

Integration Rules

Example 1: Find the indefinite Integral $\int \frac{3}{(x+4)^4} dx$

Let $u = x + 4$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Rewriting integral in terms of u and du :

$$\int \frac{3}{(x+4)^4} dx = 3 \int \frac{1}{(u)^4} du = 3 \int u^{-4} du = 3 \left[\frac{u^{-3}}{-3} \right] = -1u^{-3} = -u^{-3} + C$$

Writing answer in terms of x :

$$\int \frac{3}{(x+4)^4} dx = -u^{-3} + C = -(x+4)^{-3} + C$$

Example 2: Find the indefinite Integral $\int x^2 \sqrt{5 + x^3} dx$

Let $u = 5 + x^3$

$$\frac{du}{dx} = 3x^2$$

$$du = 3x^2 dx$$

$$\frac{1}{3} du = \frac{1}{3} \cdot 3x^2 dx$$

$$\frac{1}{3} du = x^2 dx$$

Rewriting integral in terms of u and du :

$$\begin{aligned} \int x^2 \sqrt{5 + x^3} dx &= \int \sqrt{5 + x^3} x^2 dx = \int \sqrt{u} \left(\frac{1}{3} du \right) = \frac{1}{3} \int u^{1/2} du \\ &= \frac{1}{3} \left[\frac{u^{3/2}}{3/2} \right] + C = \frac{1}{3} \cdot \frac{2}{3} [u^{3/2}] + C = \frac{2}{9} [u^{3/2}] + C \end{aligned}$$

Writing answer in terms of x :

$$\int x^2 \sqrt{5 + x^3} dx = \frac{2}{9} [u^{3/2}] + C = \frac{2}{9} [(5 + x^3)^{3/2}] + C$$

Example 3: Find the indefinite Integral

$$\int \left[2x - \frac{4}{(2x+5)^3} \right] dx$$

$$\text{Note: } \int \left[2x - \frac{4}{(2x+5)^3} \right] dx = \int [2x] dx - \int \left[\frac{4}{(2x+5)^3} \right] dx$$

$$\int [2x] dx = 2x^2 + C$$

$$\text{For } \int \left[\frac{4}{(2x+5)^3} \right] dx:$$

$$\text{Let } u = 2x + 5$$

$$\frac{du}{dx} = 2$$

$$du = 2dx$$

$$\frac{1}{2} du = \frac{1}{2} \cdot 2dx$$

$$\frac{1}{2} du = dx$$

Rewriting integral in terms of u and du :

$$\begin{aligned} \int \left[\frac{4}{(2x+5)^3} \right] dx &= \int \left[\frac{4}{(u)^3} \right] \cdot \frac{1}{2} du = \int \left[\frac{2}{(u)^3} \right] du = 2 \int [u^{-3}] du \\ &= \frac{2u^{-2}}{-2} + C = -u = -(2x+5)^{-2} + C \end{aligned}$$

Writing answer in terms of x :

$$\begin{aligned} \int \left[2x - \frac{4}{(2x+5)^3} \right] dx &= \int [2x] dx - \int \left[\frac{4}{(2x+5)^3} \right] dx \\ &= 2x^2 - (2x+5)^{-2} + C \end{aligned}$$

Example 4: Find the indefinite Integral

$$\int \left[\frac{4x}{x+2} \right] dx$$

Note: Degree of numerator is 1; degree of denominator is 1;

When degree of numerator is equal or greater than that of denominator

try to divide expression using long division.

$$\frac{4x}{x+2} \Leftrightarrow \begin{array}{r} \\ x+2 \overline{)4x} \end{array}$$

$$\text{Hence, } \frac{4x}{x+2} = 4 + \frac{-8}{x+2}$$

$$\int \left[\frac{4x}{x+2} \right] dx = \int \left[4 + \frac{-8}{x+2} \right] dx = \int [4] dx + \int \left[\frac{-8}{x+2} \right] dx$$

$$\text{Note: } \int [4] dx = 4x$$

$$\text{For } \int \left[\frac{-8}{x+2} \right] dx :$$

$$\int \left[\frac{-8}{x+2} \right] dx = -8 \int \left[\frac{1}{x+2} \right] dx$$

$$\text{Recall: } \int \left[\frac{1}{x} \right] dx = \ln |x| ; \quad \int \left[\frac{1}{x+a} \right] dx = \ln |x+a|$$

$$\int \left[\frac{-8}{x+2} \right] dx = -8 \int \left[\frac{1}{x+2} \right] dx = -8 \ln |x+2| + C$$

Writing Answer:

$$\begin{aligned} \int \left[\frac{4x}{x+2} \right] dx &= \int \left[4 + \frac{-8}{x+2} \right] dx = \int [4] dx + \int \left[\frac{-8}{x+2} \right] dx \\ &= 4x - 8 \ln |x+2| + C \end{aligned}$$

Example 5: Find the indefinite Integral

$$\int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx$$

$$\text{Note: } \int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx = \int \left[\frac{1}{3x+2} \right] dx - \int \left[\frac{4}{5x+2} \right] dx$$

$$\text{Recall: } \int \left[\frac{1}{x} \right] dx = \ln |x|; \quad \int \left[\frac{1}{x+a} \right] dx = \ln |x+a| \quad \int \left[\frac{1}{bx+a} \right] dx = \frac{1}{b} \ln |bx+a|$$

$$\begin{aligned} \int \left[\frac{1}{3x+2} - \frac{4}{5x+2} \right] dx &= \int \left[\frac{1}{3x+2} \right] dx - \int \left[\frac{4}{5x+2} \right] dx \\ &= \frac{1}{3} \ln |3x+2| - \frac{4}{5} \ln |5x+2| + C \end{aligned}$$

Example 6: Find the indefinite Integral

$$\int [\csc \pi x \cot \pi x] dx$$

$$\text{Recall: } \int [\csc u \cot u] du = -\csc u + C$$

Let $u = \pi x$

$$\frac{du}{dx} = \pi$$

$$du = \pi dx$$

$$\frac{1}{\pi} du = \frac{1}{\pi} \cdot \pi dx$$

$$\frac{1}{\pi} du = dx$$

Rewriting integral in terms of u and du :

$$\int [\csc \pi x \cot \pi x] dx = \int [\csc u \cot u] \frac{1}{\pi} du$$

$$= \frac{1}{\pi} \int [\csc u \cot u] du$$

$$= \frac{1}{\pi} [-\csc u] + C$$

$$= \frac{1}{\pi} [-\csc \pi x] + C$$

Writing answer in terms of x

Example 7: Find the indefinite Integral

$$\int \left[\csc^2 x e^{\cot x} \right] dx$$

Let $u = \cot x$

$$\frac{du}{dx} = -\csc^2 x$$

$$du = -\csc^2 x dx$$

$$-1du = (-1)(-\csc^2 x dx)$$

$$-1du = \csc^2 x dx$$

Rewriting integral in terms of u and du :

$$\int \left[\csc^2 x e^{\cot x} \right] dx = \int \left[e^{\cot x} \right] \csc^2 x dx$$

$$= \int \left[e^u \right] (-1) du$$

$$= -1 \int \left[e^u \right] du$$

$$= -1e^u + C$$

$$= -1e^{\cot x} + C$$

Writing answer in terms of x

Example 8: Find the indefinite Integral

$$\int [\tan x] [\ln(\cos x)] dx$$

Recall: $\int [\tan u] du = -\ln |\cos u| + C$

Let $u = \ln(\cos x)$

$$\frac{du}{dx} = \frac{1}{\cos x} (-\sin x)$$

Note: $D_x(\ln u) = \frac{1}{u} \cdot u'$

$$du = -\frac{\sin x}{\cos x} dx$$

$$dx = -\tan x dx$$

$$-1 du = (-1)(-\tan x dx)$$

$$-1 du = \tan x dx$$

Rewriting integral in terms of u and du :

$$\int [\tan x] [\ln(\cos x)] dx = \int [\ln(\cos x)] \tan x dx$$

$$= \int [u] (-1) du$$

$$= -1 \int [u] du$$

$$= -1 u^2 / 2 + C$$

$$= -1 [\ln(\cos x)]^2 / 2 + C \quad \text{Writing answer in terms of } x$$

Example 9: Find the indefinite Integral

$$\int_0^4 \frac{1}{\sqrt{36-x^2}} dx$$

$$\text{Recall: } \int \frac{1}{\sqrt{a^2-u^2}} du = \arcsin \frac{u}{a} + C$$

$$\text{Let } a^2 = 36 \quad \Rightarrow \quad a = 6$$

$$\text{Let } u^2 = x^2 \quad \Rightarrow \quad u = x$$

$$\frac{du}{dx} = 1$$

$$du = dx$$

Rewriting integral in terms of u and du :

$$\int_0^4 \frac{1}{\sqrt{36-x^2}} dx = \int_0^4 \frac{1}{\sqrt{a^2-u^2}} du$$

$$= \arcsin \frac{u}{a}$$

$$= \arcsin \frac{x}{6} \Big|_0^4$$

$$= \arcsin \frac{4}{6} - \arcsin \frac{0}{6}$$

$$= 0.7297276562269663 - 0 = 0.7297276562269663$$

Example 10: Find the indefinite Integral

$$\int_0^4 \frac{1}{9+16x^2} dx = \int_0^4 \frac{1}{16(9/16+x^2)} dx = \frac{1}{16} \int_0^4 \frac{1}{(9/16+x^2)} dx$$

$$\text{Recall: } \int \frac{1}{a^2+u^2} du = \frac{1}{a} \arctan \frac{u}{a} + C$$

$$\text{Let } a^2 = 9/16 \Rightarrow a = 3/4$$

$$\text{Let } u^2 = x^2 \Rightarrow u = x$$

$$\frac{du}{dx} = 1 \Rightarrow du = dx$$

Rewriting integral in terms of u and du :

$$\int_0^4 \frac{1}{9+16x^2} dx = \frac{1}{16} \left[\int \frac{1}{a^2+u^2} du \right] = \frac{1}{16} \cdot \frac{1}{a} \arctan \frac{u}{a} = \frac{1}{16} \cdot \frac{1}{3/4} \arctan \frac{x}{3/4}$$

$$= \frac{1}{12} \left[\arctan \frac{4x}{3} \right]_0^4$$

$$= \frac{1}{12} \left[\arctan \frac{16}{3} - \arctan \frac{0}{3} \right]$$

$$= 0.11545403139993349$$

Example 11: Find the indefinite Integral

$$\int \frac{4x}{\sqrt{x^2 - 9}} dx$$

$$\text{Let } u = x^2 - 9$$

$$\frac{du}{dx} = 2x$$

$$du = 2x dx$$

$$2du = 2 \cdot 2x dx$$

$$2du = 4x dx$$

Rewriting integral in terms of u and du :

$$\int \frac{4x}{\sqrt{x^2 - 9}} dx = \int \frac{1}{\sqrt{x^2 - 9}} 4x dx = \int \frac{1}{\sqrt{u}} 2du$$

$$= 2 \int \frac{1}{u^{1/2}} du$$

$$= 2 \int u^{-1/2} du$$

$$= 2 \frac{u^{1/2}}{1/2}$$

$$= 4u^{1/2} + C$$

$$= 4(x^2 - 9)^{1/2} + C \quad \text{Writing answer in terms of } x$$

$$12) \int x\sqrt{4-x} dx$$

Let $u = 4 - x$; Hence, $\frac{du}{dx} = -1$ and $du = -1dx$

Also, $u = 4 - x \Rightarrow x = 4 - u$

Hence,

$$\begin{aligned} \int x\sqrt{4-x} dx &= \int (4-u)\sqrt{u} (-1) du = (-1) \int (4-u)u^{1/2} du = \\ &= (-1) \int (4u^{1/2} - u^{3/2}) du = (-1) \left[\frac{4u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2} \right] \\ &= \frac{-8}{3} (4-x)^{3/2} + \frac{2}{5} (4-x)^{5/2} + C \end{aligned}$$