Section 8.4 Trigonometric Substitution

Trigonometric Identities:

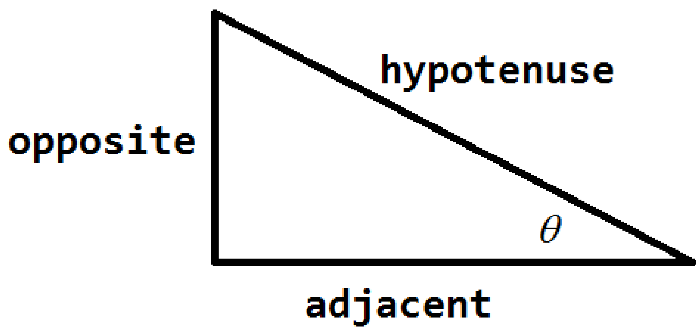
$$\sin^2 x + \cos^2 x = 1; \quad \sin^2 x = 1 - \cos^2 x; \quad \cos^2 x = 1 - \sin^2 x$$

$$\sec^2 x = 1 + \tan^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$$

For integrals containing $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$

For integrals containing $\sqrt{a^2 + u^2}$, let $u = a\tan\theta$

For integrals containing $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$



$$(hypotenuse)^2 = (opposite)^2 + (adjacent)^2$$

$$hypotenuse = \sqrt{(opposite)^2 + (adjacent)^2}$$

$$(opposite)^2 = (hypotenuse)^2 - (adjacent)^2$$

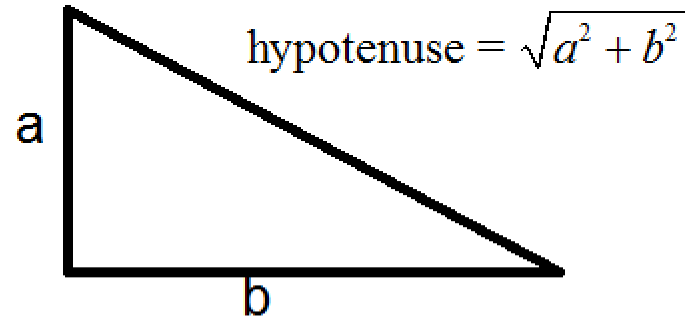
$$opposite = \sqrt{(hypotenuse)^2 - (adjacent)^2}$$

$$(adjacent)^2 = (hypotenuse)^2 - (opposite)^2$$

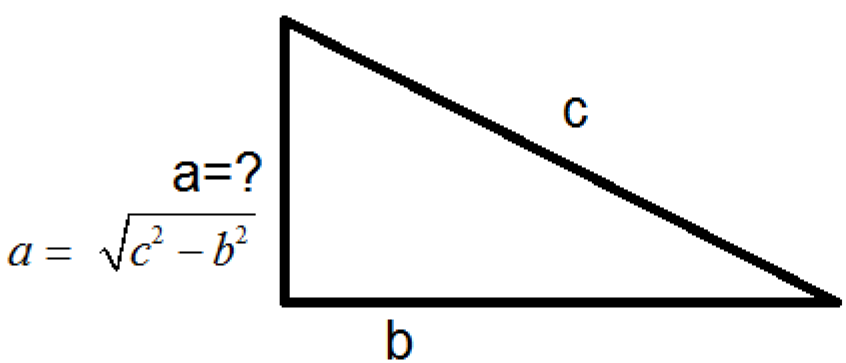
$$adjacent = \sqrt{(hypotenuse)^2 - (opposite)^2}$$

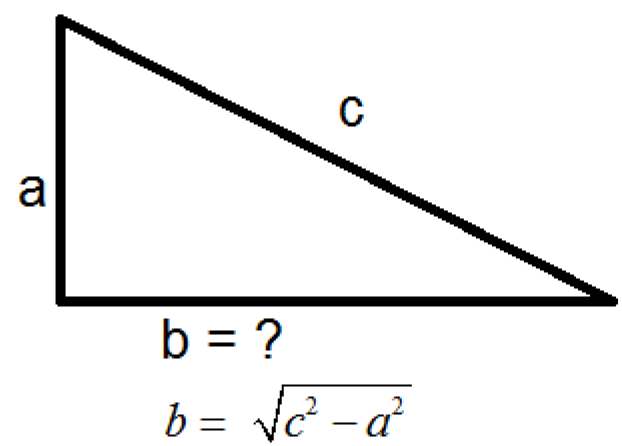
Case 1: hypotenuse is missing

$$\text{hypotenuse} = \sqrt{a^2 + b^2}$$



Case 2: One leg of triangle is missing





Example1: Find the indefinite integral $\int \frac{\sqrt{x^2-25}}{x}dx.$

Note: Integral contains Form u^2-a^2.

Let $u^2 = x^2$ and $a^2 = 25$

$\Rightarrow \quad u = x$ and $a = 5$

If we let $u = a\sec\theta,$ then $x = 5\sec\theta$

Hence, $\frac{dx}{d\theta} = 5\sec\theta\tan\theta \Rightarrow dx = 5\sec\theta\tan\theta d\theta$

$$\sqrt{x^2-25} = \sqrt{(5\sec\theta)^2-25} = \sqrt{25(\sec\theta)^2-25}$$

$$= \sqrt{25\left[(\sec\theta)^2-1\right]} = \sqrt{25\tan^2\theta} = 5\tan\theta$$

$$\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{5\tan\theta}{5\sec\theta}(5\sec\theta\tan\theta d\theta) = \int(5\tan\theta)(\tan\theta d\theta)$$

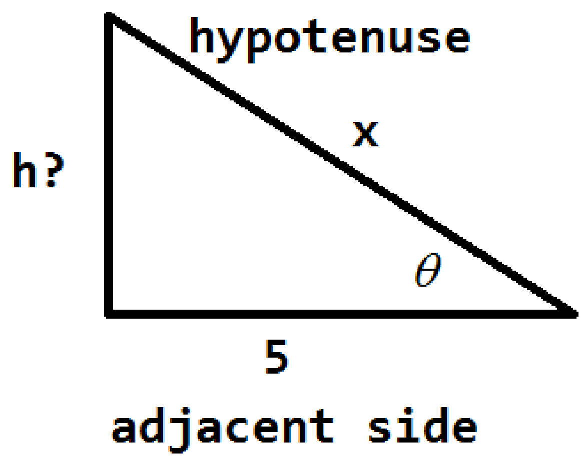
$$= 5\int \tan^2\theta d\theta = 5\int(\sec^2\theta-1)d\theta$$

Note: $\tan^2\theta = \sec^2\theta - 1$ and $\int \sec^2\theta d\theta = \tan\theta$

$$= 5[\tan\theta - \theta]$$

$Note: \; x = 5\sec\theta$

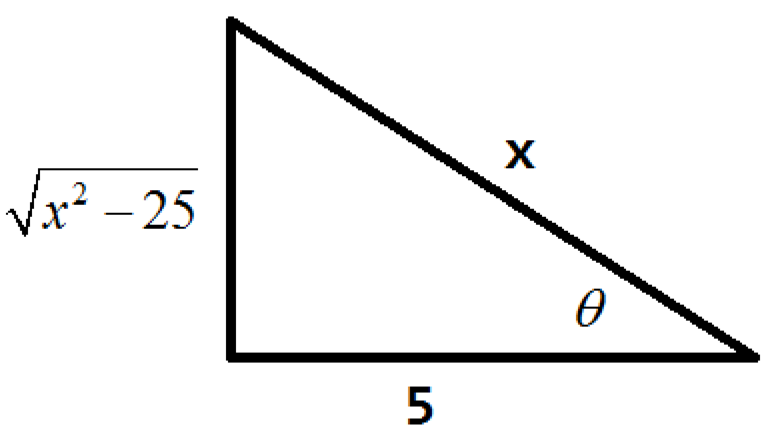
$$\sec\theta = \frac{x}{5} = \frac{hypotenuse}{adjacent}$$



$Note: \; h^2 + 5^2 = x^2$

$h^2 = x^2 - 25$

$h = \sqrt{x^2 - 25}$



$Note: \left(\sqrt{x^2 - 25}\right)^2 + 5^2 = x^2$

From triangle:

$$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{x}{5} \Rightarrow \theta = \sec^{-1}\left(\frac{x}{5}\right)$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{\sqrt{x^2-25}}{5}$$

$$\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{5\tan\theta}{5\sec\theta}\left(5\sec\theta\tan\theta d\theta\right) = \int\left(5\tan\theta\right)\left(\tan\theta d\theta\right)$$

$$=5\int \tan^2\theta d\theta$$

$$= 5\left[-\theta + \tan\theta\right]$$ Using Formula #63

$$= 5\left[-\sec^{-1}\frac{x}{5} + \frac{\sqrt{x^2-25}}{x}\right] + C$$

Example 2: Find the indefinite integral $\int \frac{x^3}{\sqrt{x^2-25}}dx.$

Note: Integral contains Form u^2-a^2.

Let $u^2 = x^2$ and $a^2 = 25$ $\Rightarrow$ $u = x$ and $a = 5$

If we let $u = a\sec\theta,$ then $x = 5\sec\theta$

Hence, $\frac{dx}{d\theta} = 5\sec\theta\tan\theta \Rightarrow dx = 5\sec\theta\tan\theta d\theta$

$$\sqrt{x^2-25} = \sqrt{(5\sec\theta)^2-25} = \sqrt{25(\sec\theta)^2-25} = \sqrt{25\left[(\sec\theta)^2-1\right]} = \sqrt{25\tan^2\theta} = 5\tan\theta$$

$$\int \frac{x^3}{\sqrt{x^2-25}}dx = \int \frac{(5\sec\theta)^3}{5\tan\theta}(5\sec\theta\tan\theta d\theta) = \int \frac{125(\sec\theta)^3}{5\tan\theta}(5\sec\theta\tan\theta d\theta) = \int 125(\sec\theta)^3(\sec\theta d\theta) = 125\int(\sec\theta)^4 d\theta$$

Note: $\int (\sec\theta)^4 d\theta = \int (\sec\theta)^2 (\sec\theta)^2 d\theta = \int \left(1+\tan^2\theta\right)(\sec\theta)^2 d\theta$

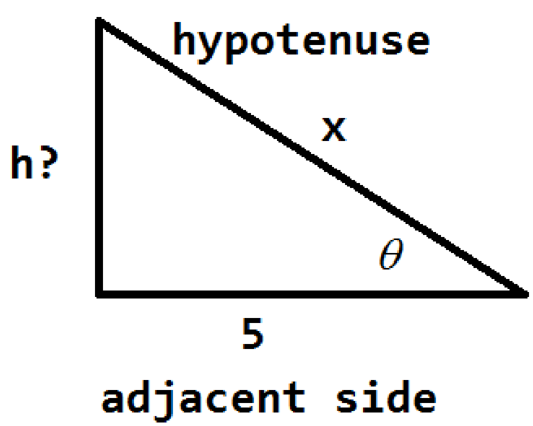
Let $u = \tan\theta$; hence $\dfrac{du}{d\theta} = \sec^2\theta$ and $du = \sec^2\theta d\theta$

$$\int (\sec\theta)^4 d\theta = \int \left(1+\tan^2\theta\right)(\sec\theta)^2 d\theta = \int \left(1+u^2\right)du = u + \frac{1}{3}u^3 = \tan\theta + \frac{1}{3}(\tan\theta)^3$$

$$\int \frac{x^3}{\sqrt{x^2-25}}dx = 125\int (\sec\theta)^4 d\theta = 125\left[\tan\theta + \frac{1}{3}(\tan\theta)^3\right] + C$$

$Note:\ x = 5\sec\theta$

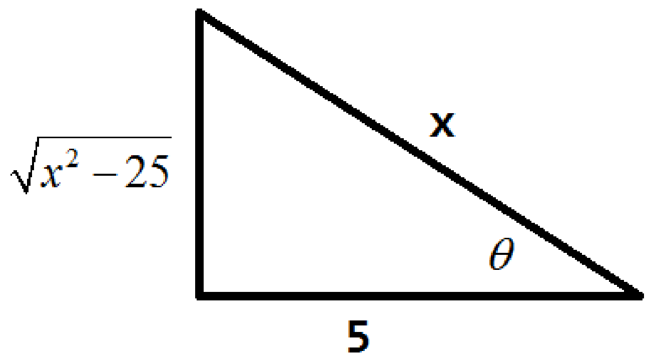
$$\sec\theta = \frac{x}{5} = \frac{hypotenuse}{adjacent}$$



$Note:\ h^2 + 5^2 = x^2$

$h^2 = x^2 - 25$

$h = \sqrt{x^2 - 25}$



$$Note:\left(\sqrt{x^2-25}\right)^2+5^2=x^2$$

From triangle:

$$\sec\theta=\frac{hypotenuse}{adjacent}=\frac{x}{5}\Rightarrow\theta=\sec^{-1}\left(\frac{x}{5}\right)$$

$$\tan\theta=\frac{hypotenuse}{opposite}=\frac{\sqrt{x^2-25}}{5}$$

$$\int\frac{x^3}{\sqrt{x^2-25}}dx=\ 125\left[\frac{\sqrt{x^2-25}}{5}+\frac{1}{3}\left(\frac{\sqrt{x^2-25}}{5}\right)^3\ \right]+C$$

Example 3: Find the indefinite integral $\int \sqrt{4+x^2}\,dx.$

Note: Integral contains Form $a^2 + u^2$.

Let $u^2 = x^2$ and $a^2 = 4$ $\Rightarrow$ $u = x$ and $a = 2$

If we let $u = a\tan\theta,$ then $x = 2\tan\theta$

Hence, $\dfrac{dx}{d\theta} = 2\sec^2\theta \Rightarrow dx = 2\sec^2\theta d\theta$

$$\sqrt{4+x^2} = \sqrt{4+(2\tan\theta)^2} = \sqrt{4+4(\tan\theta)^2} = \sqrt{4\left[1+(\tan\theta)^2\right]} = 2\sqrt{\left[1+(\tan\theta)^2\right]} = 2\sqrt{(\sec\theta)^2} = 2\sec\theta$$

$$\int \sqrt{4+x^2}\,dx = \int 2\sec\theta\left(2\sec^2\theta d\theta\right) = 4\int \sec^3\theta d\theta$$

Note: $\sec^2\theta = 1+\tan^2\theta$

For $\int \sec^3\theta d\theta = \int \sec\theta\sec^2\theta d\theta$ we can use Integration by Parts

Let $u = \sec\theta$ $\qquad dv = \sec^2\theta d\theta$

$$\frac{du}{d\theta} = \sec\theta\tan\theta \qquad \int dv = \int \sec^2\theta d\theta$$

$$du = \sec\theta\tan\theta d\theta \qquad v = \tan\theta$$

$$\int \sec^3\theta d\theta = \int \sec\theta\sec^2\theta d\theta = uv - \int vdu = \left(\sec\theta\right)\left(\tan\theta\right) - \int \tan\theta\sec\theta\tan\theta d\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta d\theta = \sec\theta\tan\theta d\theta - \int \sec\theta\left(\sec^2\theta - 1\right)d\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \left(\sec^3\theta - \sec\theta\right)d\theta = \sec\theta\tan\theta - \int \left(\sec^3\theta\right)d\theta + \int \left(\sec\theta\right)d\theta$$

$$\int \sec^3\theta d\theta = \sec\theta\tan\theta - \int \left(\sec^3\theta\right)d\theta + \ln\left|\sec\theta + \tan\theta\right|$$

$$\int \sec^3\theta d\theta + \int \left(\sec^3\theta\right)d\theta = \sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|$$

$$2\int \left(\sec^3\theta\right)d\theta = \sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|$$

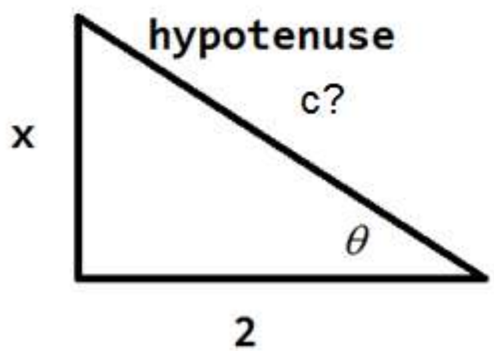
$$\int \left(\sec^3\theta\right)d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln\left|\sec\theta + \tan\theta\right|$$

$$4\int\left(\sec^3\theta\right)d\theta = 2\sec\theta\tan\theta + 2\ln\left|\sec\theta+\tan\theta\right|$$

$$\int\sqrt{4+x^2}\,dx = 4\int\sec^3\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln\left|\sec\theta+\tan\theta\right|$$

$$Note:\ x = 2\tan\theta$$

$$\tan\theta = \frac{x}{2} = \frac{opposite}{adjacent}$$



$$c^2 = x^2 + 2^2$$

$$c = \sqrt{x^2+4}$$

$$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{\sqrt{x^2+4}}{2}$$

$$\int\sqrt{4+x^2}\,dx = 4\int\sec^3\theta\,d\theta = 2\sec\theta\tan\theta + 2\ln\left|\sec\theta+\tan\theta\right| = 2\left(\frac{\sqrt{x^2+4}}{2}\right)\left(\frac{x}{2}\right) + 2\left[\ln\left|\frac{\sqrt{x^2+4}}{2}+\frac{x}{2}\right|\right] + C$$

Example 4: Find the indefinite integral $\int \sqrt{25x^2 - 1}dx$.

Note: Integral contains Form $u^2 - a^2$.

Let $u^2 = 25x^2$ and $a^2 = 1$ $\Rightarrow$ $u = 5x$ and $a = 1$

$$\frac{du}{dx} = 5 \quad \Rightarrow \quad du = 5dx \quad \Rightarrow \quad \frac{1}{5}du = dx$$

Using Formula for Table of Integration:

$$\int \sqrt{25x^2 - 1}dx = \int \sqrt{u^2 - a^2}\left(\frac{1}{5}du\right) = \frac{1}{5}\int \sqrt{u^2 - a^2}\,du$$

$$= \frac{1}{5}\left[\frac{1}{2}\left[u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right]\right] + C$$

$$= \frac{1}{10}\left[\left[5x\sqrt{(5x)^2 + 1} + \ln\left|5x + \sqrt{(5x)^2 + 1}\right|\right]\right] + C$$

Example 5: Find the indefinite integral $\int \frac{x^2}{\sqrt{36-x^2}}dx$.

Note: Integral contains Form $a^2 - u^2$.

Let $u^2 = x^2$ and $a^2 = 36$ $\Rightarrow$ $u = x$ and $a = 6$

Let $u = a\sin\theta$ $\Rightarrow x = a\sin\theta = 6\sin\theta$

$$\frac{dx}{d\theta} = 6\cos\theta \Rightarrow dx = 6\cos\theta d\theta$$

$$\sqrt{36-x^2} = \sqrt{36-(6\sin\theta)^2} = \sqrt{36-36(\sin\theta)^2} = \sqrt{36\left[1-(\sin\theta)^2\right]} = \sqrt{36\cos^2\theta} = 6\cos\theta$$

$$\int \frac{x^2}{\sqrt{36-x^2}}dx = \int \frac{(6\sin\theta)^2}{6\cos\theta}6\cos\theta d\theta = 36\int(\sin\theta)^2\, d\theta$$

$$Note: \int(\sin\theta)^2\, d\theta = \int\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\theta - \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta$$

$$\int \frac{x^2}{\sqrt{36-x^2}}dx = \int \frac{(6\sin\theta)^2}{6\cos\theta}6\cos\theta d\theta = 36\int(\sin\theta)^2\, d\theta = 36\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right] = 18\theta - 9\sin 2\theta$$

$$Note: \quad x = 6\sin\theta$$

$$\sin\theta = \frac{x}{6} = \frac{opposite}{hypotenuse}; \qquad \text{and } \theta = \sin^{-1}\left(\frac{x}{6}\right)$$

$$adjacent = \sqrt{(hypotenuse)^2 - (opposite)^2} = \sqrt{36 - x^2}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{\sqrt{36 - x^2}}{6}$$

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\left(\frac{x}{6}\right)\left(\frac{\sqrt{36 - x^2}}{6}\right)$$

$$\int \frac{x^2}{\sqrt{36 - x^2}}dx = 18\theta - 9\sin 2\theta = 18\sin^{-1}\left(\frac{x}{6}\right) - 9\sin 2\theta = 18\sin^{-1}\left(\frac{x}{6}\right) - \left[9 \cdot 2\left(\frac{x}{6}\right)\left(\frac{\sqrt{36 - x^2}}{6}\right)\right] + C$$

$$\int \frac{x^2}{\sqrt{36 - x^2}}dx = 18\theta - 9\sin 2\theta = 18\sin^{-1}\left(\frac{x}{6}\right) - 9\sin 2\theta = 18\sin^{-1}\left(\frac{x}{6}\right) - 18\left(\frac{x}{6}\right)\left(\frac{\sqrt{36 - x^2}}{6}\right) + C$$

Example 6: Find the indefinite integral $\int \frac{\sqrt{25x^2+4}}{x^4}dx.$

Note: Integral contains Form a^2+u^2.

Let $u^2 = 25x^2$ and $a^2 = 4$ $\Rightarrow$ $u = 5x$ and $a = 2$

Let $u = a\tan\theta \Rightarrow 5x = a\tan\theta \Rightarrow x = \frac{2\tan\theta}{5}$

$$\frac{dx}{d\theta} = \frac{2}{5}\sec^2\theta \Rightarrow dx = \frac{2}{5}\sec^2\theta d\theta$$

$$\sqrt{25x^2+4} = \sqrt{25\left(\frac{2\tan\theta}{5}\right)^2+4} = \sqrt{25\cdot\frac{4}{25}(\tan\theta)^2+4} = \sqrt{4(\tan\theta)^2+4} = \sqrt{4\left[(\tan\theta)^2+1\right]} = \sqrt{4\left[(\sec\theta)^2\right]} = 2\sec\theta$$

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = \int \frac{2\sec\theta}{\left(\frac{2\tan\theta}{5}\right)^4}\frac{2}{5}\sec^2\theta d\theta = \frac{2\cdot\frac{2}{5}}{\left(\frac{2}{5}\right)^4}\int \frac{\sec^3\theta}{(\tan\theta)^4}d\theta = 31.25\int \frac{\left(\frac{1}{\cos^3\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)^4}d\theta = 31.25\cos\theta(\sin\theta)^{-4}\, d\theta$$

$$Note: \frac{\left(\frac{1}{\cos^3\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)^4}\frac{\left(\frac{1}{\cos^3\theta}\right)}{\frac{(\sin\theta)^4}{(\cos\theta)^4}} = \left(\frac{1}{\cos^3\theta}\right)\frac{(\cos\theta)^4}{(\sin\theta)^4} = \frac{\cos\theta}{(\sin\theta)^4} = \cos\theta(\sin\theta)^{-4}$$

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = \ 31.25\int \cos\theta\left(\sin\theta\right)^{-4} d\theta$$

Let $u = \sin\theta$; $\dfrac{du}{d\theta} = \cos\theta$; $du = \cos\theta d\theta$

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = \ 31.25\int \cos\theta\left(\sin\theta\right)^{-4} d\theta$$

$$=31.25\int \left(\sin\theta\right)^{-4} \cos\theta d\theta = 31.25\int \left(u\right)^{-4} d\theta$$

$$=31.25\left[\frac{u^{-3}}{-3}\right] = 31.25\left[\frac{\left(\sin\theta\right)^{-3}}{-3}\right]$$

$$=31.25\left[\frac{u^{-3}}{-3}\right] = 31.25\left[\frac{\left(\sin\theta\right)^{-3}}{-3}\right]$$

$$Note:\; x=\frac{2\tan\theta}{5} \Rightarrow \frac{5}{2}x=\frac{5}{2}\cdot\frac{2\tan\theta}{5} \quad \Rightarrow \tan\theta=\frac{5x}{2}=\frac{opposite}{adjacent}$$

$Note:$

$$(hypotenuse)^2=(opposite)^2+(adjacent)^2$$

$$(hypotenuse)^2=(5x)^2+(2)^2=25x^2+4$$

$$hypotenuse=\sqrt{25x^2+4}$$

$$\sin\theta=\frac{opposite}{hypotenuse}=\frac{5x}{\sqrt{25x^2+4}}$$

$$\cos\theta=\frac{adjacent}{hypotenuse}=\frac{2}{\sqrt{25x^2+4}}$$

$$\int\frac{\sqrt{25x^2+4}}{x^4}dx= \; 31.25\left[\frac{(\sin\theta)^{-3}}{-3}\right] =31.25\left[\frac{\left(\frac{5x}{\sqrt{25x^2+4}}\right)^{-3}}{-3}\right]+C$$

Example 7: Find the indefinite integral $\int \frac{1}{x\sqrt{9x^2+1}}dx.$

Formula from Table of Integration: $\int \frac{1}{u\sqrt{u^2+1}}(du) = \frac{-1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$

Let $u^2 = 9x^2$ and $a^2 = 1 \Rightarrow u = 3x$ and $a = 1$

Also, $\frac{du}{dx} = 3$; $du = 3dx$; $\frac{1}{3}du = dx$ and $u = 3x \Rightarrow x = \frac{1}{3}u$

$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{1}{\frac{1}{3}u\sqrt{u^2+1}}\left(\frac{1}{3}du\right) = \int \frac{1}{u\sqrt{u^2+1}}du = \frac{-1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C$$

$$= \frac{-1}{1}\ln\left|\frac{1+\sqrt{9x^2+1}}{3x}\right| + C = -\ln\left|\frac{1+\sqrt{9x^2+1}}{3x}\right| + C$$

Example 8: Find the indefinite integral $\int \frac{x^2}{\left(1+x^2\right)^2}dx.$

Integral contains Form a^2+u^2.

Let $u^2 = x^2$ and $a^2 = 1 \quad \Rightarrow \quad u = x; \quad a = 1$

$Also,$ let $u = a\tan\theta \Rightarrow x = a\tan\theta \Rightarrow x = \tan\theta$

$$\frac{dx}{d\theta} = \left(\sec\theta\right)^2 \Rightarrow dx = \left(\sec\theta\right)^2 d\theta$$

$$\left(1+x^2\right)^2 = \left(1+\left(\tan\theta\right)^2\right)^2 = \left(\left(\sec\theta\right)^2\right)^2 = \left(\sec\theta\right)^4$$

$$\int \frac{x^2}{\left(1+x^2\right)^2}dx = \int \frac{\left(\tan\theta\right)^2}{\left(\sec\theta\right)^4}\left(\sec\theta\right)^2 d\theta = \int \frac{\left(\tan\theta\right)^2}{\left(\sec\theta\right)^2}d\theta \quad = \int \frac{\left(\frac{\sin^2\theta}{\cos^2\theta}\right)}{\frac{1}{\cos^2\theta}}d\theta = \int \sin^2\theta d\theta$$

$$= \int\left(\frac{1}{2}-\frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}\theta - \frac{1}{2}\left(\frac{1}{2}\sin 2\theta\right) = \frac{1}{2}\theta - \frac{1}{2}\left(\frac{1}{2}\cdot 2\sin\theta\cos\theta\right)$$

$$= \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right]$$

$$Note: \quad x = \tan\theta$$

$$\tan\theta = \frac{x}{1} = \frac{opposite}{adjacent}; \qquad \text{and} \qquad \theta = \tan^{-1} x$$

$$hypotenuse = \sqrt{(opposite)^2 + (adjacent)^2}$$

$$hypotenuse = \sqrt{(x)^2 + (1)^2} = \sqrt{x^2+1}$$

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{x}{\sqrt{x^2+1}}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{1}{\sqrt{x^2+1}}$$

$$\int \frac{x^2}{\left(1+x^2\right)^2} dx = \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right]$$

$$= \frac{1}{2}\left[\tan^{-1} x - \frac{x}{\sqrt{x^2+1}}\frac{1}{\sqrt{x^2+1}}\right] + C$$