Trigonometric Substitution

Trigonometric Identities:

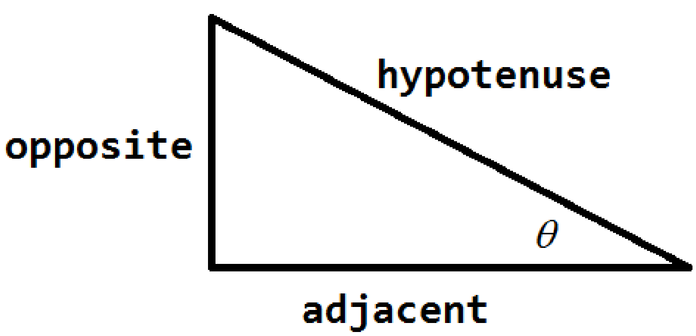
$$\sin^2 x + \cos^2 x = 1; \quad \sin^2 x = 1 - \cos^2 x; \quad \cos^2 x = 1 - \sin^2 x$$

$$\sec^2 x = 1 + \tan^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$$

For integrals containing $\sqrt{a^2 - u^2}$, let $u = a\sin\theta$

For integrals containing $\sqrt{a^2 + u^2}$, let $u = a\tan\theta$

For integrals containing $\sqrt{u^2 - a^2}$, let $u = a\sec\theta$



$$(hypotenuse)^2 = (opposite)^2 + (adjacent)^2$$

$$hypotenuse = \sqrt{(opposite)^2 + (adjacent)^2}$$

$$(opposite)^2 = (hypotenuse)^2 - (adjacent)^2$$

$$opposite = \sqrt{(hypotenuse)^2 - (adjacent)^2}$$

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Example1: Find the indefinite integral $\int \frac{\sqrt{x^2-25}}{x}dx$.

Note: Integral contains Form $u^2 - a^2$.

Let $u^2 = x^2$ and $a^2 = 25$

$\Rightarrow$ $u = x$ and $a = 5$

If we let $u = a\sec\theta$, then $x = 5\sec\theta$

Hence, $\frac{dx}{d\theta} = 5\sec\theta\tan\theta \Rightarrow dx = 5\sec\theta\tan\theta d\theta$

$$\sqrt{x^2-25} = \sqrt{(5\sec\theta)^2 - 25} = \sqrt{25(\sec\theta)^2 - 25}$$

$$= \sqrt{25\left[(\sec\theta)^2 - 1\right]} = \sqrt{25\tan^2\theta} = 5\tan\theta$$

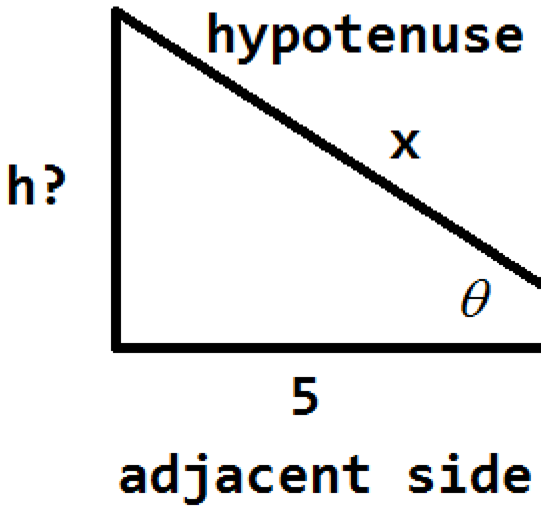
$$\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{5\tan\theta}{5\sec\theta}(5\sec\theta\tan\theta d\theta) = \int (5\tan\theta)(\tan\theta d\theta)$$

$$=5\int \tan^2\theta d\theta$$

$$= 5\left[-\theta + \tan\theta\right]$$ Using Formula #63

$$Note:\ x = 5\sec\theta$$

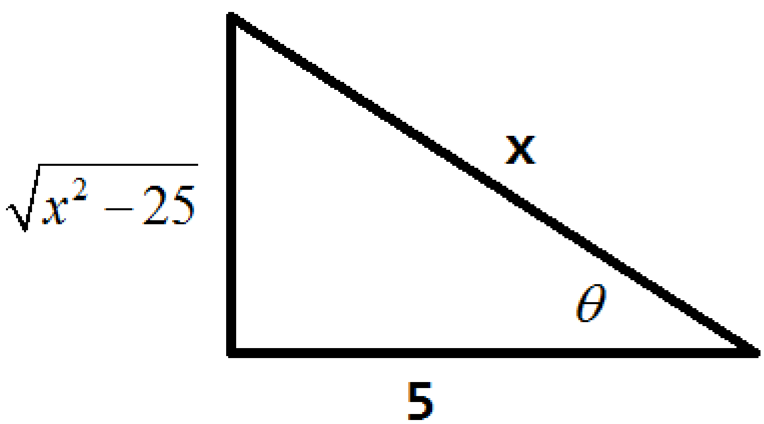
$$\sec\theta = \frac{x}{5} = \frac{hypotenuse}{adjacent}$$



$$Note:\ h^2 + 5^2 = x^2$$

$$h^2 = x^2 - 25$$

$$h = \sqrt{x^2 - 25}$$



$$Note:\left(\sqrt{x^2 - 25}\right)^2 + 5^2 = x^2$$

From triangle:

$$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{x}{5} \Rightarrow \theta = \sec^{-1}\left(\frac{x}{5}\right)$$

$$\tan\theta = \frac{opposite}{adjacent} = \frac{\sqrt{x^2-25}}{5}$$

$$\int \frac{\sqrt{x^2-25}}{x}dx = \int \frac{5\tan\theta}{5\sec\theta}\left(5\sec\theta\tan\theta d\theta\right) = \int\left(5\tan\theta\right)\left(\tan\theta d\theta\right)$$

$$=5\int \tan^2\theta d\theta$$

$$= 5\left[-\theta + \tan\theta\right]$$ Using Formula #63

$$= 5\left[-\sec^{-1}\frac{x}{5} + \frac{\sqrt{x^2-25}}{x}\right] + C$$

Example 4: Find the indefinite integral $\int \sqrt{25x^2 - 1}dx$.

Note: Integral contains Form $u^2 - a^2$.

Let $u^2 = 25x^2$ and $a^2 = 1$

$\Rightarrow \quad u = 5x$ and $a = 1$

$$\frac{du}{dx} = 5 \Rightarrow du = 5dx \Rightarrow \frac{1}{5}du = dx$$

Using Formula 26:

$$\int \sqrt{25x^2 - 1}dx = \int \sqrt{u^2 - a^2}\left(\frac{1}{5}du\right) = \frac{1}{5}\int \sqrt{u^2 - a^2}du$$

$$= \frac{1}{5}\left[\frac{1}{2}\left[u\sqrt{u^2 + a^2} + a^2 \ln\left|u + \sqrt{u^2 + a^2}\right|\right]\right] + C$$

$$= \frac{1}{10}\left[\left[5x\sqrt{(5x)^2 + 1} + \ln\left|5x + \sqrt{(5x)^2 + 1}\right|\right]\right] + C$$

Example 2: Find the indefinite integral $\int \frac{x^3}{\sqrt{x^2-25}}dx.$

Note: Integral contains Form $u^2 - a^2$.

Let $u^2 = x^2$ and $a^2 = 25$

$\Rightarrow \quad u = x$ and $a = 5$

If we let $u = a\sec\theta,$ then $x = 5\sec\theta$

Hence, $\frac{dx}{d\theta} = 5\sec\theta\tan\theta \Rightarrow dx = 5\sec\theta\tan\theta d\theta$

$$\sqrt{x^2-25} = \sqrt{(5\sec\theta)^2 - 25} = \sqrt{25(\sec\theta)^2 - 25}$$

$$= \sqrt{25\left[(\sec\theta)^2 - 1\right]} = \sqrt{25\tan^2\theta} = 5\tan\theta$$

$$\int \frac{x^3}{\sqrt{x^2-25}}dx = \int \frac{(5\sec\theta)^3}{5\tan\theta}(5\sec\theta\tan\theta d\theta)$$

$$= \int \frac{125(\sec\theta)^3}{5\tan\theta}(5\sec\theta\tan\theta d\theta)$$

$$= \int 125(\sec\theta)^3(\sec\theta d\theta)$$

$$= 125\int (\sec\theta)^4 d\theta$$

$$\int \frac{x^3}{\sqrt{x^2-25}}dx = 125\int (\sec\theta)^4 d\theta$$

Using Formula #69 with $n = 4$:

$$\int (\sec\theta)^4 d\theta = \frac{\sec^2\theta\tan\theta}{3} + \frac{2}{3}\int \sec^2\theta d\theta$$

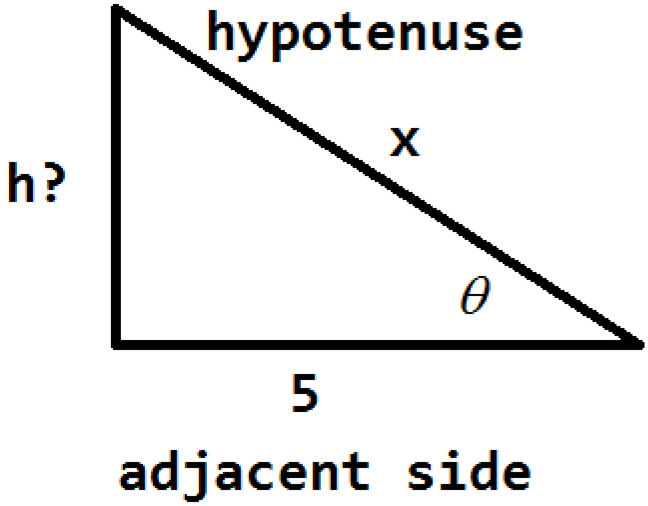
$$= \frac{\sec^2\theta\tan\theta}{3} + \frac{2}{3}[\tan\theta] \quad \text{Using Formula #65}$$

$$\int \frac{x^3}{\sqrt{x^2-25}}dx = 125\int (\sec\theta)^4 d\theta$$

$$=125\left[\frac{\sec^2\theta\tan\theta}{3} + \frac{2}{3}[\tan\theta] \right] + C$$

$$Note:\ x = 5\sec\theta$$

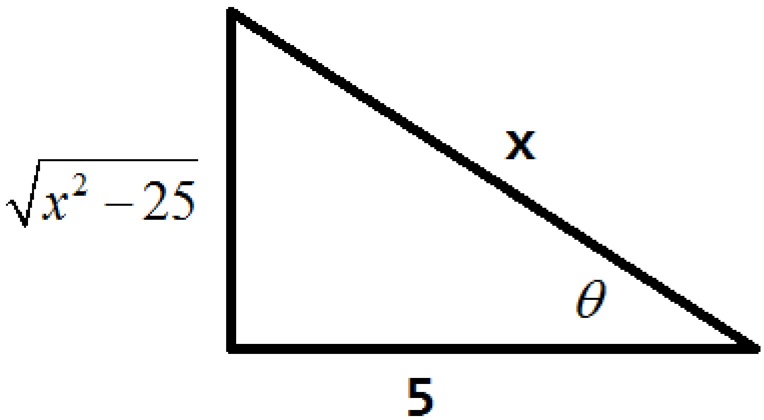
$$\sec\theta = \frac{x}{5} = \frac{hypotenuse}{adjacent}$$



$$Note: \ h^2 + 5^2 = x^2$$

$$h^2 = x^2 - 25$$

$$h = \sqrt{x^2 - 25}$$



$$Note: \left(\sqrt{x^2 - 25}\right)^2 + 5^2 = x^2$$

From triangle:

$$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{x}{5} \Rightarrow \theta = \sec^{-1}\left(\frac{x}{5}\right)$$

$$\tan\theta = \frac{hypotenuse}{opposite} = \frac{\sqrt{x^2 - 25}}{5}$$

$$\int \frac{x^3}{\sqrt{x^2-25}}dx = 125\left[\frac{\sec^2\theta\tan\theta}{3} + \frac{2}{3}\left[\tan\theta\right]\right] + C$$

$$= 125\left[\frac{\left(\frac{x}{5}\right)^2\left(\frac{\sqrt{x^2-25}}{5}\right)}{3} + \frac{2}{3}\left[\frac{\sqrt{x^2-25}}{5}\right]\right] + C$$

Example 3: Find the indefinite integral $\int \sqrt{4+x^2}\,dx.$

Note: Integral contains Form $a^2 + u^2$.

Let $u^2 = x^2$ and $a^2 = 4$

$\Rightarrow \quad u = x$ and $a = 2$

If we let $u = a\tan\theta,$ then $x = 2\tan\theta$

Hence, $\dfrac{dx}{d\theta} = 2\sec^2\theta \Rightarrow dx = 2\sec^2\theta d\theta$

$$\sqrt{4+x^2} = \sqrt{4+\left(2\tan\theta\right)^2} = \sqrt{4+4\left(\tan\theta\right)^2}$$

$$= \sqrt{4\left[1+\left(\tan\theta\right)^2\right]} = 2\sqrt{\left[1+\left(\tan\theta\right)^2\right]}$$

$$= 2\sqrt{\left(\sec\theta\right)^2} = 2\sec\theta$$

$$\int \sqrt{4+x^2}\,dx = \int 2\sec\theta\left(2\sec^2\theta d\theta\right) = 4\int \sec^3\theta d\theta$$

Using Formula #69 with $n = 3$:

$$\int \sec^3\theta d\theta = \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\int \sec\theta d\theta$$

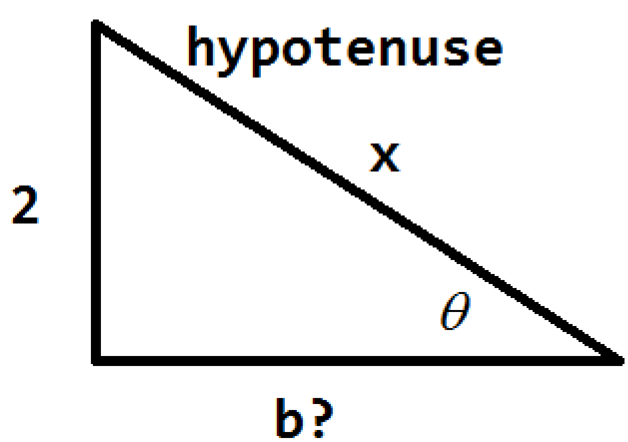
$$= \frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\left[\ln\left|\sec\theta + \tan\theta\right|\right] \quad \text{Using Formula \#61}$$

$$\int \sqrt{4+x^2}\,dx = 4\int \sec^3\theta d\theta$$

$$= 4\left[\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\left[\ln\left|\sec\theta + \tan\theta\right|\right]\right]$$

$Note:\ x = 2\tan\theta$

$$\tan\theta = \frac{x}{2} = \frac{hypotenuse}{opposite}$$



$$b^2 + 2^2 = x^2$$

$$b^2 + 4 = x^2$$

$$b^2 = x^2 - 4$$

$$b = \sqrt{x^2 - 4}$$

$$\sec\theta = \frac{hypotenuse}{adjacent} = \frac{x}{\sqrt{x^2-4}}$$

$$\int \sqrt{4+x^2}\,dx = 4\left[\frac{\sec\theta\tan\theta}{2} + \frac{1}{2}\left[\ln\left|\sec\theta + \tan\theta\right|\right]\right]$$

$$= 4\left[\frac{\left(\frac{x}{\sqrt{x^2-4}}\right)\left(\frac{x}{2}\right)}{2} + \frac{1}{2}\left[\ln\left|\frac{x}{\sqrt{x^2-4}} + \frac{x}{2}\right|\right]\right] + C$$

Example 5: Find the indefinite integral $\int \frac{x^2}{\sqrt{36-x^2}}dx.$

Note: Integral contains Form $a^2 - u^2$.

Let $u^2 = x^2$ and $a^2 = 36$

$\Rightarrow$ $u = x$ and $a = 6$

Let $u = a\sin\theta \quad \Rightarrow x = a\sin\theta = 6\sin\theta$

$$\frac{dx}{d\theta} = 6\cos\theta \Rightarrow dx = 6\cos\theta d\theta$$

$$\sqrt{36-x^2} = \sqrt{36-(6\sin\theta)^2} = \sqrt{36-36(\sin\theta)^2}$$

$$= \sqrt{36\left[1-(\sin\theta)^2\right]} = \sqrt{36\cos^2\theta} = 6\cos\theta$$

$$\int \frac{x^2}{\sqrt{36-x^2}}dx = \int \frac{(6\sin\theta)^2}{6\cos\theta}6\cos\theta d\theta = 36\int (\sin\theta)^2\, d\theta$$

$$\int \frac{x^2}{\sqrt{36-x^2}}dx = \int \frac{(6\sin\theta)^2}{6\cos\theta}6\cos\theta d\theta = 36\int (\sin\theta)^2\, d\theta$$

Using Formula 48: $\int (\sin\theta)^2\, d\theta = \frac{1}{2}[\theta - \sin\theta\cos\theta]$

$$\int \frac{x^2}{\sqrt{36-x^2}}dx = \int \frac{(6\sin\theta)^2}{6\cos\theta}6\cos\theta d\theta = 36\int (\sin\theta)^2\, d\theta$$

$$= 36\left[\frac{1}{2}[\theta - \sin\theta\cos\theta]\right] = 18[\theta - \sin\theta\cos\theta]$$

$Note: \;\; x = 6\sin\theta$

$\sin\theta = \frac{x}{6} = \frac{opposite}{hypotenuse};$ and $\theta = \sin^{-1}\left(\frac{x}{6}\right)$

$$adjacent = \sqrt{(hypotenuse)^2 - (opposite)^2} = \sqrt{36-x^2}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{\sqrt{36-x^2}}{6}$$

$$\int \frac{x^2}{\sqrt{36-x^2}}dx = 18[\theta - \sin\theta\cos\theta]$$

$$= 18\left[\sin^{-1}\left(\frac{x}{6}\right) - \left(\frac{x}{6}\right)\left(\frac{\sqrt{36-x^2}}{6}\right)\right] + C$$

Example 6: Find the indefinite integral $\int \frac{\sqrt{25x^2+4}}{x^4}dx.$

Note: Integral contains Form a^2+u^2.

Let $u^2 = 25x^2$ and $a^2 = 4$

$\Rightarrow \quad u = 5x$ and $a = 2$

Let $u = a\tan\theta \quad \Rightarrow 5x = a\tan\theta \Rightarrow x = \frac{2\tan\theta}{5}$

$$\frac{dx}{d\theta} = \frac{2}{5}\sec^2\theta \Rightarrow dx = \frac{2}{5}\sec^2\theta d\theta$$

$$\sqrt{25x^2+4} = \sqrt{25\left(\frac{2\tan\theta}{5}\right)^2+4} = \sqrt{25\cdot\frac{4}{25}(\tan\theta)^2+4}$$

$$= \sqrt{4(\tan\theta)^2+4} = \sqrt{4\left[(\tan\theta)^2+1\right]} = \sqrt{4\left[(\sec\theta)^2\right]} = 2\sec\theta$$

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = \int \frac{2\sec\theta}{\left(\frac{2\tan\theta}{5}\right)^4}\frac{2}{5}\sec^2\theta d\theta = \frac{2\cdot\frac{2}{5}}{\left(\frac{2}{5}\right)^4}\int \frac{\sec^3\theta}{(\tan\theta)^4}d\theta$$

$$= 31.25\int \frac{\left(\frac{1}{\cos^3\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)^4}d\theta = 31.25\cos\theta(\sin\theta)^{-4}\,d\theta$$

$$Note: \frac{\left(\frac{1}{\cos^3\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)^4}\frac{\left(\frac{1}{\cos^3\theta}\right)}{\frac{(\sin\theta)^4}{(\cos\theta)^4}} = \left(\frac{1}{\cos^3\theta}\right)\frac{(\cos\theta)^4}{(\sin\theta)^4} = \frac{\cos\theta}{(\sin\theta)^4} = \cos\theta(\sin\theta)^{-4}$$

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = 31.25\int \cos\theta(\sin\theta)^{-4}\,d\theta$$

$$\text{Let } u = \sin\theta \;\; ; \quad \frac{du}{d\theta} = \cos\theta; \quad du = \cos\theta d\theta$$

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = 31.25\int \cos\theta(\sin\theta)^{-4}\,d\theta$$

$$=31.25\int (\sin\theta)^{-4}\cos\theta d\theta = 31.25\int (u)^{-4}\,d\theta$$

$$=31.25\left[\frac{u^{-3}}{-3}\right] = 31.25\left[\frac{(\sin\theta)^{-3}}{-3}\right]$$

$$=31.25\left[\frac{u^{-3}}{-3}\right] = 31.25\left[\frac{(\sin\theta)^{-3}}{-3}\right]$$

$$Note:\ x=\frac{2\tan\theta}{5}\Rightarrow\frac{5}{2}x=\frac{5}{2}\cdot\frac{2\tan\theta}{5}$$

$$\Rightarrow\tan\theta=\frac{5x}{2}=\frac{opposite}{adjacent}$$

$Note:$

$$(hypotenuse)^2=(opposite)^2+(adjacent)^2$$

$$(hypotenuse)^2=(5x)^2+(2)^2=25x^2+4$$

$$hypotenuse=\sqrt{25x^2+4}$$

$$\sin\theta=\frac{opposite}{hypotenuse}=\frac{5x}{\sqrt{25x^2+4}}$$

$$\cos\theta=\frac{adjacent}{hypotenuse}=\frac{2}{\sqrt{25x^2+4}}$$

$$\int \frac{\sqrt{25x^2+4}}{x^4}dx = 31.25\left[\frac{(\sin\theta)^{-3}}{-3}\right]$$

$$=31.25\left[\frac{\left(\frac{5x}{\sqrt{25x^2+4}}\right)^{-3}}{-3}\right]+C$$

Example 7: Find the indefinite integral $\int \frac{1}{x\sqrt{9x^2+1}}dx.$

Formula 32: $\int \frac{1}{u\sqrt{u^2+a^2}}du.$

Let $u^2 = 9x^2$ and $a^2 = 1$

$\Rightarrow u = 3x$ and $a = 1$

Also, $\frac{du}{dx} = 3;\ \ du = 3dx;\ \ \frac{1}{3}du = dx$

and $u = 3x \ \Rightarrow \ x = \frac{1}{3}u$

$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{1}{\frac{1}{3}u\sqrt{u^2+1}}\left(\frac{1}{3}du\right) = \int \frac{1}{u\sqrt{u^2+1}}(du)$$

$$\int \frac{1}{x\sqrt{9x^2+1}}dx = \int \frac{1}{\frac{1}{3}u\sqrt{u^2+1}}\left(\frac{1}{3}du\right)$$

$$= \int \frac{1}{u\sqrt{u^2+1}}(du) = \frac{-1}{a}\ln\left|\frac{a+\sqrt{u^2+a^2}}{u}\right| + C \quad \text{Formula \#32}$$

$$= \frac{-1}{1}\ln\left|\frac{1+\sqrt{9x^2+1}}{3x}\right| + C$$

$$= -\ln\left|\frac{1+\sqrt{9x^2+1}}{3x}\right| + C$$

Example 8: Find the indefinite integral $\int \frac{x^2}{\left(1+x^2\right)^2}dx.$

Integral contains Form a^2+u^2.

Let $u^2 = x^2$ and $a^2 = 1$

$\Rightarrow u = x; \quad a = 1$

Also, let $u = a\tan\theta \Rightarrow x = a\tan\theta \Rightarrow x = \tan\theta$

$$\frac{dx}{d\theta} = \left(\sec\theta\right)^2 \Rightarrow dx = \left(\sec\theta\right)^2 d\theta$$

$$\left(1+x^2\right)^2 = \left(1+\left(\tan\theta\right)^2\right)^2 = \left(\left(\sec\theta\right)^2\right)^2 = \left(\sec\theta\right)^4$$

$$\int \frac{x^2}{\left(1+x^2\right)^2}dx = \int \frac{\left(\tan\theta\right)^2}{\left(\sec\theta\right)^4}\left(\sec\theta\right)^2 d\theta$$

$$\int \frac{x^2}{\left(1+x^2\right)^2}dx = \int \frac{\left(\tan\theta\right)^2}{\left(\sec\theta\right)^4}\left(\sec\theta\right)^2 d\theta = \int \frac{\left(\tan\theta\right)^2}{\left(\sec\theta\right)^2}d\theta$$

$$= \int \frac{\left(\dfrac{\sin^2\theta}{\cos^2\theta}\right)}{\dfrac{1}{\cos^2\theta}}d\theta = \int \sin^2\theta d\theta$$

$$= \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right]$$ Formula #48

$$Note: \quad x = \tan\theta$$

$$\tan\theta = \frac{x}{1} = \frac{opposite}{adjacent}; \qquad \text{and} \qquad \theta = \tan^{-1} x$$

$$hypotenuse = \sqrt{(opposite)^2 + (adjacent)^2}$$

$$hypotenuse = \sqrt{(x)^2 + (1)^2} = \sqrt{x^2 + 1}$$

$$\sin\theta = \frac{opposite}{hypotenuse} = \frac{x}{\sqrt{x^2+1}}$$

$$\cos\theta = \frac{adjacent}{hypotenuse} = \frac{1}{\sqrt{x^2+1}}$$

$$\int \frac{x^2}{\left(1+x^2\right)^2} dx = \frac{1}{2}\left[\theta - \sin\theta\cos\theta\right]$$

$$= \frac{1}{2}\left[\tan^{-1} x - \frac{x}{\sqrt{x^2+1}} \frac{1}{\sqrt{x^2+1}}\right] + C$$