Integration Rules

Example 1: Find the indefinite Integral $\int \frac{3}{(x+4)^4}\,dx$

Let $u = x + 4$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Rewriting integral in terms of u and du:

$$\int \frac{3}{(x+4)^4}\,dx = 3\int \frac{1}{(u)^4}\,du = 3\int u^{-4}du = 3\left[\frac{u^{-3}}{-3}\right] = -1u^{-3} = -u^{-3} + C$$

Writing answer in terms of x:

$$\int \frac{3}{(x+4)^4}\,dx = -u^{-3} + C = -(x+4)^{-3} + C = -\frac{1}{(x+4)^3} + C$$

Example 2: Find the indefinite Integral $\int x^2\sqrt{5+x^3}\,dx$

Let $u = 5 + x^3$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2 dx$$

$$\frac{1}{3}du = \frac{1}{3}\cdot 3x^2 dx$$

$$\frac{1}{3}du = x^2 dx$$

Rewriting integral in terms of u and du:

$$\int x^2\sqrt{5+x^3}\,dx = \int \sqrt{5+x^3}\,x^2 dx = \int \sqrt{u}\left(\frac{1}{3}du\right) = \frac{1}{3}\int u^{1/2} du$$

$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right] + C = \frac{1}{3}\cdot\frac{2}{3}\left[u^{3/2}\right] + C = \frac{2}{9}\left[u^{3/2}\right] + C$$

Writing answer in terms of x:

$$\int x^2\sqrt{5+x^3}\,dx = \frac{2}{9}\left[u^{3/2}\right] + C = \frac{2}{9}\left[\left(5+x^3\right)^{3/2}\right] + C$$

Example 3: Find the indefinite Integral

$$\int\left[2x-\frac{4}{(2x+5)^3}\right]dx$$

Note: $\int\left[2x-\frac{4}{(2x+5)^3}\right]dx=\int[2x]dx \quad - \quad \int\left[\frac{4}{(2x+5)^3}\right]dx$

$$\int[2x]dx = 2x^2\cdot\frac{1}{2} = x^2$$

For $\int\left[\frac{4}{(2x+5)^3}\right]dx$:

Let $u=2x+5$

$$\frac{du}{dx}=2 \quad \Rightarrow \quad du=2dx \quad \Rightarrow \quad \frac{1}{2}du=\frac{1}{2}\cdot 2dx \quad \Rightarrow \quad \frac{1}{2}du=dx$$

Rewriting integral in terms of u and du:

$$\int\left[\frac{4}{(2x+5)^3}\right]dx=\int\left[\frac{4}{(u)^3}\right]\cdot\frac{1}{2}du=\int\left[\frac{2}{(u)^3}\right]du = 2\int\left[u^{-3}\right]du$$

$$=\frac{2u^{-2}}{-2}+C=-u^{-2}=-(2x+5)^{-2}=-\frac{1}{(2x+5)^2}$$

Writing answer in terms of x:

$$\int\left[2x-\frac{4}{(2x+5)^3}\right]dx=\int[2x]dx \quad - \quad \int\left[\frac{4}{(2x+5)^3}\right]dx$$

$$=x^2-\left(-\frac{1}{(2x+5)^2}\right) = x^2+\frac{1}{(2x+5)^2}+C$$

Example 4: Find the indefinite Integral

$$\int \left[\frac{4x}{x+2} \right] dx$$

Note: Degree of numerator is 1; degree of denominator is 1;

When degree of numerator is equal or greater than that of denominator

try to divide expression using long division.

$$\frac{4x}{x+2} \quad \Leftrightarrow \quad x+2\overline{\smash{)}4x}$$

Hence, $\dfrac{4x}{x+2} = 4 + \dfrac{-8}{x+2}$

$$\int \left[\frac{4x}{x+2} \right] dx = \int \left[4 + \frac{-8}{x+2} \right] dx = \int [4]\, dx \; + \; \int \left[\frac{-8}{x+2} \right] dx$$

Note: $\int [4]\, dx = 4x$

For $\int \left[\dfrac{-8}{x+2} \right] dx$:

$$\int \left[\frac{-8}{x+2} \right] dx = -8 \int \left[\frac{1}{x+2} \right] dx$$

Recall: $\int \left[\dfrac{1}{x} \right] dx = \ln |x|$; $\qquad \int \left[\dfrac{1}{x+a} \right] dx = \ln |x+a|$

$$\int \left[\frac{-8}{x+2} \right] dx = -8 \int \left[\frac{1}{x+2} \right] dx = -8 \ln |x+2| + C$$

Writing Answer:

$$\int \left[\frac{4x}{x+2} \right] dx = \int \left[4 + \frac{-8}{x+2} \right] dx = \int [4]\, dx \; + \; \int \left[\frac{-8}{x+2} \right] dx$$

$$= 4x - 8 \ln |x+2| + C$$

Example 5: Find the indefinite Integral

$$\int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx$$

Note: $\int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx = \int \left[\frac{1}{3x+2} \right] dx - \int \left[\frac{4}{5x+2} \right] dx$

Recall: $\int \left[\frac{1}{x} \right] dx = \ln|x|$; $\quad \int \left[\frac{1}{x+a} \right] dx = \ln|x+a| \quad \int \left[\frac{1}{bx+a} \right] dx = \frac{1}{b} \ln|bx+a|$

$$\int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx = \int \left[\frac{1}{3x+2} \right] dx - \int \left[\frac{4}{5x+2} \right] dx$$

$$= \frac{1}{3} \ln|3x+2| \; - \; \frac{1}{5} \ln|5x+2| \; + \; C$$

Example 6: Find the indefinite Integral

$$\int [\csc \pi x \cot \pi x]\, dx$$

Recall: $\int [\csc u \cot u]\, du = -\csc u + C$

Let $u = \pi x$

$$\frac{du}{dx} = \pi$$

$$du = \pi dx$$

$$\frac{1}{\pi} du = \frac{1}{\pi} \cdot \pi dx$$

$$\frac{1}{\pi} du = dx$$

Rewriting integral in terms of u and du:

$$\int [\csc \pi x \cot \pi x]\, dx = \int [\csc u \cot u] \frac{1}{\pi} du$$

$$= \frac{1}{\pi} \int [\csc u \cot u]\, du$$

$$= \frac{1}{\pi} [-\csc u] + \text{C}$$

$$= \frac{1}{\pi} [-\csc \pi x] + \text{C}$$ Writing answer in terms of x

Example 7: Find the indefinite Integral

$$\int \left[\csc^2 x e^{\cot x} \right] dx$$

Let $u = \cot x$

$$\frac{du}{dx} = -\csc^2 x$$

$$du = -\csc^2 x dx$$

$$-1du = (-1)\left(-\csc^2 xdx\right)$$

$$-1du = \csc^2 dx$$

Rewriting integral in terms of u and du:

$$\int \left[\csc^2 x e^{\cot x} \right] dx = \int \left[e^{\cot x} \right] \csc^2 xdx$$

$$= \int \left[e^u \right] (-1) du$$

$$= -1 \int \left[e^u \right] du$$

$$= -1e^u + C$$

$$= -1e^{\cot x} + C$$ Writing answer in terms of x

Example 8: Find the indefinite Integral

$$\int [\tan x][\ln(\cos x)]\,dx$$

Recall: $\int [\tan u]\,du = -\ln|\cos u| + C$

Let $u = \ln(\cos x)$

$$\frac{du}{dx} = \frac{1}{\cos x}(-\sin x) \qquad \text{Note: } D_x(\ln u) = \frac{1}{u}\cdot u'$$

$$du = -\frac{\sin x}{\cos x}\,dx$$

$$du = -\tan x\,dx \;\Rightarrow\; -1du = (-1)(-\tan x\,dx) \;\Rightarrow\; -1du = \tan x\,dx$$

Rewriting integral in terms of u and du:

$$\int [\tan x][\ln(\cos x)]\,dx = \int [\ln(\cos x)]\tan x\,dx$$

$$= \int [u](-1)\,du$$

$$= -1\int [u]\,du$$

$$= -1u^2 \cdot \frac{1}{2} + C$$

$$= -\frac{1}{2}[\ln(\cos x)]^2 + \text{C}$$ Writing answer in terms of x

Example 9: Find the indefinite Integral

$$\int_0^4 \frac{1}{\sqrt{36-x^2}}\,dx$$

Recall: $\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$

Let $a^2 = 36 \quad \Rightarrow \quad a = 6$

Let $u^2 = x^2 \quad \Rightarrow \quad u = x$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Rewriting integral in terms of u and du:

$$\int_0^4 \frac{1}{\sqrt{36-x^2}}\,dx = \int_0^4 \frac{1}{\sqrt{a^2-u^2}}\,du$$

$$= \arcsin\frac{u}{a}$$

$$= \arcsin\frac{x}{6}\Big]_0^4$$

$$= \arcsin\frac{4}{6} - \arcsin\frac{0}{6}$$

= 0.7297276562269663 - 0 = 0.7297276562269663

Example 10: Find the indefinite Integral $\int_0^4 \frac{1}{9+16x^2}\,dx$

Recall: $\int \frac{1}{a^2+b^2u^2}\,du = \frac{1}{ab}\arctan\frac{bu}{a} + C$

Let $a^2 = 9 \quad \Rightarrow \quad a = 3$; $\quad b^2 = 16 \quad \Rightarrow \quad b = 4$

Let $u^2 = x^2 \quad \Rightarrow \quad u = x$

Rewriting integral in terms of u and du:

$$\int_0^4 \frac{1}{9+16x^2}\,dx = \frac{1}{(3)(4)}\arctan\frac{4x}{3} = \frac{1}{12}\arctan\frac{4x}{3}$$

$$= \frac{1}{12}\left[\arctan\frac{4x}{3}\right]_0^4$$

$$= \frac{1}{12}\left[\arctan\frac{16}{3} - \arctan\frac{0}{3}\right]$$

$= 0.11545403139993349$

Example 11: Find the indefinite Integral

$$\int \frac{4x}{\sqrt{x^2-9}}\,dx$$

Let $u = x^2 - 9$

$$\frac{du}{dx} = 2x$$

$$du = 2xdx \quad \Rightarrow \quad 2du = 2 \cdot 2xdx \quad \Rightarrow \quad 2du = 4xdx$$

Rewriting integral in terms of u and du:

$$\int \frac{4x}{\sqrt{x^2-9}}\,dx = \int \frac{1}{\sqrt{x^2-9}}\,4xdx = \int \frac{1}{\sqrt{u}}\,2dx$$

$$= 2\int \frac{1}{u^{1/2}}\,dx$$

$$= 2\int u^{-1/2}dx$$

$$= 2u^{1/2} \cdot 2$$

$$= 4u^{1/2} + \text{C}$$

$$= 4\left(x^2-9\right)^{1/2} + \text{C}$$ Writing answer in terms of x

12) Find $\int x\sqrt{4-x}dx$

Let $u = 4 - x$. Note: $x = 4 - u$

$$\frac{du}{dx} = -1 \quad \Rightarrow \quad du = -1dx \quad \Rightarrow \quad -1du = dx$$

Hence, $\int x\sqrt{4-x}dx = \int (4-u)\sqrt{u}(-1)du = (-1)\int (4-u)\cdot u^{1/2}du$

$$= (-1)\int \left(4\cdot u^{1/2} - u\cdot u^{1/2}\right)du = (-1)\int \left(4\cdot u^{1/2} - u^{3/2}\right)du$$

$$= (-1)\left[4u^{3/2}\cdot\frac{2}{3} - u^{5/2}\cdot\frac{2}{5}\right]$$

$$= -\frac{8}{3}u^{3/2} + \frac{2}{5}u^{5/2}$$

$$= \frac{-8}{3}(4-x)^{3/2} + \frac{2}{5}(4-x)^{5/2} + C$$