L'Hopital's Rule

1) Find $\lim\limits_{x\to\infty}\dfrac{4x+3}{5x^2+1}$.

Since $\lim\limits_{x\to\infty}\dfrac{4x+3}{5x^2+1}=\dfrac{\lim\limits_{x\to\infty}(4x+3)}{\lim\limits_{x\to\infty}(5x^2+1)}=\dfrac{\infty}{\infty}$, we use L'Hopital's Rule

$$\lim_{x\to\infty}\frac{4x+3}{5x^2+1}=\frac{\lim\limits_{x\to\infty}(4x+3)}{\lim\limits_{x\to\infty}(5x^2+1)}=\frac{\lim\limits_{x\to\infty}(4)}{\lim\limits_{x\to\infty}(10x)}=\frac{4}{\infty}=0$$

2) Find $\lim\limits_{x\to-4}\dfrac{2x^2+13x+20}{x+4}$.

Since $\lim\limits_{x\to-4}\dfrac{2x^2+13x+20}{x+4}=\dfrac{\lim\limits_{x\to-4}(2x^2+13x+20)}{\lim\limits_{x\to-4}(x+4)}=\dfrac{\infty}{\infty}$, we use L'Hopital's Rule

$$\lim_{x\to-4}\frac{2x^2+13x+20}{x+4}=\frac{\lim\limits_{x\to-4}(2x^2+13x+20)}{\lim\limits_{x\to-4}(x+4)}=\frac{\lim\limits_{x\to-4}(4x+13)}{\lim\limits_{x\to-4}(1)}=\frac{-3}{1}=-3$$

3) Find $\lim\limits_{x\to 5^-}\dfrac{\sqrt{25-x^2}}{x-5}$.

Since $\lim\limits_{x\to 5^-}\dfrac{\sqrt{25-x^2}}{x-5}=\dfrac{\lim\limits_{x\to 5^-}\sqrt{25-x^2}}{\lim\limits_{x\to 5^-}(x-5)}=\dfrac{0}{0}$, we use L'Hopital's Rule

$$\lim_{x\to 5^-}\frac{\sqrt{25-x^2}}{x-5}=\frac{\lim\limits_{x\to 5^-}\sqrt{25-x^2}}{\lim\limits_{x\to 5^-}(x-5)}=\frac{\lim\limits_{x\to 5^-}\left[\dfrac{-x}{\left(25-x^2\right)^{1/2}}\right]}{\lim\limits_{x\to 5^-}(1)}=\frac{\dfrac{-5}{(0)^{1/2}}}{1}=\frac{-5}{(0)^{1/2}}=\infty$$

$$\boxed{\begin{aligned}&\textit{Note}:\ \ D_x\left(\sqrt{25-x^2}\right)=D_x\left[\left(25-x^2\right)^{1/2}\right]=\left[\frac{1}{2}\left(25-x^2\right)^{-1/2}(-2x)\right]\\&\qquad\qquad=\left[-x\left(25-x^2\right)^{-1/2}\right]=\frac{-x}{\left(25-x^2\right)^{1/2}}\end{aligned}}$$

4) Find $\lim\limits_{x\to 1}\dfrac{\ln x^3}{x^2-1}$.

Recall: $\ln(1)=0$; $\quad \ln x^p = p\cdot\ln x$

Since $\lim\limits_{x\to 1}\dfrac{\ln x^3}{x^2-1}=\lim\limits_{x\to 1}\dfrac{\ln x^3}{x^2-1}=\lim\limits_{x\to 1}\dfrac{3\ln x}{x^2-1}=\dfrac{0}{0}$, we use L'Hopital's Rule

$$\lim_{x\to 1}\frac{\ln x^3}{x^2-1}=\lim_{x\to 1}\frac{\ln x^3}{x^2-1}=\lim_{x\to 1}\frac{3\ln x}{x^2-1}=\lim_{x\to 1}\frac{3(1/x)}{2x}=\frac{3}{2}$$

$$\boxed{\textit{Note}:\ \ D_x(\ln x)=\frac{1}{x}}$$

5) Find $\lim_{x \to 0} \frac{\sin 6x}{4x}$.

Since $\lim_{x \to 0} \frac{\sin 6x}{4x} = \frac{\lim_{x \to 0} (\sin 6x)}{\lim_{x \to 0} (4x)} = \frac{0}{0}$, we use L'Hopital's Rule.

$$\lim_{x \to 0} \frac{\sin 6x}{4x} = \frac{\lim_{x \to 0} (\sin 6x)}{\lim_{x \to 0} (4x)} = \frac{\lim_{x \to 0} (6 \cos 6x)}{\lim_{x \to 0} (4)} = \frac{6 \cdot 1}{4} = \frac{3}{2}$$

$$\boxed{Note: \ D_x (\sin 6x) = \cos 6x \cdot 6 = 6 \cos 6x}$$

6) Find $\lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}}$.

Since $\lim\limits_{x\to\infty} \dfrac{x^3}{e^{x^2}} = \dfrac{\lim\limits_{x\to\infty}\left(x^3\right)}{\lim\limits_{x\to\infty}\left(e^{x^2}\right)} = \dfrac{\infty}{\infty}$, we use L'Hopital's Rule.

$$\boxed{Note:\ \ D_x\left(e^{x^2}\right) = e^{x^2}\cdot D_x\left(x^2\right) = e^{x^2}\cdot(2x)}$$

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \frac{\lim\limits_{x\to\infty}\left(x^3\right)}{\lim\limits_{x\to\infty}\left(e^{x^2}\right)} = \frac{\lim\limits_{x\to\infty}\left(3x^2\right)}{\lim\limits_{x\to\infty}\left(e^{x^2}\cdot(2x)\right)} = \frac{\infty}{\infty},\ \text{we use L'Hopital's Rule again.}$$

$$\boxed{\begin{aligned} Note:\ \ D_x\left(e^{x^2}\cdot(2x)\right) &= e^{x^2}\cdot D_x(2x) + 2x\cdot D_x\left(e^{x^2}\right) \quad \text{Product Rule for derivative} \\ &= e^{x^2}\cdot(2) + 2x\cdot\left(e^{x^2}\cdot(2x)\right) \\ &= 2e^{x^2} + 4x^2\cdot\left(e^{x^2}\right) \end{aligned}}$$

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \frac{\lim\limits_{x\to 0}\left(x^3\right)}{\lim\limits_{x\to 0}\left(e^{x^2}\right)} = \frac{\lim\limits_{x\to\infty}\left(3x^2\right)}{\lim\limits_{x\to\infty}\left(e^{x^2}\cdot(2x)\right)}$$

$$= \frac{\lim\limits_{x\to\infty}(6x)}{\lim\limits_{x\to\infty}\left[2e^{x^2} + 4x^2\cdot\left(e^{x^2}\right)\right]} = \frac{\infty}{\infty},\ \text{we use L'Hopital's Rule again}$$

$$\boxed{Note:\ \ D_x\left(2e^{x^2}\right) = 4xe^{x^2};\qquad D_x\left(2e^{x^2} + 4x^2\cdot\left(e^{x^2}\right)\right) = 8x^3e^{x^2} + 8xe^{x^2}}$$

$$\lim_{x\to\infty} \frac{x^3}{e^{x^2}} = \frac{\lim\limits_{x\to 0}\left(x^3\right)}{\lim\limits_{x\to 0}\left(e^{x^2}\right)} = \frac{\lim\limits_{x\to\infty}\left(3x^2\right)}{\lim\limits_{x\to\infty}\left(e^{x^2}\cdot(2x)\right)}$$

$$= \frac{\lim\limits_{x\to\infty}(6x)}{\lim\limits_{x\to\infty}\left[2e^{x^2} + 4x^2\cdot\left(e^{x^2}\right)\right]} == \frac{\lim\limits_{x\to\infty}(6)}{\lim\limits_{x\to\infty}\left[8x^3e^{x^2} + 8xe^{x^2}\right]} = \frac{6}{\infty} = 0$$

7) Find $\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}$.

Note: If $x\to\infty$, $\sqrt{x^2}=\lvert x\rvert=x$; If $x\to-\infty$, $\sqrt{x^2}=\lvert x\rvert=-x$

For this problem, L'Hopital's Rule is not useful.

$$\lim_{x\to\infty}\frac{x}{\sqrt{x^2+1}}=\lim_{x\to\infty}\frac{x/\sqrt{x^2}}{\sqrt{x^2+1}/\sqrt{x^2}}$$ divide numerator and denominator by $\sqrt{x^2}$

$$=\lim_{x\to\infty}\frac{x/x}{\sqrt{\left(x^2+1\right)/x^2}}=\lim_{x\to\infty}\frac{1}{\sqrt{1+1/x^2}}=\frac{\lim_{x\to\infty}1}{\lim_{x\to\infty}\sqrt{1+1/x^2}}=\frac{1}{\sqrt{1+0}}=1$$

1) Find $\lim_{x\to\infty}\frac{\cos x}{x}$.

For this problem, L'Hopital's Rule is not useful.

Note: $-1\le\cos x\le 1$

$$\frac{-1}{x}\le\frac{\cos x}{x}\le\frac{1}{x}$$

$$\lim_{x\to\infty}\frac{-1}{x}\le\lim_{x\to\infty}\frac{\cos x}{x}\le\lim_{x\to\infty}\frac{1}{x}$$

Because $\lim_{x\to\infty}\frac{-1}{x}=0$ and $\lim_{x\to\infty}\frac{1}{x}=0$, $\lim_{x\to\infty}\frac{\cos x}{x}=0$ by Squeeze Theorem

8) Find $\lim_{x\to 0^+} x^{1/x}$.

Let $y = \lim_{x\to 0^+} x^{1/x}$

$$\ln(y) = \ln\left(\lim_{x\to 0^+} x^{1/x}\right)$$

$$\ln(y) = \lim_{x\to 0^+} \left(\ln x^{1/x}\right)$$

$$\ln(y) = \lim_{x\to 0^+} \left(\frac{1}{x}\cdot \ln x\right) \qquad \text{Note:} \quad \ln x^p = p\cdot \ln x$$

$$\ln(y) = \lim_{x\to 0^+} \left(\frac{1}{x}\right)\cdot \lim_{x\to 0^+} (\ln x) = (\infty)(-\infty) = -\infty$$

Note: As $x \to 0$ from the right, $\ln x \to -\infty$

$$\ln(y) = -\infty$$

$\log_e(y) = -\infty$ Note: ln is the same $\log_e$

$y = e^{-\infty}$ Note: Property of log: $\log_b y = x \Leftrightarrow y = b^x$

$$y = 0$$

$$\lim_{x\to 0^+} \left(\ln x^{1/x}\right) = 0$$

9) Find $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Let $y=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$$\ln(y)=\ln\left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right)$$

$$\ln(y)=\lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^x\right)$$

$$\ln(y)=\lim_{x\to\infty}\left(x\cdot\ln\left(1+\frac{1}{x}\right)\right)$$ Note: $\ln x^p = p\cdot\ln x$

$$\boxed{Note: x\cdot\ln\left(1+\frac{1}{x}\right)=\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}}$$

$$\ln(y)=\frac{\lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)\right)}{\lim_{x\to\infty}(1/x)}=\frac{\ln(1)}{0}=\frac{0}{0}$$ we can use L'Hopital's Rule

$$\ln(y)=\frac{\lim_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)\right)}{\lim_{x\to\infty}(1/x)}=\frac{\lim_{x\to\infty}\left(\frac{1}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right)\right)}{\lim_{x\to\infty}\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}\frac{\left(\frac{1}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right)\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)}=\frac{1}{1}=1$$

$$\ln(y)=1$$

$\log_e(y)=1$ Note: ln is the same $\log_e$

$y=e^1=e$ Note: Property of log: $\log_b y=x \Leftrightarrow y=b^x$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$