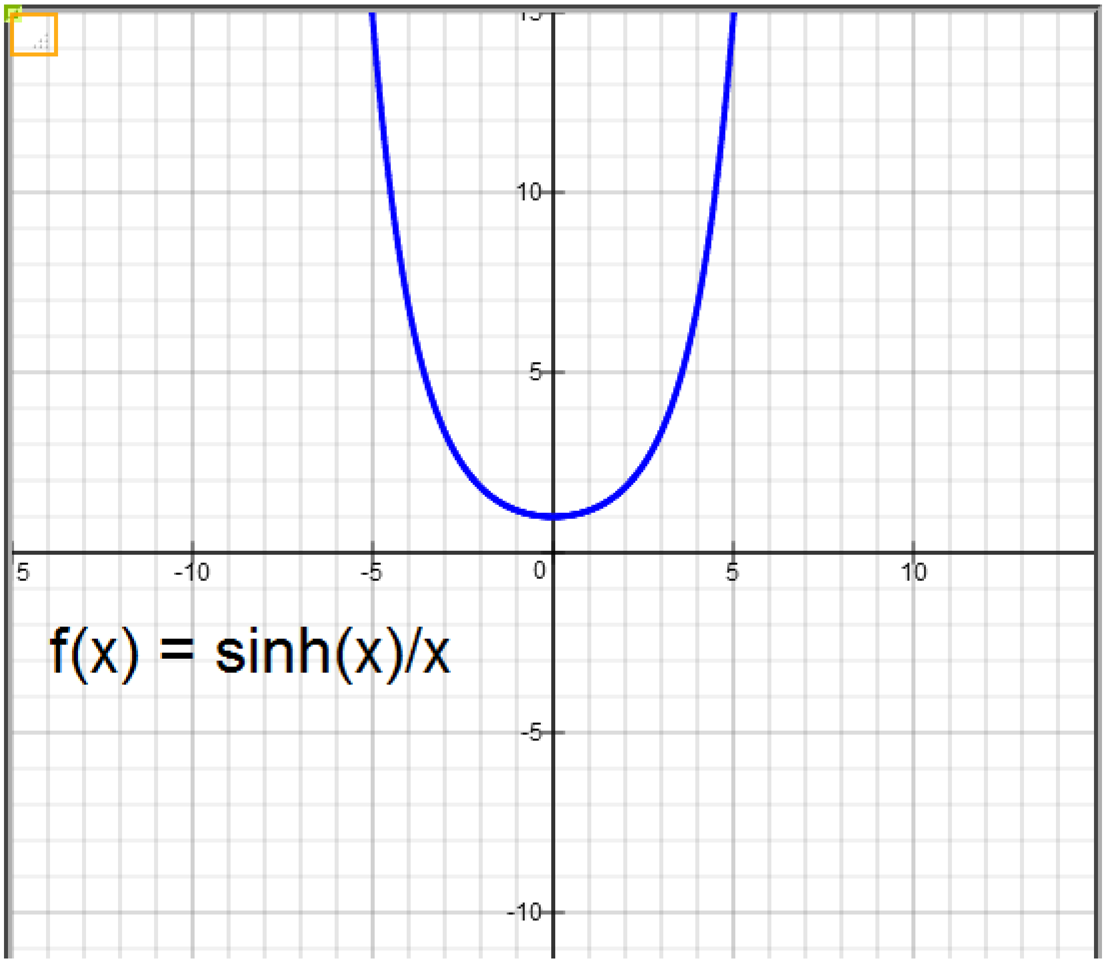
$$\lim_{x \to \infty} \sinh x = \infty$$



$$\lim_{x \to \infty} \operatorname{csch} x = \infty$$



Find $\lim_{x \to 0} \frac{\sinh x}{x} = 1$



Let $f(x) = \sinh(7x)$.   Find $f'(x)$.

Chain Rule: $D_x\left[\sinh\left(\text{Expr}\right)\right] = \cosh\left(\text{Expr}\right) D_x\left(\text{Expr}\right)$

$$f'(x) = \cosh\left(7x\right) D_x\left[7x\right] = \cosh\left(7x\right)\cdot\left(7\right)$$

Let $f(x) = \operatorname{sech}(5x^2+8)$. Find $f'(x)$.

Chain Rule:

$$D_x\left[\operatorname{sech}\left(\text{Expr}\right)\right] = -\operatorname{sech}\left(\text{Expr}\right)\cdot\tanh\left(\text{Expr}\right)\cdot D_x\left(\text{Expr}\right)$$

$$f'(x) = -\operatorname{sech}\left(5x^2+8\right)\cdot\tanh\left(5x^2+8\right)\cdot D_x\left(5x^2+8\right)$$

$$f'(x) = -\operatorname{sech}\left(5x^2+8\right)\cdot\tanh\left(5x^2+8\right)\cdot\left(10x\right)$$

Let $h(x) = \dfrac{1}{2}\cosh(15x) - \dfrac{x}{2}$. Find $h'(x)$.

$$h(x) = \frac{1}{2}\cosh(15x) - \frac{x}{2} = \frac{1}{2}\cosh(15x) - \frac{1}{2}x$$

Chain Rule: $D_x\left[\cosh\left(\text{Expr}\right)\right] = \sinh\left(\text{Expr}\right) D_x\left(\text{Expr}\right)$

$$h'(x) = \frac{1}{2}\left[\sinh\left(\text{Expr}\right) D_x\left(\text{Expr}\right)\right] - \frac{1}{2}$$

$$h'(x) = \frac{1}{2}\left[\sinh\left(15x\right) D_x\left(15x\right)\right] - \frac{1}{2}$$

$$h'(x) = \frac{1}{2}\left[\sinh\left(15x\right)\left(15\right)\right] - \frac{1}{2}$$

$$h'(x) = \frac{15}{2}\sinh\left(15x\right) - \frac{1}{2}$$

Let $y = \arctan\left(\sinh x\right)$. Find equation of tangent line (0, 0).

Chain Rule: $D_x\left[\arctan\left(\text{Expr}\right)\right] = \dfrac{1}{1+\left(\text{Expr}\right)^2} \cdot D_x\left(\text{Expr}\right)$

Chain Rule: $D_x\left[\sinh\left(\text{Expr}\right)\right] = \cosh\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right)$

a ) $y' = \dfrac{1}{1+\left(\text{Expr}\right)^2} \cdot D_x\left(\text{Expr}\right) = \dfrac{1}{1+\left(\sinh x\right)^2} \cdot D_x\left(\sinh x\right)$

$$= \frac{1}{1+\left(\sinh x\right)^2} \cdot \left(\cosh x\right)$$

b) Slope of tangent line = $y'(0) = \dfrac{1}{1+\left(\sinh x\right)^2} \cdot \left(\cosh x\right)$

$$= \frac{1}{1+\left(\sinh 0\right)^2} \cdot \left(\cosh 0\right) = \frac{1}{1+0} \cdot \left(1\right) = 1$$

c) Equation of tangent line: $y - y_1 = m(x - x_1)$

$$y - 0 = 1(x - 0)$$

Let $y = \cosh\left(9 - x^2\right)$ Find equation of tangent line (3, 1).

Chain Rule: $D_x\left[\cosh\left(\text{Expr}\right)\right] = \sinh\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right)$

a ) $y' = \sinh\left(\text{Expr}\right) \cdot D_x\left(\text{Expr}\right) = \sinh\left(9 - x^2\right) \cdot D_x\left(9 - x^2\right)$

$= \sinh\left(9 - x^2\right) \cdot \left(-2x\right)$

b) Slope of tangent line = $y'(3) = \sinh\left(9 - x^2\right) \cdot \left(-2x\right)$

$= \sinh\left(0\right) \cdot \left(-6\right) = 0$

c) Equation of tangent line: $y - y_1 = m(x - x_1)$

$y - 1 = 0(x - 3)$

# Anti Derivative for Hyperbolic Functions

$$\int \cosh u\,du = \sinh u + C$$

$$\int \sinh u\,du = \cosh u + C$$

$$\int \operatorname{sech}^2 u\,du = \tanh u + C$$

$$\int \operatorname{csch}^2 u\,du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u\,du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u\,du = -\operatorname{csch} u + C$$

Find $\int \sinh 2x dx$.

Let $u = 2x$

$$\frac{du}{dx} = 2 \quad \Rightarrow \quad du = 2dx \quad \Rightarrow \quad \frac{1}{2}du = dx$$

$$\int \sinh 2x dx = \int \sinh u \frac{1}{2} du = \frac{1}{2}\int \sinh u du = \frac{1}{2}[\cosh u]$$

$$= \frac{1}{2}[\cosh 2x] + C$$

Find $\int \operatorname{sech}^2(1 - x)dx$.

Hint: Let $u = 1 - x$

$$\frac{du}{dx} = -1 \quad \Rightarrow \quad du = -1dx \quad \Rightarrow \quad -1du = dx$$

$$\int \operatorname{sech}^2(1 - x)dx = \int \operatorname{sech}^2(u)\left(-1\right)du = \left(-1\right)\int \operatorname{sech}^2(u)du$$

$$= \left(-1\right)\left[\tanh u\right] = \left(-1\right)\left[\tanh(1 - x)\right] + C$$

$$\int \operatorname{csch}^2(5x-10)\,dx.$$

Hint: Let $u = 5x - 10$

$$\frac{du}{dx} = 5 \Rightarrow du = 5dx \quad \Rightarrow \quad \frac{1}{5}du = dx$$

$$\int \operatorname{csch}^2(5x-10)\,dx = \int \operatorname{csch}^2(u)\frac{1}{5}du = \frac{1}{5}\int \operatorname{csch}^2(u)\,du$$

$$= \frac{1}{5}\left[-\coth u\right] = \frac{1}{5}\left[-\coth\left(5x-10\right)\right] + C$$

Find $\displaystyle\int_0^4 \frac{1}{\sqrt{9-x^2}}\,dx.$

Recall: $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right) + C$

Let $a^2 = 9; \quad a = 3;$ Let $u^2 = x^2; \quad u = x$

$$\frac{du}{dx} = 1; \quad du = dx$$

$$\int \frac{1}{\sqrt{9-x^2}}\,dx = \int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\left(\frac{u}{a}\right) = \arcsin\left(\frac{x}{3}\right)$$

$$\int_0^4 \frac{1}{\sqrt{9-x^2}}\,dx = \left[\arcsin\left(\frac{x}{3}\right)\right]_0^4 = \arcsin\left(\frac{4}{3}\right) - \arcsin(0)$$

# Inverse Hyperbolic Functions

$$f(x) = \sinh x \qquad f^{-1}(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$f(x) = \cosh x \qquad f^{-1}(x) = \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right)$$

$$f(x) = \tanh x \qquad f^{-1}(x) = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$f(x) = \coth x \qquad f^{-1}(x) = \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right)$$

$$f(x) = \operatorname{sech} x \qquad f^{-1}(x) = \operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)$$

$$f(x) = \operatorname{csch} x \qquad f^{-1}(x) = \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{x^2+1}}{|x|}\right)$$

Prove: $y = \sinh^{-1} = \ln\left(x + \sqrt{x^2 + 1}\right)$

$$y = \sinh x = \frac{e^x - e^{-x}}{2}$$

Finding inverse function:

$$x = \frac{e^y - e^{-y}}{2} \quad \Rightarrow \quad e^y - e^{-y} = 2x \quad \Rightarrow \quad e^y - 2x - e^{-y} = 0$$

$$\Rightarrow \quad e^y e^y - 2xe^y - e^{-y}e^y = 0 \quad \Rightarrow \quad e^{2y} - 2xe^y - e^0 = 0$$

$\left(e^y\right)^2 - 2xe^y - 1 = 0$ is of quadratic form.

Prove: $y = \sinh^{-1} = \ln\left(x + \sqrt{x^2 + 1}\right)$ con't

Using Quadratic Formula with $a = 1$; $b = -2x$; $c = -1$:

$$e^y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = \frac{2x \pm 2\sqrt{\left(x^2 + 1\right)}}{2}$$

$$e^y = x \pm \sqrt{x^2 + 1}$$

$\ln e^y = \ln\left(x \pm \sqrt{x^2 + 1}\right)$; Note $e^y > 0$; "$-$" is extraneous.

$$y = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$f^{-1}(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

Let $f(x) = \sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$. Find $f'(x)$.

$$D_x\left[\sinh^{-1} x\right] = D_x\left[\ln\left(x+\sqrt{x^2+1}\right)\right] = \frac{1}{x+\sqrt{x^2+1}} D_x\left(x+\sqrt{x^2+1}\right)$$

$$= \frac{1}{x+\sqrt{x^2+1}}\left(1+\frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{x+\sqrt{x^2+1}}\left(\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\left(\sqrt{x^2+1}+x\right)}{\left(x+\sqrt{x^2+1}\right)\left(\sqrt{x^2+1}\right)} = \frac{1}{\sqrt{x^2+1}}$$

Derivative for inverse hyperbolic functions:

$$D_x\left[\sinh^{-1} u\right] = \frac{1}{\sqrt{u^2+1}} D_x(u) \qquad D_x\left[\cosh^{-1} u\right] = \frac{1}{\sqrt{u^2-1}} D_x(u)$$

$$D_x\left[\tanh^{-1} u\right] = \frac{1}{1-u^2} D_x(u) \qquad D_x\left[\coth^{-1} u\right] = \frac{1}{1-u^2} D_x(u)$$

$$D_x\left[\sec h^{-1} u\right] = \frac{-1}{u\sqrt{1-u^2}} D_x(u) \qquad D_x\left[\csc h^{-1} u\right] = \frac{-1}{|u|\sqrt{1+u^2}} D_x(u)$$

$$y = \sinh^{-1}(15x)$$

Chain Rule: $D_x\left[\sinh^{-1}(\text{Expr})\right] = \dfrac{1}{\sqrt{(\text{Expr})^2 + 1}} \cdot D_x(\text{Expr})$

Find $y' = \dfrac{1}{\sqrt{(\text{Expr})^2 + 1}} \cdot D_x(\text{Expr}) = \dfrac{1}{\sqrt{(15x)^2 + 1}} \cdot D_x(15x)$

$$= \frac{1}{\sqrt{225x^2 + 1}} \cdot (15)$$

Let $y = 7x \cdot \coth^{-1}(5x)$. Find $y'$.

Chain Rule: $D_x\left[\coth^{-1}(\text{Expr})\right] = \dfrac{1}{1-(\text{Expr})^2} \cdot D_x(\text{Expr})$

$$y' = (7x)\,D_x\left(\coth^{-1}(5x)\right) + \left(\coth^{-1}(5x)\right)D_x(7x)$$

$$y' = (7x)\left[\frac{1}{1-(\text{Expr})^2} \cdot D_x(\text{Expr})\right] + \left(\coth^{-1}(5x)\right)(7)$$

$$y' = (7x)\left[\frac{1}{1-(5x)^2} \cdot D_x(5x)\right] + \left(\coth^{-1}(5x)\right)(7)$$

$$y' = (7x)\left[\frac{5}{1-(5x)^2}\right] + 7\coth^{-1}(5x)$$

Find $\lim_{x\to\infty}\left(\sinh x\right)=\infty$

Find $\lim_{x\to 0}\left(\sinh x\right)=0$

Find $\lim_{x\to\infty}\left(\cosh x\right)=\infty$

Find $\lim_{x\to 0}\left(\sinh x\right)=1$

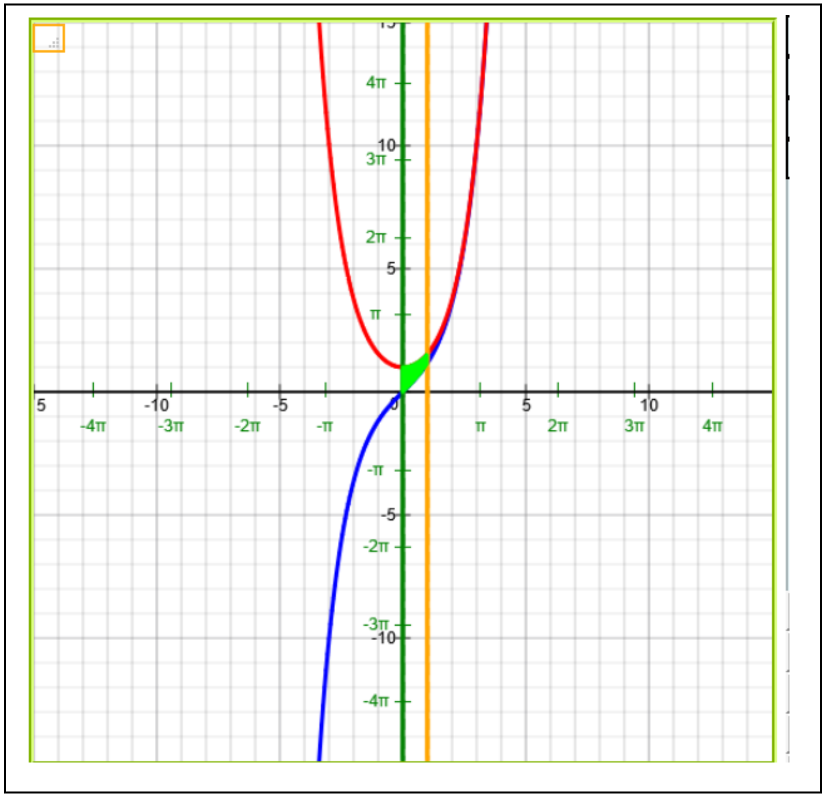
Find $\lim_{x\to 0}\left(\dfrac{\sinh x}{x}\right)=1$

Find the area bounded by $f(x) = \sinh(x);\quad g(x) = \cosh(x);\quad x = 0;\qquad x = 1$

$$\text{Area} = \int_0^1 (\cosh x - \sinh x)\,dx$$

$$= (\sinh x - \cosh x)\Big|_0^1$$

$$= [\sinh 1 - \cosh 1] - [\sinh 0 - \cosh 0] = 0.632120$$



Find the area bounded by $f(x) = \cosh(x) + 2; \quad g(x) = x - 1; \quad x = 0; \qquad x = 1$

$$\text{Area} = \int_0^1 \left([\text{top graph}] - [\text{bottom graph}]\right) dx = \int_0^1 \left([\cosh(x) + 2] - [x - 1]\right) dx$$

$$= \int_0^1 \left([\cosh(x) - x + 3]\right) dx$$

$$= \left(\sinh x - \frac{1}{2}x^2 + 3x\right)\Bigg|_0^1 = \left[\sinh 1 - \frac{1}{2}(1)^2 + 3(1)\right] - \left[\sinh 0 - \frac{1}{2}(0)^2 + 3(0)\right] = 3.6752011$$