Trigonometric Integrals

Trigonometric Identities:

$$\sin^2 x + \cos^2 x = 1;$$

$$\sin^2 x = 1 - \cos^2 x$$

$$\cos^2 x = 1 - \sin^2 x$$

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2}\left(1-\cos 2x\right) = \frac{1}{2} - \frac{1}{2}\cos 2x$$

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2}\left(1+\cos 2x\right) = \frac{1}{2} + \frac{1}{2}\cos 2x$$

$$1+\tan^2 x = \sec^2 x \quad \text{or} \quad \tan^2 x = \sec^2 x - 1$$

Example1: Find the indefinite integral $\int \cos^3 x \sin^4 x dx$.

Note: Power of the cosine is *odd*.

$$\int \cos^3 x \sin^4 x dx = \int \cos^2 x \cdot \cos x \sin^4 x dx = \int \cos^2 x \cdot \sin^4 x \cdot \cos x dx = \int (1\text{-}\sin^2 x) \cdot \sin^4 x \cdot \cos x dx$$

Now let $u = \sin x \Rightarrow \dfrac{du}{dx} = \cos x \Rightarrow du = \cos x dx$

$$\int \cos^3 x \sin^4 x dx = \int \cos^2 x \cdot \cos x \sin^4 x dx$$

$$= \int \cos^2 x \cdot \sin^4 x \cdot \cos x dx$$

$$= \int (1\text{-}\sin^2 x) \cdot \sin^4 x \cdot \cos x dx$$

$$= \int (1\text{-}u^2) \cdot u^4 \cdot du$$

$$= \int (u^4\text{-}u^6) \cdot du \ = \frac{u^5}{5} - \frac{u^7}{7} + C$$

$$= \frac{(\sin x)^5}{5} - \frac{(\sin x)^7}{7} + C$$

Example 2: Find the indefinite integral $\int \sin^3 3x dx$.

Note: Power of the sine is *odd*.

$$\int \sin^3 3x dx = \int \sin^2 3x \cdot \sin 3x dx = \int \left(1-\cos^2 3x\right) \cdot \sin 3x dx$$

Now let $u = \cos 3x \Rightarrow \dfrac{du}{dx} = -\sin 3x \cdot (3) = -3 \cdot \sin 3x$

$$\Rightarrow du = -3\sin 3x dx$$

$$\Rightarrow -\frac{1}{3} du = \sin 3x dx$$

$$\int \sin^3 3x dx = \int \sin^2 3x \cdot \sin 3x dx = \int \left(1-\cos^2 3x\right) \cdot \sin 3x dx$$

$$= \int \left(1-u^2\right) \cdot \left(-\frac{1}{3} du\right) = -\frac{1}{3} \int \left(1-u^2\right) \cdot du$$

$$= -\frac{1}{3}\left(u - \frac{u^3}{3}\right) + C$$

$$= -\frac{1}{3}\left(\cos 3x - \frac{\left(\cos 3x\right)^3}{3}\right) + C$$

Example 3: Find the indefinite integral $\int \cos^3 \frac{x}{3} dx.$ *Note*: Power of the cosine is *odd*.

$$\int \cos^3 \frac{x}{3} dx = \int \cos^2 \frac{x}{3} \cdot \cos \frac{x}{3} dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cdot \cos \frac{x}{3} dx$$

Now let $u = \sin \frac{x}{3} \Rightarrow \frac{du}{dx} = \cos \frac{x}{3} \cdot D_x \left(\frac{x}{3}\right) = \cos \frac{x}{3} \cdot \left(\frac{1}{3}\right)$

$$\Rightarrow du = \left(\frac{1}{3}\right) \cdot \cos \frac{x}{3} dx$$

$$\Rightarrow 3du = \cos \frac{x}{3} dx$$

$$\int \cos^3 \frac{x}{3} dx = \int \cos^2 \frac{x}{3} \cdot \cos \frac{x}{3} dx = \int \left(1 - \sin^2 \frac{x}{3}\right) \cdot \cos \frac{x}{3} dx$$

$$= \int \left(1 - u^2\right) \cdot (3du) = 3 \int \left(1 - u^2\right) \cdot du = 3\left(u - \frac{u^3}{3}\right) + C$$

$$= 3\left(\sin \frac{x}{3} - \frac{\left(\sin \frac{x}{3}\right)^3}{3}\right) + C$$

Example 4: Find the indefinite integral $\int \frac{\cos^5 x}{\sqrt{\sin x}}dx.$

Note: Power of the cosine is *odd*.

$$\int \frac{\cos^5 x}{\sqrt{\sin x}}dx = \int \cos^5 x \frac{1}{\sqrt{\sin x}}dx = \int \cos^2 x \cdot \cos^2 x \cdot \cos x \frac{1}{\sqrt{\sin x}}dx$$

$$= \int \left(1-\sin^2 x\right)\left(1-\sin^2 x\right) \cdot \cos x \frac{1}{\left(\sin x\right)^{1/2}}dx$$

$$= \int \left(1-\sin^2 x\right)\left(1-\sin^2 x\right)\left(\sin x\right)^{-1/2} \cdot \cos x dx$$

Now let $u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow du = \cos x dx$

$$\int \frac{\cos^5 x}{\sqrt{\sin x}}dx = \int \left(1-\sin^2 x\right)\left(1-\sin^2 x\right)\left(\sin x\right)^{-1/2} \cdot \cos x dx$$

$$= \int \left(1-u^2\right)\left(1-u^2\right)\left(u\right)^{-1/2} \cdot \left(du\right)$$

$$= \int \left(1-2u^2+u^4\right)\left(u\right)^{-1/2} \cdot \left(du\right) \quad = \int \left(u^{-1/2} - 2u^{3/2} + u^{7/2}\right) \cdot \left(du\right)$$

$$= u^{1/2}\left(2\right) - 2u^{5/2}\left(2/5\right) + u^{9/2}\left(2/9\right) + C \;\; = 2u^{1/2} - \frac{4}{5}u^{5/2} + \frac{2}{9}u^{9/2} + C$$

$$= 2\left(\sin x\right)^{1/2} - \frac{4}{5}\left(\sin x\right)^{5/2} + \frac{2}{9}\left(\sin x\right)^{9/2} + C$$

Example 5: Find the indefinite integral $\int \sin^4 x dx$.

Recall: $\sin^2 x = \frac{1}{2}(1-\cos 2x) = \frac{1}{2} - \frac{1}{2}\cos 2x$

$$\int \sin^4 x dx = \int \sin^2 x \sin^2 x dx = \int \left[\frac{1}{2}(1-\cos 2x)\right]\left[\frac{1}{2}(1-\cos 2x)\right]dx$$

$$= \frac{1}{4}\int (1-\cos 2x)(1-\cos 2x)dx = \frac{1}{4}\int \left(1-2\cos 2x + (\cos 2x)^2\right)dx$$

Note: $\int (\cos 2x)^2 dx = \int \left(\cos^2 2x\right)dx$

Let $u = 2x \Rightarrow \frac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2}du = dx$

$$\int (\cos 2x)^2 dx = \int \left(\cos^2 2x\right)dx = \int \cos^2 u \cdot \left(\frac{1}{2}du\right)$$

$$= \frac{1}{2}\int \cos^2 u \cdot du = \frac{1}{2}\int \left(\frac{1}{2} + \frac{1}{2}\cos 2u\right)\cdot du = \frac{1}{2}\left[\frac{1}{2}u + \frac{1}{2}\left(\frac{1}{2}\sin 2u\right)\right]$$

Note: $\int \cos bu du = \frac{1}{b}[\sin 2u]$

$$= \frac{1}{4}u + \frac{1}{8}\sin 2u$$

$$= \frac{1}{4}(2x) + \frac{1}{8}\sin(4x) = \frac{1}{2}x + \frac{1}{8}\sin(4x) + C$$

For $\int(\cos 2x)dx$:

Let $u = 2x \Rightarrow \dfrac{du}{dx} = 2 \Rightarrow du = 2dx \Rightarrow \dfrac{1}{2}du = dx$

$$\int(\cos 2x)dx = \int(\cos u)\left(\frac{1}{2}du\right) = \frac{1}{2}\int(\cos u)\,du$$

$$= \frac{1}{2}[\sin u] = \frac{1}{2}[\sin 2x]$$

Hence,

$$\int \sin^4 x dx = \int \sin^2 x \sin^2 x dx$$

$$= \frac{1}{4}\int\left(1 - 2\cos 2x + (\cos 2x)^2\right)dx$$

$$= \frac{1}{4}\left(x - \sin 2x + \left[\frac{1}{2}x + \frac{1}{8}\sin(4x) + \right]\right) + C$$

Example 6: Find the indefinite integral $I = \int x^2 \sin^2 x dx$.

Recall: $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$$I = \int x^2 \sin^2 x dx = \int x^2 \left[\frac{1}{2}(1 - \cos 2x)\right] dx = \frac{1}{2}\int \left(x^2 - x^2 \cos 2x\right) dx = \underbrace{\frac{1}{2}\int \left(x^2\right) dx}_{I_2} - \frac{1}{2}\underbrace{\int \left(x^2 \cos 2x\right) dx}_{I_3}$$

$$I_2 = \frac{1}{2}\int \left(x^2\right) dx = \frac{1}{2}\left[x^3 \cdot \frac{1}{3}\right] = \frac{1}{6}x^3$$

To find $I_3 = \int \left(x^2 \cos 2x\right) dx$, we can use Integration by Parts.

Let $u = x^2$ $\qquad$ $dv = \cos 2x dx$

$\frac{du}{dx} = 2x$ $\qquad$ $\int dv = \int \cos 2x dx$

$du = 2x dx$ $\qquad$ $v = \frac{1}{2}\sin 2x$

Hence, $\int u dv = uv - \int v du$

$$I_3 = \int \left(x^2 \cos 2x\right) dx = \left(x^2\right)\left(\frac{1}{2}\sin 2x\right) - \int \frac{1}{2}\sin 2x (2x)\, dx$$

$$I_3 = \int \left(x^2 \cos 2x\right) dx = \frac{1}{2}\left(x^2\right)\left(\sin 2x\right) - \int x \sin 2x dx$$

For $\int x\sin 2xd$ we can use Integration by Parts.

Let $u = x$ $\qquad dv = \sin 2xdx$

$$\frac{du}{dx} = 1 \qquad \int dv = \int \sin 2xdx$$

$$du = dx \qquad v = -\frac{1}{2}\cos 2x$$

Hence, $\int udv = uv - \int vdu$

$$\int (x\sin 2x)dx = (x)\left(-\frac{1}{2}\cos 2x\right) - \int \frac{1}{2}\cos 2xdx$$

$$\int (x\sin 2x)dx = -\frac{1}{2}x(\cos 2x) - \left[\frac{1}{2}\left(\frac{1}{2}\sin 2x\right)\right]$$

$$\int (x\sin 2x)dx = -\frac{1}{2}x(\cos 2x) - \frac{1}{4}\sin 2x$$

$$I_3 = \frac{1}{2}\left(x^2\right)(\sin 2x) - \left[-\frac{1}{2}x(\cos 2x) - \frac{1}{4}\sin 2x\right]$$

$$I_3 = \frac{1}{2}\left(x^2\right)(\sin 2x) + \frac{1}{2}x(\cos 2x) + \frac{1}{4}\sin 2x$$

Therefore, $I = \int x^2 \sin^2 x dx = I_2 - \frac{1}{2} I_3$

$$I = \int x^2 \sin^2 x dx = \frac{1}{6}x^3 - \frac{1}{2}\left[\frac{1}{2}\left(x^2\right)\left(\sin 2x\right) + \frac{1}{2}x\left(\cos 2x\right) + \frac{1}{4}\sin 2x\right] + C$$

Example 7: $\int_0^{\pi/2} \cos^9 x dx$ Note: $n = 9$

Using Wallis's Formula:

$$\int_0^{\pi/2} \cos^9 x dx = \left(\frac{2}{3}\right)\left(\frac{4}{5}\right)\left(\frac{6}{7}\right)\left(\frac{8}{9}\right) = \frac{384}{945}$$

Example 8:

$\int \sec^4 2x dx$ Note: power of the secant is even

Trigonometric Identity: $1+\tan^2 x = \sec^2 x$

$$\int \sec^4 2x dx = \int \left(\sec^2 2x\right)\left(\sec^2 2x\right) dx$$

$$= \int \left(1+\tan^2 2x\right)\left(\sec^2 2x\right) dx$$

Now Let $u = \tan 2x \Rightarrow \dfrac{du}{dx} = \sec^2 2x \cdot D_x\left(2x\right) = \sec^2 2x \cdot \left(2\right)$

$$\Rightarrow du = 2\sec^2 2x dx \quad \Rightarrow \quad \frac{1}{2} du = \sec^2 2x dx$$

$$\int \sec^4 2x dx = \int \left(1+\tan^2 2x\right)\left(\sec^2 2x\right) dx$$

$$= \int \left(1+u^2\right)\left(\frac{1}{2} du\right) = \frac{1}{2}\int \left(1+u^2\right) du$$

$$= \frac{1}{2}\left[u + \frac{u^3}{3}\right] = \frac{1}{2}\left[\tan 2x + \frac{\left(\tan 2x\right)^3}{3}\right] + C$$

Example 9:

$\int \tan^6 3x\,dx$ Note: power of the tangent is even

Trigonometric Identity: $1+\tan^2 x = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$

$$\int \tan^6 3x\,dx = \int \left(\tan^4 3x\right)\left(\tan^2 3x\right)dx = \int \left(\tan^4 3x\right)\left(\sec^2 3x - 1\right)dx$$

$$= \int \left(\tan^4 3x \sec^2 3x - \tan^4 3x\right)dx = \int \left(\tan^4 3x \sec^2 3x\right)dx - \int \left(\tan^4 3x\right)dx$$

For $\int \left(\tan^4 3x \sec^2 3x\right)dx$:

Let $u = \tan 3x \Rightarrow \dfrac{du}{dx} = \left(\sec^2 3x\right)D_x(3x) = \left(\sec^2 3x\right)\cdot 3 \Rightarrow du = 3\left(\sec^2 3x\right)dx \Rightarrow \dfrac{1}{3}du = \left(\sec^2 3x\right)dx$

$$\int \left(\tan^4 3x \sec^2 3x\right)dx = \int \left(\tan^4 3x\right)\left(\sec^2 3x\right)dx$$

$$= \int u^4\left(\frac{1}{3}du\right) = \frac{1}{3}\int u^4 du = \frac{1}{3}\left[\frac{u^5}{5}\right] = \frac{1}{15}\left[u^5\right] = \frac{1}{15}\left[\left(\tan 3x\right)^5\right]$$

For $\int \tan^4 3x dx$:

$$\int \tan^4 3x dx = \int \left(\tan^2 3x\right)\left(\tan^2 3x\right) dx = \int \left(\sec^2 3x - 1\right)\left(\tan^2 3x\right) dx = \int \left(\sec^2 3x \cdot \tan^2 3x - \tan^2 3x\right) dx$$

$$= \underbrace{\int \left(\sec^2 3x \cdot \tan^2 3x\right) dx}_{I_3} - \underbrace{\int \left(\tan^2 3x\right) dx}_{I_4}$$

$$I_3 = \int \left(\sec^2 3x \cdot \tan^2 3x\right) dx, \quad \text{let } u = \tan x \quad \Rightarrow \frac{du}{dx} = \sec^2 3x \cdot 3$$

$$\Rightarrow du = 3\sec^2 3x dx \Rightarrow \frac{1}{3} du = \sec^2 3x dx$$

$$I_3 = \int \left(\sec^2 3x \cdot \tan^2 3x\right) dx = \int \left(\tan^2 3x \cdot \sec^2 3x\right) dx = \int u^2 \cdot \frac{1}{3} du = \frac{1}{3} u^3 \cdot 3 = u^3 = \left(\tan x\right)^3$$

$$I_4 = \int \left(\tan^2 3x\right) dx = \int \left(\sec^2 3x - 1\right) dx = \frac{1}{3} \tan 3x - x \qquad \text{Note: } \int \left(\sec^2 bx\right) dx = \frac{1}{b} \tan bx$$

$$\text{So } \int \tan^4 3x dx = I_3 - I_4 = \left(\tan x\right)^3 - \left[\frac{1}{3} \tan 3x - x\right] = \left(\tan x\right)^3 - \frac{1}{3} \tan 3x + x$$

Therefore:

$$\int \tan^6 3x\,dx = \int\left(\tan^4 3x \sec^2 3x\right)dx - \int\left(\tan^4 3x\right)dx$$

$$= \left[\frac{1}{15}\left[\left(\tan 3x\right)^5\right]\right] - \left[\frac{1}{3}\left[\frac{\tan^3 3x}{3} - \left[-3x + \tan 3x\right]\right]\right] + C$$

Example 10:

$$\int \tan^3 3x dx = \int \left(\tan^2 3x\right)\left(\tan 3x\right) dx = \int \left(\sec^2 3x - 1\right)\left(\tan 3x\right) dx = \int \left(\sec^2 3x \cdot \tan 3x - \tan 3x\right) dx$$

$$= \underbrace{\int \left(\sec^2 3x \cdot \tan 3x\right) dx}_{I_3} - \underbrace{\int \left(\tan 3x\right) dx}_{I_4}$$

$$I_3 = \int \left(\sec^2 3x \cdot \tan 3x\right) dx, \quad \text{let } \; u = \tan x \quad \Rightarrow \frac{du}{dx} = \sec^2 3x \cdot 3 \;\Rightarrow\; du = 3\sec^2 3x dx \;\Rightarrow\; \frac{1}{3} du = \sec^2 3x dx$$

$$I_3 = \int \left(\sec^2 3x \cdot \tan 3x\right) dx = \int \left(\tan 3x \cdot \sec^2 3x\right) dx = \int u \cdot \frac{1}{3} du = \frac{1}{3} u^2 \cdot \frac{1}{2} = \frac{1}{6} u^3 = \frac{1}{6}\left(\tan x\right)^2$$

$$I_4 = \int \left(\tan 3x\right) dx = -\frac{1}{3} \ln\left|\cos 3x\right|$$

Note: $\int \left(\tan bx\right) dx = -\frac{1}{b} \ln\left|\cos bx\right|$

So $\int \tan^3 3x dx = I_3 - I_4 = \frac{1}{6}\left(\tan x\right)^2 - \left[-\frac{1}{3}\ln\left|\cos 3x\right|\right] = \frac{1}{6}\left(\tan x\right)^2 + \frac{1}{3}\ln\left|\cos 3x\right| + C$