Integration By Parts

Let $u = f(x)$ and $v = g(x)$

$Example: \quad u = 3x^2$ and $v = 4x + 5$

Hence, $u' = f'(x)$ and $v' = g'(x)$

$$\frac{du}{dx} = f'(x) \qquad \frac{dv}{dx} = f'(x)$$

$$du = dx \cdot f'(x) \qquad dv = dx \cdot f'(x)$$

$$du = u'dx \qquad dv = v'dx$$

$$\frac{d}{dx}[u \cdot v] = u\frac{d}{dx}[v] + v\frac{d}{dx}[u] = u\frac{dv}{dx} + v\frac{du}{dx}$$ Product Rule for Derivative

$$\frac{d}{dx}[u \cdot v] = u \cdot v' + v \cdot u'$$

$$\int \frac{d}{dx}[u \cdot v] dx = \int u \cdot v'dx + \int v \cdot u'dx$$

$$u \cdot v = \int u \cdot dv + \int v \cdot du$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$ Integration By Parts Formula

Example 1:

$$\int x \cdot \ln x dx$$

Note: $\int x \cdot \ln x dx = \int \ln x \cdot x dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{x dx}_{dv}$

Let $u = \ln x$ and $dv = x dx$

$$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int x dx$$

$$du = \frac{1}{x} dx \qquad \int 1 dv = \int x dx$$

$$v = \frac{x^2}{2}$$

$$\int \ln x \cdot x dx = \int u dv = uv - \int v du = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x \cdot dx$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \frac{x^2}{2}$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{4} x^2 + C$$

Example 2:

$$\int x \cdot e^x dx$$

Note: $\int x \cdot e^x dx = \int \underbrace{x}_{u} \cdot \underbrace{e^x dx}_{dv}$

Let $u = x$ and $dv = e^x dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int e^x dx$$

$$du = dx \qquad \int 1 dv = \int e^x dx$$

$$v = e^x$$

Hence,

$$\int xe^x dx = \int udv = uv - \int vdu = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Example 3:

$$\int x \cdot e^{4x} dx$$

Note: $\int x \cdot e^{4x} dx = \int \underbrace{x}_{u} \cdot \underbrace{e^{4x} dx}_{dv}$

Let $u = x$ and $dv = e^{4x} dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int e^{4x} dx$$

$$du = dx \qquad \int 1 dv = \frac{1}{4} \int e^{4x} dx$$

$$v = \frac{1}{4} e^{4x}$$

Side Note:

For $\int e^{4x} dx$:

Let $u = 4x$

Hence, $\frac{du}{dx} = 4$; $du = 4dx$; $\frac{1}{4} du = dx$

$$\int e^{4x} dx = \int e^{u} \cdot \frac{1}{4} du = \frac{1}{4} \int e^{u} \cdot du = \frac{1}{4} e^{u} = \frac{1}{4} e^{4x}$$

Hence,

$$\int xe^{4x}dx = \int udv = uv - \int vdu = xe^{4x} - \int \frac{1}{4}e^{4x}dx$$

$$= x\left(\frac{1}{4}e^{4x}\right) - \frac{1}{4}\left[\frac{1}{4}e^{4x}\right] + C$$

$$= \frac{1}{4}xe^{4x} - \frac{1}{16}e^{4x} + C$$

Example 4:

$$\int x\sqrt{x+8}\,dx$$

Note: $\int x\sqrt{x+8}\,dx = \int \underbrace{x}_{u} \cdot \underbrace{\sqrt{x+8}\,dx}_{dv}$

Let $u = x$ and $dv = \sqrt{x+8}\,dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int \sqrt{x+8}\,dx$$

$$du = dx \qquad \int 1\,dv = \int \sqrt{x+8}\,dx$$

$$v = \frac{2}{3}(x+8)^{3/2}$$

Side Note:

$$\int \sqrt{x+8}\,dx = \int (x+8)^{1/2}\,dx$$

Let $u = x+8$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int \sqrt{x+8}\,dx = \int (x+8)^{1/2}\,dx = \int (u)^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x+8)^{3/2} + C$$

Hence,

$$\int x\sqrt{x+8}dx = \int udv = uv - \int vdu = x\left(\frac{2}{3}(x+8)^{3/2}\right) - \int \frac{2}{3}(x+8)^{3/2}\,dx$$

Note:

$$\int \frac{2}{3}(x+8)^{3/2}\,dx = \frac{2}{3}\int (x+8)^{3/2}\,dx$$

Let $u = x+8$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int \frac{2}{3}(x+8)^{3/2}\,dx = \frac{2}{3}\int (x+8)^{3/2}\,dx = \frac{2}{3}\int (u)^{3/2}\,du = \frac{2}{3}\frac{u^{5/2}}{5/2} + C$$

$$= \frac{2}{3}\cdot\frac{2}{5}u^{5/2} + C = \frac{4}{15}(x+8)^{5/2} + C$$

Hence,

$$\int x\sqrt{x+8}dx = \int udv = uv - \int vdu = x\left(\frac{2}{3}(x+8)^{3/2}\right) - \int \frac{2}{3}(x+8)^{3/2}\,dx$$

$$= x\left(\frac{2}{3}(x+8)^{3/2}\right) - \frac{4}{15}(x+8)^{5/2} + C$$

$$= \frac{2}{3}x(x+8)^{3/2} - \frac{4}{15}(x+8)^{5/2} + C$$

Example 5:

$$\int x \sin x dx$$

Note: $\int x \sin x dx = \int \underbrace{x}_{u} \cdot \underbrace{\sin x dx}_{dv}$

Let $u = x$ and $dv = \sin x dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int \sin x dx$$

$$du = dx \qquad \int 1 dv = \int \sin x dx$$

$$v = -\cos x$$

$$\int x \sin x dx = \int u dv = uv - \int v du$$

$$= x(-\cos x) - \int -\cos x dx$$

$$= -x \cos x + \int \cos x dx$$

$$= -x \cos x + \sin x + C$$

Example 6:

$$\int x^2 \cos x dx$$

Note: $\int x^2 \cos x dx = \int \underbrace{x^2}_{u} \cdot \underbrace{\cos x dx}_{dv}$

Let $u = x^2$ and $dv = \cos x dx$

$$\frac{du}{dx} = 2x \qquad \int dv = \int \cos x dx$$

$$du = 2x dx \qquad \int 1 dv = \int \cos x dx$$

$$v = \sin x$$

$$\int x^2 \cos x dx = \int u dv = uv - \int v du$$

$$= x^2 \sin x - \int \sin x \cdot 2x dx$$

$Note:$ $\int \sin x \cdot 2x dx = 2\int x \cdot \sin x dx = 2\left(-x \cos x + \sin x + C\right)$

$$\int x^2 \cos x dx = \int u dv = uv - \int v du$$

$$= x^2 \sin x - \int \sin x \cdot 2x dx$$

$$= x^2 \sin x - 2\left(-x \cos x + \sin x + C\right)$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x + C$$

Example 7:

$$\int \arccos x dx = \int \cos^{-1} x dx$$

Note: $\int \cos^{-1} x dx = \int \underbrace{\cos^{-1} x}_{u} \cdot \underset{dv}{dx}$

Let $u = \cos^{-1} x$ and $dv = dx$

$$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}} \qquad \int dv = \int dx$$

$$du = \frac{-1}{\sqrt{1-x^2}} dx \qquad \int 1 dv = \int 1 dx$$

$$v = x$$

$$\int \arccos x dx = \int \cos^{-1} x dx = \int u dv = uv - \int v du$$

$$= \left(\cos^{-1} x\right)\left(x\right) - \int x \frac{-1}{\sqrt{1-x^2}} dx$$

$$Side\ Note: \int x \frac{-1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\left(-xdx\right)$$

$$\text{Let } u = 1 - x^2 \Rightarrow \quad \frac{du}{dx} = -2x; \quad du = -2xdx; \quad \frac{1}{2}du = -xdx$$

$$\int x \frac{-1}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\left(-xdx\right) = \int \frac{1}{\sqrt{u}}\left(\frac{1}{2}du\right) = \frac{1}{2}\int u^{-1/2}\left(du\right)$$

$$= \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) = u^{1/2} = \left(1-x^2\right)^{1/2}$$

$$Therefore:$$

$$\int \arccos x dx = \int \cos^{-1} x dx = \int u dv = uv - \int v du$$

$$= \left(\cos^{-1} x\right)\left(x\right) - \left(1-x^2\right)^{1/2} + C$$

Example 9:

$$\int x \sin 3x dx$$

Note: $\int x \sin 3x dx = \int \underbrace{x}_{u} \cdot \underbrace{\sin 3x dx}_{dv}$

Let $u = x$ and $dv = \sin 3x dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int \sin 3x dx$$

$$du = dx \qquad \int 1 dv = \int \sin 3x dx$$

$$v = -\frac{1}{3} \cos 3x$$

$$\int x \sin 3x dx = \int u dv = uv - \int v du$$

$$= x\left(-\frac{1}{3}\cos 3x\right) - \int -\frac{1}{3}\cos 3x dx$$

$$= -\frac{1}{3} x \cos 3x + \frac{1}{3}\int \cos 3x dx$$

$$= -\frac{1}{3} x \cos 3x + \frac{1}{3}\left[\frac{1}{3}(\sin 3x)\right] + C$$

Example 10:

$$\int e^{3x}\cos 4x dx =$$

Note: $\int e^{3x}\cos 4x dx = \int \cos 4x e^{3x} dx = \int \underbrace{\cos 4x}_{u} \cdot \underbrace{e^{3x}dx}_{dv}$

Let $u = \cos 4x$ and $dv = e^{3x}dx$

$$\frac{du}{dx} = -4\sin 4x \qquad \int dv = \int e^{3x}dx$$

$$du = -4\sin 4x dx \qquad \int 1 dv = \int e^{3x}dx$$

$$v = \frac{1}{3}e^{3x}$$

$$\int e^{3x}\cos 4x dx = \int \cos 4x e^{3x} dx = \int \underbrace{\cos 4x}_{u} \cdot \underbrace{e^{3x}dx}_{dv}$$

$$\int e^{3x}\cos 4x dx = \int u dv = uv - \int v du$$

$$= (\cos 4x)\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(-4\sin 4x dx)$$

$$= \frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{3}\int e^{3x}(\sin 4x dx)$$

$$\int e^{3x}\left(\sin 4x dx\right)$$

Note: $\int e^{3x} \sin 4x dx = \int \sin 4x e^{3x} dx = \int \underbrace{\sin 4x}_{u} \cdot \underbrace{e^{3x} dx}_{dv}$

Let $u = \sin 4x$ and $dv = e^{3x} dx$

$$\frac{du}{dx} = 4\cos 4x \qquad \int dv = \int e^{3x} dx$$

$$du = 4\cos 4x dx \qquad \int 1 dv = \int e^{3x} dx$$

$$v = \frac{1}{3} e^{3x}$$

$$\int x^2 \cos x dx = \int u dv = uv - \int v du$$

$$= \left(\sin 4x\right)\left(\frac{1}{3} e^{3x}\right) - \int \frac{1}{3} e^{3x}\left(4\cos 4x dx\right)$$

$$= \frac{1}{3}\left(\sin 4x\right)\left(e^{3x}\right) - \frac{4}{3}\int e^{3x}\left(\cos 4x dx\right)$$

$$\int e^{3x}\cos 4x dx = \int \cos 4x e^{3x} dx = \int \underbrace{\cos 4x}_{u} \cdot \underbrace{e^{3x} dx}_{dv}$$

$$\int e^{3x}\cos 4x dx = \int u dv = uv - \int v du$$

$$= \quad (\cos 4x)\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(-4\sin 4x dx)$$

$$= \quad \frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{3}\int e^{3x}(\sin 4x dx) \qquad (**)$$

Now we need to find $\int e^{3x}(\sin 4x dx)$:

$$\int e^{3x}(\sin 4x dx) = \int \underbrace{\sin 4x}_{u} \cdot \underbrace{e^{3x} dx}_{dv}$$

$$u = \sin 4x \Rightarrow \frac{du}{dx} = 4\cos 4x; \quad du = 4\cos 4x dx$$

$$dv = e^{3x} dx \Rightarrow \int dv = \int e^{3x} dx; \quad v = \frac{1}{3}e^{3x}$$

$$\int e^{3x}(\sin 4x dx) = \int u dv = uv - \int v du$$

$$= \quad (\sin 4x)\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(4\cos 4x dx)$$

$$= \quad \frac{1}{3}(\sin 4x)\left(e^{3x}\right) - \frac{4}{3}\int e^{3x}(\cos 4x dx)$$

Hence: Continue from (**)

$$\int e^{3x}\cos 4x dx = \frac{1}{3}(\cos 4x)\left(e^{3x}\right)+\frac{4}{3}\int e^{3x}(\sin 4x dx) \qquad \text{(from above)}$$

$$\int e^{3x}\cos 4x dx = \frac{1}{3}(\cos 4x)\left(e^{3x}\right)+\frac{4}{3}\left[\frac{1}{3}(\sin 4x)\left(e^{3x}\right)-\frac{4}{3}\int e^{3x}(\cos 4x dx)\right]$$

$$\int e^{3x}\cos 4x dx = \frac{1}{3}(\cos 4x)\left(e^{3x}\right)+\frac{4}{9}(\sin 4x)\left(e^{3x}\right)-\frac{16}{9}\int e^{3x}(\cos 4x dx)$$

$$1\int e^{3x}\cos 4x dx+\frac{16}{9}\int e^{3x}(\cos 4x dx) = \frac{1}{3}(\cos 4x)\left(e^{3x}\right)+\frac{4}{9}(\sin 4x)\left(e^{3x}\right)$$

$$\left(1+\frac{16}{9}\right)\int e^{3x}\cos 4x dx = \frac{1}{3}(\cos 4x)\left(e^{3x}\right)+\frac{4}{9}(\sin 4x)\left(e^{3x}\right)$$

$$\left(\frac{25}{9}\right)\int e^{3x}\cos 4x dx = \frac{1}{3}(\cos 4x)\left(e^{3x}\right)+\frac{4}{9}(\sin 4x)\left(e^{3x}\right)$$

$$\int e^{3x}\cos 4x dx = \left(\frac{9}{25}\right)\left[\frac{1}{3}(\cos 4x)\left(e^{3x}\right)+\frac{4}{9}(\sin 4x)\left(e^{3x}\right)\right]$$