Integration By Parts

Derivation of Integration by Parts Formula:

Let $u = f(x)$ and $v = g(x)$

Hence, $u' = f'(x)$ and $v' = g'(x)$

$$\frac{du}{dx} = f'(x) \qquad\qquad \frac{dv}{dx} = f'(x)$$

$$du = dx \cdot f'(x) \qquad\qquad dv = dx \cdot f'(x)$$

$$du = u'dx \qquad\qquad dv = v'dx$$

$$\frac{d}{dx}[u \cdot v] = u\frac{d}{dx}[v] + v\frac{d}{dx}[u] = u\frac{dv}{dx} + v\frac{du}{dx}$$ Product Rule for Derivative

$$\frac{d}{dx}[u \cdot v] = u \cdot v' + v \cdot u'$$

$$\int \frac{d}{dx}[u \cdot v]dx = \int u \cdot v'dx + \int v \cdot u'dx$$

$$u \cdot v = \int u \cdot dv + \int v \cdot du$$

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$ Integration By Parts Formula

Example 1: Find $\int x \cdot \ln x dx$ by using Integration by Parts Method

Note: $\int x \cdot \ln x dx = \int \ln x \cdot x dx = \int \underbrace{\ln x}_{u} \cdot \underbrace{x dx}_{dv}$

Let $u = \ln x$ and $dv = x dx$

$$\frac{du}{dx} = \frac{1}{x} \qquad \int dv = \int x dx$$

$$du = \frac{1}{x} dx \qquad \int 1 dv = \int x dx$$

$$v = \frac{x^2}{2}$$

Hence,

$$\int u dv = uv - \int v du$$

$$\int \underbrace{\ln x}_{u} \cdot \underbrace{x dx}_{dv} = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \cdot \frac{1}{x} dx$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \int x \cdot dx$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{2} \frac{x^2}{2}$$

$$= \ln x \cdot \frac{x^2}{2} - \frac{1}{4} x^2 + C$$

Example 2: Find $\int x \cdot e^x dx$ by using Integration by Parts Method

Note: $\int x \cdot e^x dx = \int \underset{u}{x} \cdot \underset{dv}{e^x dx}$

Let $u = x$ and $dv = e^x dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int e^x dx$$

$$du = dx \qquad \int 1 dv = \int e^x dx$$

$$v = e^x$$

Hence,

$$\int u dv = uv - \int v du$$

$$\int \underset{u}{x} \cdot \underset{dv}{e^x dx} = xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

Example 3: Find $\int x \cdot e^{4x} dx$ by using Integration by Parts Method

Note: $\int x \cdot e^{4x} dx = \int \underset{u}{x} \cdot \underset{dv}{e^{4x} dx}$

Let $u = x$ and $dv = e^{4x} dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int e^{4x} dx$$

$$du = dx \qquad \int 1 dv = \frac{1}{4} \int e^{4x} dx$$

$$v = \frac{1}{4} e^{4x}$$

Side Note:

Formula: $\int e^{bx} dx = \frac{1}{b} e^{bx}$

Hence, $\int e^{4x} dx = \frac{1}{4} e^{4x}$

Hence,

$$\int u dv = uv - \int v du$$

$$\int \underset{u}{x} \cdot \underset{dv}{e^{4x} dx} = xe^{4x} - \int \frac{1}{4} e^{4x} dx$$

$$= x\left(\frac{1}{4} e^{4x}\right) - \frac{1}{4}\left[\frac{1}{4} e^{4x}\right] + C$$

$$= \frac{1}{4} xe^{4x} - \frac{1}{16} e^{4x} + C$$

Example 4: Find $\int x\sqrt{x+8}dx$ by using Integration by Parts Method

Note: $\int x\sqrt{x+8}dx = \int \underbrace{x}_{u} \cdot \underbrace{\sqrt{x+8}dx}_{dv}$

Let $u = x$ and $dv = \sqrt{x+8}dx$

$\frac{du}{dx} = 1$ $\int dv = \int \sqrt{x+8}dx$

$du = dx$ $\int 1dv = \int \sqrt{x+8}dx$

$$v = \frac{2}{3}(x+8)^{3/2}$$

Side Note:

$$\int \sqrt{x+8}dx = \int (x+8)^{1/2} dx$$

Let $u = x+8$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int \sqrt{x+8}dx = \int (x+8)^{1/2} dx = \int (u)^{1/2} du = u^{3/2} \cdot \frac{2}{3} + C = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x+8)^{3/2} + C$$

Hence,

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u} \cdot \underbrace{\sqrt{x+8}\,dx}_{dv} = x\left(\frac{2}{3}(x+8)^{3/2}\right) - \int \frac{2}{3}(x+8)^{3/2}\,dx$$

Note:

$$\int \frac{2}{3}(x+8)^{3/2}\,dx = \frac{2}{3}\int (x+8)^{3/2}\,dx$$

Let $u = x+8;\qquad \frac{du}{dx} = 1;\qquad du = dx$

$$\int \frac{2}{3}(x+8)^{3/2}\,dx = \frac{2}{3}\int (x+8)^{3/2}\,dx = \frac{2}{3}\int (u)^{3/2}\,du = \frac{2}{3}u^{5/2}\cdot\frac{2}{5} + C$$

$$= \frac{2}{3}\cdot\frac{2}{5}u^{5/2} + C = \frac{4}{15}(x+8)^{5/2} + C$$

Hence,

$$\int x\sqrt{x+8}\,dx = x\left(\frac{2}{3}(x+8)^{3/2}\right) - \int \frac{2}{3}(x+8)^{3/2}\,dx$$

$$= x\left(\frac{2}{3}(x+8)^{3/2}\right) - \frac{4}{15}(x+8)^{5/2} + C$$

$$= \frac{2}{3}x(x+8)^{3/2} - \frac{4}{15}(x+8)^{5/2} + C$$

Example 5: Find $\int x\sin x dx$ by using Integration by Parts Method

Note: $\int x\sin x dx = \int \underbrace{x}_{u} \cdot \underbrace{\sin x dx}_{dv}$

Let $u = x$ and $dv = \sin x dx$

$\frac{du}{dx} = 1$ $\int dv = \int \sin x dx$

$du = dx$ $\int 1 dv = \int \sin x dx$

$v = -\cos x$

$$\int u dv = uv - \int v du$$

$$\int \underbrace{x}_{u} \cdot \underbrace{\sin x dx}_{dv} = x(-\cos x) - \int -\cos x dx$$

$$= -x\cos x + \int \cos x dx$$

$$= -x\cos x + \sin x + C$$

Example 6: Find $\int x^2 \cos x dx$ by using Integration by Parts Method

Note: $\int x^2 \cos x dx = \int \underbrace{x^2}_{u} \cdot \underbrace{\cos x dx}_{dv}$

Let $u = x^2$ and $dv = \cos x dx$

$\frac{du}{dx} = 2x$ $\int dv = \int \cos x dx$

$du = 2x dx$ $\int 1 dv = \int \cos x dx$

$v = \sin x$

Hence,

$\int u dv = uv - \int v du$

$\int \underbrace{x^2}_{u} \cdot \underbrace{\cos x dx}_{dv} = x^2 \sin x - \int \sin x \cdot 2x dx$

Note: $\int \sin x \cdot 2x dx = 2\int x \cdot \sin x dx = 2(-x \cos x + \sin x + C)$ From Example #5

Therefore,

$\int x^2 \cos x dx = x^2 \sin x - \int \sin x \cdot 2x dx$

$= x^2 \sin x - 2(-x \cos x + \sin x + C)$

$= x^2 \sin x + 2x \cos x - 2 \sin x + C$

Example 7:

Find $\int \arccos x dx = \int \cos^{-1} x dx$ by using Integration by Parts Method

Note: $\int \cos^{-1} x dx = \int \underbrace{\cos^{-1} x}_{u} \cdot \underbrace{dx}_{dv}$

Let $u = \cos^{-1} x$ and $dv = dx$

$$\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}} \qquad \int dv = \int dx$$

$$du = \frac{-1}{\sqrt{1-x^2}} dx \qquad \int 1 dv = \int 1 dx$$

$$v = x$$

Hence,

$$\int u dv = uv - \int v du$$

$$\int \underbrace{\cos^{-1} x}_{u} \cdot \underbrace{dx}_{dv} = \left(\cos^{-1} x\right)(x) - \int x \frac{-1}{\sqrt{1-x^2}} dx$$

$$\textit{Side Note}: \int x \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}(xdx)$$

Let $u = 1 - x^2 \Rightarrow \quad \frac{du}{dx} = -2x; \quad du = -2xdx; \quad -\frac{1}{2}du = xdx$

$$\int x \frac{-1}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{\sqrt{1-x^2}}(xdx) = -\int \frac{1}{\sqrt{u}}\left(-\frac{1}{2}du\right) = \frac{1}{2}\int u^{-1/2}(du)$$

$$= \frac{1}{2}\left(\frac{u^{1/2}}{1/2}\right) = u^{1/2} = \left(1-x^2\right)^{1/2}$$

Therefore:

$$\int \cos^{-1} xdx = \left(\cos^{-1} x\right)(x) - \int x \frac{-1}{\sqrt{1-x^2}}\,dx$$

$$\int \cos^{-1} xdx = \left(\cos^{-1} x\right)(x) - \left(1-x^2\right)^{1/2} + C$$

Example 8: Find $\int x\sin 3x\,dx$ by using Integration by Parts Method

Note: $\int x\sin 3x\,dx = \int \underbrace{x}_{u} \cdot \underbrace{\sin 3x\,dx}_{dv}$

Let $u = x$ and $dv = \sin 3x\,dx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int \sin 3x\,dx$$

$$du = dx \qquad \int 1\,dv = \int \sin 3x\,dx$$

$$v = -\frac{1}{3}\cos x$$

Side Note:

Formula: $\int \sin bx\,dx = -\frac{1}{b}\cos bx$

Hence, $\int \sin 3x\,dx = -\frac{1}{3}\cos 3x$

Hence,

$$\int u\,dv = uv - \int v\,du$$

$$\int \underbrace{x}_{u} \cdot \underbrace{\sin 3x\,dx}_{dv} = x\left(-\frac{1}{3}\cos 3x\right) - \int -\frac{1}{3}\cos 3x\,dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\int \cos 3x\,dx$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{3}\left[\frac{1}{3}(\sin 3x)\right] + C$$

$$= -\frac{1}{3}x\cos 3x + \frac{1}{9}\sin 3x + C$$

Side Note:

Formula: $\int \cos bx\,dx = \frac{1}{b}\sin bx$

Hence, $\int \cos 3x\,dx = -\frac{1}{3}\sin 3x$

Example 9:

Find $\mathrm{I} = \int e^{3x}\cos 4x dx$

Note: $\int e^{3x}\cos 4x dx = \int \cos 4x e^{3x} dx = \int \underbrace{\cos 4x}_{u} \cdot \underbrace{e^{3x} dx}_{dv}$

Let $u = \cos 4x$ and $dv = e^{3x} dx$

$$\frac{du}{dx} = -4\sin 4x \qquad \int dv = \int e^{3x} dx$$

$$du = -4\sin 4x dx \qquad \int 1 dv = \int e^{3x} dx$$

$$v = \frac{1}{3}e^{3x}$$

Hence,

$$\int u dv = uv - \int v du$$

$$\int \underbrace{\cos 4x}_{u} \cdot \underbrace{e^{3x} dx}_{dv} = (\cos 4x)\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(-4\sin 4x dx)$$

$$= \frac{1}{3}(\cos 4x)(e^{3x}) + \frac{4}{3}\underbrace{\int e^{3x}(\sin 4x dx)}_{\mathrm{I}_2}$$

Finding $\mathrm{I}_2 = \int e^{3x}\left(\sin 4x dx\right)$ by using Integration by Parts Method

Note: $\int e^{3x} \sin 4x dx = \int \sin 4x e^{3x} dx = \int \underbrace{\sin 4x}_{u} \cdot \underbrace{e^{3x} dx}_{dv}$

Let $u = \sin 4x$ and $dv = e^{3x} dx$

$$\frac{du}{dx} = 4\cos 4x \qquad \int dv = \int e^{3x} dx$$

$$du = 4\cos 4x dx \qquad \int 1 dv = \int e^{3x} dx$$

$$v = \frac{1}{3} e^{3x}$$

Hence,

$$\int u dv = uv - \int v du$$

$$\mathrm{I}_2 = \int e^{3x} \sin 4x dx = (\sin 4x)\left(\frac{1}{3} e^{3x}\right) - \int \frac{1}{3} e^{3x}\left(4\cos 4x dx\right)$$

$$\mathrm{I}_2 = \frac{1}{3}(\sin 4x)\left(e^{3x}\right) - \frac{4}{3}\int e^{3x}\left(\cos 4x dx\right)$$

$$\mathrm{I}_2 = \frac{1}{3}(\sin 4x)\left(e^{3x}\right) - \frac{4}{3} \cdot \mathrm{I}$$

Note: $\mathrm{I} = \int e^{3x}\left(\cos 4x dx\right)$

Therefore,

$$\int e^{3x}\cos 4x dx = (\cos 4x)\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(-4\sin 4x dx)$$

$$\text{I} = (\cos 4x)\left(\frac{1}{3}e^{3x}\right) - \int \frac{1}{3}e^{3x}(-4\sin 4x dx) \qquad \text{Note: } \text{I} = \int e^{3x}\cos 4x dx =$$

$$\text{I} = \frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{3}\int e^{3x}(\sin 4x dx)$$

$$\text{I} = \frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{3}\left[\frac{1}{3}(\sin 4x)\left(e^{3x}\right) - \frac{4}{3}\int e^{3x}(\cos 4x dx)\right]$$

$$\text{I} = \frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{9}(\sin 4x)\left(e^{3x}\right) - \frac{16}{9}\cdot \text{I}$$

$$\left(\text{I} + \frac{16}{9}\text{I}\right) = \frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{9}(\sin 4x)\left(e^{3x}\right)$$

$$\left(\frac{9}{25}\right)\text{I} = \frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{9}(\sin 4x)\left(e^{3x}\right)$$

$$\text{I} = \left(\frac{25}{9}\right)\left[\frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{9}(\sin 4x)\left(e^{3x}\right)\right]$$

$$\int e^{3x}\cos 4x dx = \left(\frac{25}{9}\right)\left[\frac{1}{3}(\cos 4x)\left(e^{3x}\right) + \frac{4}{9}(\sin 4x)\left(e^{3x}\right)\right]$$

Tabular Method works well for integrals of the form:

$\int x^n \sin bx dx$ $\quad\quad$ $\int x^n \cos bx dx$ $\quad\quad$ $\int x^n e^{ax} dx$

Example 10: Find $\int x^3 \cos 4x dx$ by using Tabular Method

Alternate Sign	u' and derivatives	v' and antiderivatives
$+$	x^3	$\cos 4x$
$-$	$\frac{d}{dx}\left[x^3\right] = 3x^2$	$\int \cos 4x dx = \frac{1}{4}\sin 4x$
$+$	$\frac{d}{dx}\left[3x^2\right] = 6x$	$\int \frac{1}{4}\sin 4x dx = \frac{-1}{16}\cos 4x$
$-$	$\frac{d}{dx}[6x] = 6$	$\int \frac{-1}{16}\cos 4x dx = \frac{-1}{64}\sin 4x$
$+$	$\frac{d}{dx}[6] = 0$	$\int \frac{-1}{64}\sin 4x dx = \frac{1}{256}\cos 4x$

Hence, $\int x^3 \cos 4x dx = x^3\left(\frac{1}{4}\sin 4x\right) - 3x^2\left(\frac{-1}{16}\cos 4x\right) + 6x\left(\frac{-1}{64}\sin 4x\right) - 6\left(\frac{1}{256}\cos 4x\right) + C$