Partial Fractions

Review:

$$\int \frac{1}{x}\,dx = \ln|x| + C; \quad \int \frac{1}{x-a}\,dx = \ln|x-a| + C; \int \frac{1}{bx-a}\,dx = \frac{1}{b}\ln|x-a| + C$$

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm cb}{bd}$$

Example 1: Find the indefinite integral $\int \frac{5}{x^2-4}\,dx$

Decompose $\frac{5}{x^2-4}$:

$$\frac{5}{x^2-4} = \frac{5}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)}$$

$$\frac{5}{(x+2)(x-2)} = \frac{A(x-2)+B(x+2)}{(x+2)(x-2)}$$

$$5 = A(x-2) + (x+2)B$$

Let $x = 2$: $5 = A(x-2) + (x+2)B \Leftrightarrow 5 = A(0) + (4)B \Leftrightarrow B = 5/4$

Let $x = -2$: $5 = A(x-2) + (x+2)B \Leftrightarrow 5 = A(-4) + (0)B \Leftrightarrow A = -5/4$

$$\frac{5}{x^2-4} = \frac{5}{(x+2)(x-2)} = \frac{A}{(x+2)} + \frac{B}{(x-2)}$$

$$\frac{5}{x^2-4} = \frac{5/4}{(x+2)} + \frac{-5/4}{(x-2)}$$

$$\int \frac{5}{x^2-4}dx = \int \frac{5/4}{(x+2)}dx + \int \frac{-5/4}{(x-2)}dx$$

$$\int \frac{5}{x^2-4}dx = \frac{5}{4}\int \frac{1}{(x+2)}dx + \frac{-5}{4}\int \frac{1}{(x-2)}dx$$

$$\int \frac{5}{x^2-4}dx = \frac{5}{4}\ln|x+2| - \frac{5}{4}\ln|x-2| + C$$

Example 2: Find the indefinite integral $\int \frac{5x}{x^2-4x+3}\,dx$

Decompose $\frac{5x}{x^2-4x+3}$:

$$\frac{5x}{x^2-4x+3}=\frac{5x}{(x-3)(x-1)}=\frac{A}{(x-3)}+\frac{B}{(x-1)}$$

$$\frac{5x}{(x-3)(x-1)}=\frac{A(x-1)+B(x-3)}{(x-3)(x-1)}$$

Let $x=1$: $\quad 5x=A(x-1)+B(x-3)\Leftrightarrow 5=A(0)+B(-2)\Leftrightarrow B=-5/2$

Let $x=3$: $\quad 5x=A(x-1)+B(x-3)\Leftrightarrow 15=A(2)+B(0)\Leftrightarrow B=15/2$

$$\frac{5x}{x^2-4x+3}=\frac{5x}{(x-3)(x-1)}=\frac{A}{(x-3)}+\frac{B}{(x-1)}$$

$$\frac{5x}{x^2-4x+3}=\frac{5x}{(x-3)(x-1)}=\frac{15/2}{(x-3)}+\frac{-5/2}{(x-1)}$$

$$\int\frac{5x}{x^2-4x+3}dx=\int\frac{15/2}{(x-3)}dx+\int\frac{-5/2}{(x-1)}dx$$

$$\int\frac{5x}{x^2-4x+3}dx=\frac{15}{2}\int\frac{1}{(x-3)}dx+\frac{-5}{2}\int\frac{1}{(x-1)}dx$$

$$\int\frac{5x}{x^2-4x+3}dx=\frac{15}{2}\ln|x-3|-\frac{5}{2}\ln|x-1|+C$$

Example 3: Find the indefinite integral $\int \frac{4x^3 + 10x^2 - x - 2}{x^2+3x+2}dx$

If degree of numerator polynomial is greater than or equal to

degree of denominator polynomial, use long division to divide first.

Using long division to divide $\frac{4x^3 + 10x^2 - x - 2}{x^2+3x+2}$:

$$\frac{4x^3 + 10x^2 - x - 2}{x^2+3x+2} = 4x - 2 + \frac{(-3x+2)}{(x^2+3x+2)}$$

Now decompose $\frac{(-3x+2)}{(x^2+3x+2)}$:

$$\frac{-3x+2}{x^2+3x+2} = \frac{-3x+2}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)}$$

$$\frac{-3x+2}{(x+2)(x+1)} = \frac{A(x+1)+B(x+2)}{(x+2)(x+1)}$$

$$-3x+2 = A(x+1)+B(x+2)$$

Let $x=-1: -3x+2 = A(x+1)+B(x+2) \Leftrightarrow 5 = A(0)+B(1) \Leftrightarrow B=5$

Let $x=-2: -3x+2 = A(x+1)+B(x+2) \Leftrightarrow 8 = A(-1)+B(0) \Leftrightarrow A=-8$

$$\frac{-3x+2}{(x+2)(x+1)} = \frac{A}{(x+2)} + \frac{B}{(x+1)} = \frac{-8}{(x+2)} + \frac{5}{(x+1)}$$

$$\int \frac{4x^3 + 10x^2 - x - 2}{x^2+3x+2} = \int 4x dx - \int 2dx + \int \frac{(-3x+2)}{(x^2+3x+2)}dx$$

$$= 4\left(\frac{x^2}{2}\right) - 2x + \int\left[\frac{-8}{(x+2)} + \frac{5}{(x+1)}\right]dx$$

$$= 2x^2 - 2x + -8\int\left[\frac{1}{(x+2)}\right]dx + 5\int\left[\frac{1}{(x+1)}\right]dx$$

$$= 2x^2 - 2x + -8\ln|x+2| + 5\ln|x+1| + C$$

Example 4: Find the indefinite integral $\int \frac{x-1}{x^3+x^2}\,dx$

Now decompose $\frac{x-1}{x^3+x^2} = \frac{x-1}{(x)^2(x+1)}$:

Note: $(x)^2$ is a repeated factor.

$$\frac{x-1}{x^3+x^2} = \frac{x-1}{(x)^2(x+1)} = \frac{A}{(x)} + \frac{B}{(x)^2} + \frac{C}{(x+1)}$$

$$\frac{A}{(x)} + \frac{B}{(x)^2} = \frac{Ax^2+Bx}{(x)x^2} = \frac{Ax+B}{x^2}$$

$$\frac{A}{(x)} + \frac{B}{(x)^2} + \frac{C}{(x+1)} = \frac{Ax+B}{x^2} + \frac{C}{(x+1)} = \frac{(Ax+B)(x+1)+Cx^2}{x^2(x+1)}$$

$$x-1 = (Ax+B)(x+1)+Cx^2$$

Let $x=-1 \Rightarrow x-1 = (Ax+B)(x+1)+Cx^2 \Rightarrow (-1)-1 = (A(-1)+B)(0)+C(-1)^2 \Rightarrow$ C = -2

Let $x=0 \Rightarrow x-1 = (Ax+B)(x+1)+Cx^2 \Rightarrow (0)-1 = (A(0)+B)(1)+C(0)^2 \Rightarrow$ -1 = B

Let $x=1 \Rightarrow x-1 = (Ax+B)(x+1)+Cx^2 \Rightarrow 0 = (A+B)(2)+C$

$$2A+2B+C = 2 \Rightarrow A = 2$$

$$\int \frac{x-1}{x^3+x^2}dx = \int \frac{x-1}{(x)^2(x+1)}dx = \int \frac{A}{(x)}dx + \int \frac{B}{(x)^2}dx + \int \frac{C}{(x+1)}dx$$

$$= \int \frac{2}{(x)}dx + \int \frac{-1}{(x)^2}dx + \int \frac{-2}{(x+1)}dx$$

$$= 2\ln|x| + \frac{1}{x} - 2\ln|x+1| + C$$

Example 5: Find the indefinite integral $\int \frac{x^2 - x + 9}{(x^2+9)^2}\,dx$

Now decompose $\frac{x^2 - x + 9}{(x^2+9)^2} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{\left(x^2+9\right)^2}$:

$$\frac{x^2 - x + 9}{(x^2+9)^2} = \frac{Ax+B}{x^2+9} + \frac{Cx+D}{\left(x^2+9\right)^2} = \frac{(Ax+B)\left(x^2+9\right)^2 + (Cx+D)\left(x^2+9\right)}{\left(x^2+9\right)\left(x^2+9\right)^2}$$

$$= \frac{(Ax+B)\left(x^2+9\right) + (Cx+D)}{\left(x^2+9\right)^2}$$

$$x^2 - x + 9 = (Ax+B)\left(x^2+9\right) + (Cx+D)$$

$$x^2 - x + 9 = Ax^3 + Bx^2 + (9A+C)x + (9B+D)$$

$$A = 0; B = 1; D = 0; C = -1$$

$$\int \frac{x^2 - x + 9}{(x^2+9)^2}\,dx = \int \frac{Ax+B}{x^2+9}\,dx + \int \frac{Cx+D}{\left(x^2+9\right)^2}dx$$

$$= \int \frac{0x+1}{x^2+9}\,dx + \int \frac{-1x+0}{\left(x^2+9\right)^2}dx$$

$$= \int \frac{1}{x^2+9}\,dx - \int \frac{x}{\left(x^2+9\right)^2}dx = \frac{1}{3}\arctan\left(\frac{x}{3}\right) + \frac{1}{2\left(x^2+9\right)} + C$$

Note:

For $\int \frac{1}{x^2+9}\,dx,$ use Formula #23.

For $\int \frac{x}{\left(x^2+9\right)^2}dx$: Let $u = x^2+9 \Rightarrow du = 2xdx \Rightarrow \frac{1}{2}du = xdx$

$$\int \frac{x}{\left(x^2+9\right)^2}dx = \int \frac{1}{\left(x^2+9\right)^2}xdx = \int \frac{1}{(u)^2}\frac{1}{2}du = \frac{1}{2}\int u^{-2}du = \frac{1}{2}\left(\frac{u^{-1}}{-1}\right) = -\frac{1}{2}\left(\frac{1}{u}\right) = -\frac{1}{2}\left(\frac{1}{x^2+9}\right)$$

Example 6: Find the indefinite integral $\int \frac{x}{16x^4-1}dx$

Now decompose $\frac{x}{16x^4-1} = \frac{x}{(2x-1)(2x+1)(4x^2+1)}$:

$$\frac{x}{16x^4-1} = \frac{x}{(2x-1)(2x+1)(4x^2-1)} = \frac{A}{(2x-1)} + \frac{B}{(2x+1)} + \frac{Cx+D}{(4x^2+1)}$$

$$= \frac{A(2x+1)(4x^2-1) + B(2x-1)(4x^2-1) + C(2x-1)(2x+1)}{(2x-1)(2x+1)(4x^2+1)}$$

$$x = A(2x+1)(4x^2-1) + B(2x-1)(4x^2+1) + C(2x-1)(2x+1)$$

Let $x = \frac{1}{2}$: $\frac{1}{2} = A(2(\tfrac{1}{2})+1)(4(\tfrac{1}{2})^2-1) + B(2(\tfrac{1}{2})-1)(4(\tfrac{1}{2})^2+1) + C(2(\tfrac{1}{2})-1)(2(\tfrac{1}{2})+1)$

$$A = \frac{1}{8}$$

Let $x = -\frac{1}{2}$: $-\frac{1}{2} = A(2(-\tfrac{1}{2})+1)(4(-\tfrac{1}{2})^2-1) + B(2(-\tfrac{1}{2})-1)(4(-\tfrac{1}{2})^2+1) + C(2(-\tfrac{1}{2})-1)(2(-\tfrac{1}{2})+1)$

$$B = \frac{1}{8}$$

Let $x = 0$: $-\frac{1}{2} = A(2(0)+1)(4(0)^2-1) + B(2(0)-1)(4(0)^2+1) + C(2(0)-1)(2(0)+1)$

$$A - B - D = 0 \Rightarrow D = 0$$

Let $x = 1$: $-\frac{1}{2} = A(2(1)+1)(4(1)^2-1) + B(2(1)-1)(4(1)^2+1) + C(2(1)-1)(2(1)+1)$

$$C = -\frac{1}{2}$$

$$\int \frac{x}{16x^4-1} = \int \frac{A}{(2x-1)}dx + \int \frac{B}{(2x+1)}dx + \int \frac{Cx+D}{(4x^2+1)}dx$$

$$= \int \frac{1/8}{(2x-1)}dx + \int \frac{1/8}{(2x+1)}dx + \int \frac{-(1/2)x+0}{(4x^2+1)}dx$$

$$= \frac{1}{8}\left(\frac{1}{2}\ln|2x-1|\right) + \frac{1}{8}\left(\frac{1}{2}\ln|2x+1|\right) - \frac{1}{2}\left(\frac{1}{8}\ln|4x^2+1|\right) + C$$

Example 7: Find the indefinite integral $\int \frac{4x^2}{x^3+x^2-x-1}\,dx$

Note: $x^3+x^2-x-1 = x^2(x+1)-1(x+1) = (x+1)(x^2-1)$

Now decompose $\frac{4x^2}{x^3+x^2-x-1} = \frac{4x^2}{(x-1)(x+1)^2} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$$= \frac{A(x+1)^2 + B(x+1)(x-1) + C(x-1)}{(x-1)(x+1)^2}$$

$4x^2 = A(x+1)^2 + B(x+1)(x-1) + C(x-1)$

Let $x=-1$: $4(-1)^2 = A(0)^2 + B(0)((-1)-1) + C((-1)-1) \Leftrightarrow C = -2$

Let $x=1$: $4(1)^2 = A(2)^2 + B(2)(0) + C(0) \Leftrightarrow A = 1$

Let $x=0: 4(0)^2 = A(1)^2 + B(1)(-1) + C(-1) \Leftrightarrow 0 = A-B-C \Leftrightarrow B = 3$

$$\int \frac{4x^2}{x^3+x^2-x-1}\,dx = \int \frac{1}{x-1}dx + \int \frac{3}{x+1}\,dx + \int \frac{-2}{(x+1)^2}\,dx$$

$$= \ln|x-1| + 3\ln|x+1| + \frac{2}{(x+1)} + C$$

Note: For $\int \frac{-2}{(x+1)^2}\,dx,$

let $u = x+1 \;\; \Rightarrow du = dx$

$$\int \frac{-2}{(x+1)^2}\,dx = \int \frac{-2}{(u)^2}\,du = -2\int u^{-2}du = -2\frac{u^{-1}}{-1} = 2u^{-1} = \frac{2}{u} = \frac{2}{x+1}$$