

Trigonometric Integrals

Trigonometric Identities:

$$\sin^2 x + \cos^2 x = 1; \quad \sin^2 x = 1 - \cos^2 x; \quad \cos^2 x = 1 - \sin^2 x$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} = \frac{1}{2}(1 - \cos 2x)$$

$$\cos^2 x = \frac{1 + \cos 2x}{2} = \frac{1}{2}(1 + \cos 2x)$$

Example 1: Find the indefinite integral $\int \cos^3 x \sin^4 x dx$.

Note: Power of the cosine is *odd*.

$$\begin{aligned}\int \cos^3 x \sin^4 x dx &= \int \cos^2 x \cdot \cos x \sin^4 x dx \\ &= \int \cos^2 x \cdot \sin^4 x \cdot \cos x dx = \int (1 - \sin^2 x) \cdot \sin^4 x \cdot \cos x dx\end{aligned}$$

Now let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x dx$

$$\begin{aligned}\int \cos^3 x \sin^4 x dx &= \int \cos^2 x \cdot \cos x \sin^4 x dx \\ &= \int \cos^2 x \cdot \sin^4 x \cdot \cos x dx \\ &= \int (1 - \sin^2 x) \cdot \sin^4 x \cdot \cos x dx \\ &= \int (1 - u^2) \cdot u^4 \cdot du \\ &= \int (u^4 - u^6) \cdot du \\ &= \frac{u^5}{5} - \frac{u^7}{7} + C \\ &= \frac{(\sin x)^5}{5} - \frac{(\sin x)^7}{7} + C\end{aligned}$$

Example 2: Find the indefinite integral $\int \sin^3 3x dx$.

Note : Power of the sine is *odd*.

$$\int \sin^3 3x dx = \int \sin^2 3x \cdot \sin 3x dx = \int (1 - \cos^2 3x) \cdot \sin 3x dx$$

$$\text{Now let } u = \cos 3x \Rightarrow \frac{du}{dx} = -\sin 3x \cdot (3) = -3 \cdot \sin 3x$$

$$\Rightarrow du = -3 \sin 3x dx$$

$$\Rightarrow -\frac{1}{3} du = \sin 3x dx$$

$$\int \sin^3 3x dx = \int \sin^2 3x \cdot \sin 3x dx = \int (1 - \cos^2 3x) \cdot \sin 3x dx$$

$$= \int (1 - u^2) \cdot \left(-\frac{1}{3} du\right) = -\frac{1}{3} \int (1 - u^2) \cdot du$$

$$= -\frac{1}{3} \left(u - \frac{u^3}{3} \right) + C$$

$$= -\frac{1}{3} \left(\cos 3x - \frac{(\cos 3x)^3}{3} \right) + C$$

Example 3: Find the indefinite integral $\int \cos^3 \frac{x}{3} dx$.

Note: Power of the cosine is *odd*.

$$\int \cos^3 \frac{x}{3} dx = \int \cos^2 \frac{x}{3} \cdot \cos \frac{x}{3} dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cdot \cos \frac{x}{3} dx$$

$$\text{Now let } u = \sin \frac{x}{3} \Rightarrow \frac{du}{dx} = \cos \frac{x}{3} \cdot D_x \left(\frac{x}{3}\right) = \cos \frac{x}{3} \cdot \left(\frac{1}{3}\right)$$

$$\Rightarrow du = \left(\frac{1}{3}\right) \cdot \cos \frac{x}{3} dx$$

$$\Rightarrow 3du = \cos \frac{x}{3} dx$$

$$\int \cos^3 \frac{x}{3} dx = \int \cos^2 \frac{x}{3} \cdot \cos \frac{x}{3} dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cdot \cos \frac{x}{3} dx$$

$$= \int (1 - u^2) \cdot (3du) = 3 \int (1 - u^2) \cdot du$$

$$= 3 \left(u - \frac{u^3}{3} \right) + C$$

$$= 3 \left(\sin \frac{x}{3} - \frac{\left(\sin \frac{x}{3}\right)^3}{3} \right) + C$$

Example 4: Find the indefinite integral $\int \frac{\cos^5 x}{\sqrt{\sin x}} dx$.

Note: Power of the cosine is *odd*.

$$\begin{aligned}\int \frac{\cos^5 x}{\sqrt{\sin x}} dx &= \int \cos^5 x \frac{1}{\sqrt{\sin x}} dx = \int \cos^2 x \cdot \cos^2 x \cdot \cos x \frac{1}{\sqrt{\sin x}} dx \\ &= \int (1 - \sin^2 x)(1 - \sin^2 x) \cdot \cos x \frac{1}{(\sin x)^{1/2}} dx \\ &= \int (1 - \sin^2 x)(1 - \sin^2 x)(\sin x)^{-1/2} \cdot \cos x dx\end{aligned}$$

Now let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x dx$

$$\begin{aligned}\int \frac{\cos^5 x}{\sqrt{\sin x}} dx &= \int (1 - \sin^2 x)(1 - \sin^2 x)(\sin x)^{-1/2} \cdot \cos x dx \\ &= \int (1 - u^2)(1 - u^2)(u)^{-1/2} \cdot (du) \\ &= \int (1 - 2u^2 + u^4)(u)^{-1/2} \cdot (du) \\ &= \int (u^{-1/2} - 2u^{3/2} + u^{7/2}) \cdot (du) \\ &= \frac{u^{1/2}}{1/2} - 2 \frac{u^{5/2}}{5/2} + \frac{u^{9/2}}{9/2} + C \\ &= 2u^{1/2} - \frac{4}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C \\ &= 2(\sin x)^{1/2} - \frac{4}{5}(\sin x)^{5/2} + \frac{2}{9}(\sin x)^{9/2} + C\end{aligned}$$

Example 5: Find the indefinite integral $\int \sin^4 x dx$.

Recall: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\begin{aligned}\int \sin^4 x dx &= \int \sin^2 x \sin^2 x dx \\ &= \int \left[\frac{1}{2}(1 - \cos 2x) \right] \left[\frac{1}{2}(1 - \cos 2x) \right] dx \\ &= \frac{1}{4} \int (1 - \cos 2x)(1 - \cos 2x) dx \\ &= \frac{1}{4} \int (1 - 2\cos 2x + (\cos 2x)^2) dx\end{aligned}$$

Note: $\int (\cos 2x)^2 dx = \int (\cos^2 2x) dx$

Let $u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2} du = dx$

$$\begin{aligned}\int (\cos 2x)^2 dx &= \int (\cos^2 2x) dx = \int \cos^2 u \cdot \left(\frac{1}{2} du \right) \\ &= \frac{1}{2} \int \cos^2 u \cdot du = \frac{1}{2} \left[\frac{1}{2}(u + \sin u \cos u) \right] \quad \text{Formula \#49} \\ &= \frac{1}{4}(u + \sin u \cos u) = \frac{1}{4}(2x + \sin 2x \cos 2x)\end{aligned}$$

For $\int (\cos 2x) dx$:

$$\text{Let } u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2} du = dx$$

$$\begin{aligned} \int (\cos 2x) dx &= \int (\cos u) \left(\frac{1}{2} du \right) = \frac{1}{2} \int (\cos u) du \\ &= \frac{1}{2} [\sin u] = \frac{1}{2} [\sin 2x] \end{aligned}$$

Hence,

$$\begin{aligned} \int \sin^4 x dx &= \int \sin^2 x \sin^2 x dx \\ &= \frac{1}{4} \int (1 - 2 \cos 2x + (\cos 2x)^2) dx \\ &= \frac{1}{4} \left[x - 2 \left[\frac{1}{2} [\sin 2x] \right] + \left[\frac{1}{4} (2x + \sin 2x \cos 2x) \right] \right] + C \end{aligned}$$

Example 6: Find the indefinite integral $\int x^2 \sin^2 x dx$.

Recall: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$\begin{aligned}\int x^2 \sin^2 x dx &= \int x^2 \left[\frac{1}{2}(1 - \cos 2x) \right] dx \\ &= \frac{1}{2} \int (x^2 - x^2 \cos 2x) dx\end{aligned}$$

Note:

Formula #55 with $n = 2$: $\int u^2 \cos u du = u^2 \sin u - 2 \int u \sin u du$

Formula 52: $\int u \sin u du = \sin u - u \cos u$

Hence, $\int u^2 \cos u du = u^2 \sin u - 2[\sin u - u \cos u]$

$$\text{Hence, } \int u^2 \cos u du = u^2 \sin u - 2[\sin u - u \cos u]$$

$$\text{For } \int (x^2 \cos 2x) dx :$$

$$\text{Let } u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2} du = dx$$

$$\text{Also, } u = 2x \Rightarrow \frac{u}{2} = x \Rightarrow \frac{u^2}{4} = x^2 \Rightarrow \frac{1}{4} u^2 = x^2$$

$$\int (x^2 \cos 2x) dx = \int \left(\frac{1}{4} u^2 \right) (\cos u) \left(\frac{1}{2} du \right)$$

$$= \frac{1}{8} \int (u^2 \cos u) du$$

$$= \frac{1}{8} [u^2 \sin u - 2[\sin u - u \cos u]] + C$$

$$= \frac{1}{8} [(2x)^2 \sin 2x - 2[\sin 2x - 2x \cos 2x]] + C$$

$$\text{Therefore, } \int x^2 \sin^2 x dx = \int x^2 \left[\frac{1}{2} (1 - \cos 2x) \right] dx$$

$$= \frac{1}{2} \int (x^2 - x^2 \cos 2x) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - \left[\frac{1}{8} [(2x)^2 \sin 2x - 2[\sin 2x - 2x \cos 2x]] \right] \right] + C$$

Example 7: $\int_0^{\pi/2} \cos^9 x dx$ Note: $n = 9$

Using Wallis's Formula on p. 526.

$$\int_0^{\pi/2} \cos^9 x dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right) = \frac{384}{945}$$

Example 8:

$$\int \sec^4 2x dx \quad \text{Note: power of the secant is even}$$

Trigonometric Identity: $1 + \tan^2 x = \sec^2 x$

$$\begin{aligned} \int \sec^4 2x dx &= \int (\sec^2 2x)(\sec^2 2x) dx \\ &= \int (1 + \tan^2 2x)(\sec^2 2x) dx \end{aligned}$$

$$\text{Now Let } u = \tan 2x \Rightarrow \frac{du}{dx} = \sec^2 2x \cdot D_x(2x) = \sec^2 2x \cdot (2)$$

$$\Rightarrow du = 2\sec^2 2x dx$$

$$\Rightarrow \frac{1}{2} du = \sec^2 2x dx$$

$$\int \sec^4 2x dx = \int (1 + \tan^2 2x)(\sec^2 2x) dx$$

$$= \int (1 + u^2) \left(\frac{1}{2} du \right) = \frac{1}{2} \int (1 + u^2) du$$

$$= \frac{1}{2} \left[u + \frac{u^3}{3} \right]$$

$$= \frac{1}{2} \left[\tan 2x + \frac{(\tan 2x)^3}{3} \right] + C$$

Example 9:

$$\int \tan^6 3x dx \quad \text{Note: power of the tangent is even}$$

Trigonometric Identity: $1 + \tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$$\begin{aligned} \int \tan^6 3x dx &= \int (\tan^4 3x)(\tan^2 3x) dx \\ &= \int (\tan^4 3x)(\sec^2 3x - 1) dx \\ &= \int (\tan^4 3x \sec^2 3x - \tan^4 3x) dx \\ &= \int (\tan^4 3x \sec^2 3x) dx - \int (\tan^4 3x) dx \end{aligned}$$

For $\int (\tan^4 3x \sec^2 3x) dx$:

$$\text{Let } u = \tan 3x \Rightarrow \frac{du}{dx} = (\sec^2 3x) D_x(3x) = (\sec^2 3x) \cdot 3$$

$$\Rightarrow du = 3(\sec^2 3x) dx$$

$$\Rightarrow \frac{1}{3} du = (\sec^2 3x) dx$$

$$\int (\tan^4 3x \sec^2 3x) dx = \int (\tan^4 3x) (\sec^2 3x) dx$$

$$= \int u^4 \left(\frac{1}{3} du \right) = \frac{1}{3} \int u^4 du$$

$$= \frac{1}{3} \left[\frac{u^5}{5} \right] = \frac{1}{15} [u^5]$$

$$= \frac{1}{15} [(\tan 3x)^5]$$

For $\int (\tan^4 3x) dx$, use Formula #67 with $n = 4$

$$\text{Let } u = 3x \Rightarrow \frac{du}{dx} = 3 \Rightarrow du = 3dx \Rightarrow \frac{1}{3} du = dx$$

$$\int (\tan^4 3x) dx = \int (\tan^4 u) \left(\frac{1}{3} du \right) = \frac{1}{3} \int (\tan^4 u) du$$

$$= \frac{1}{3} \left[\frac{\tan^3 u}{3} - \int \tan^2 u du \right]$$

$$= \frac{1}{3} \left[\frac{\tan^3 u}{3} - [-u + \tan u] \right] \quad \text{Using Formula #63}$$

$$= \frac{1}{3} \left[\frac{\tan^3 3x}{3} - [-3x + \tan 3x] \right]$$

Therefore:

$$\int \tan^6 3x dx = \int (\tan^4 3x \sec^2 3x) dx - \int (\tan^4 3x) dx$$

$$= \left[\frac{1}{15} [(\tan 3x)^5] \right] - \left[\frac{1}{3} \left[\frac{\tan^3 3x}{3} - [-3x + \tan 3x] \right] \right] + C$$

Example 10: $\int (\tan^3 3x) dx$

For $\int (\tan^3 3x) dx$, use Formula #67 with $n = 3$

$$\text{Let } u = 3x \Rightarrow \frac{du}{dx} = 3 \Rightarrow du = 3dx \Rightarrow \frac{1}{3} du = dx$$

$$\int (\tan^3 3x) dx = \int (\tan^3 u) \left(\frac{1}{3} du \right) = \frac{1}{3} \int (\tan^3 u) du$$

$$= \frac{1}{3} \left[\frac{\tan^3 u}{2} - \int \tan u du \right]$$

$$= \frac{1}{3} \left[\frac{\tan^3 u}{2} - [\ln |\cos u|] \right] \quad \text{Using Formula \#59}$$

$$= \frac{1}{3} \left[\frac{\tan^3 3x}{2} - [\ln |\cos 3x|] \right] + C$$