L'Hopital's Rule

1) Find $\lim_{x\to\infty}\frac{4x+3}{5x^2+1}$.

Since $\lim_{x\to\infty}\frac{4x+3}{5x^2+1}=\frac{\lim_{x\to\infty}(4x+3)}{\lim_{x\to\infty}(5x^2+1)}=\frac{\infty}{\infty}$, we use L'Hopital's Rule

$$\lim_{x\to\infty}\frac{4x+3}{5x^2+1}=\lim_{x\to\infty}\frac{D_x(4x+3)}{D_x(5x^2+1)}=\lim_{x\to\infty}\frac{(4)}{(10x)}=\frac{4}{\infty}=0$$

2) Find $\lim_{x\to-4}\frac{2x^2+13x+20}{x+4}$.

Since $\lim_{x\to-4}\frac{2x^2+13x+20}{x+4}=\frac{\lim_{x\to-4}(2x^2+13x+20)}{\lim_{x\to-4}(x+4)}=\frac{\infty}{\infty}$, we use L'Hopital's Rule

$$\lim_{x\to-4}\frac{2x^2+13x+20}{x+4}=\lim_{x\to-4}\frac{D_x(2x^2+13x+20)}{D_x(x+4)}=\lim_{x\to-4}\frac{(4x+13)}{(1)}=\frac{-3}{1}=-3$$

3) Find $\lim\limits_{x\to 5^-}\dfrac{\sqrt{25-x^2}}{x-5}$.

Since $\lim\limits_{x\to 5^-}\dfrac{\sqrt{25-x^2}}{x-5}=\dfrac{\lim\limits_{x\to 5^-}\sqrt{25-x^2}}{\lim\limits_{x\to 5^-}(x-5)}=\dfrac{0}{0}$, we use L'Hopital's Rule

$$\lim_{x\to 5^-}\frac{\sqrt{25-x^2}}{x-5}=\lim_{x\to 5^-}\frac{D_x\left[\sqrt{25-x^2}\right]}{D_x(x-5)}=\lim_{x\to 5^-}\frac{\left[\dfrac{-x}{\left(25-x^2\right)^{1/2}}\right]}{1}=\frac{\dfrac{-5}{(\to 0)^{1/2}}}{1}=\frac{-5}{(\to 0)^{1/2}}=\infty$$

$$\boxed{\begin{aligned}\mathit{Note}:\ D_x\left(\sqrt{25-x^2}\right)&=D_x\left[\left(25-x^2\right)^{1/2}\right]=\left[\frac{1}{2}\left(25-x^2\right)^{-1/2}(-2x)\right]\\&=\left[-x\left(25-x^2\right)^{-1/2}\right]=\frac{-x}{\left(25-x^2\right)^{1/2}}\end{aligned}}$$

4) Find $\lim_{x\to 1}\dfrac{\ln x^3}{x^2-1}$.

Recall: $\ln(1) = 0$; $\ln x^p = p\cdot\ln x$

Since $\lim_{x\to 1}\dfrac{\ln x^3}{x^2-1} = \lim_{x\to 1}\dfrac{\ln x^3}{x^2-1} = \lim_{x\to 1}\dfrac{3\ln x}{x^2-1} = \dfrac{0}{0}$, we use L'Hopital's Rule

$$\lim_{x\to 1}\frac{\ln x^3}{x^2-1} = \lim_{x\to 1}\frac{\ln x^3}{x^2-1} = \lim_{x\to 1}\frac{3\ln x}{x^2-1} = \lim_{x\to 1}\frac{D_x\left(3\ln x\right)}{D_x\left(x^2-1\right)} = \lim_{x\to 1}\frac{3\left(1/x\right)}{2x} = \frac{3}{2}$$

$$\boxed{\textit{Note}:\ \ D_x\left(\ln x\right) = \frac{1}{x}}$$

5) Find $\lim_{x\to 0}\dfrac{\sin 6x}{4x}$.

Since $\lim_{x\to 0}\dfrac{\sin 6x}{4x} = \dfrac{\lim_{x\to 0}\left(\sin 6x\right)}{\lim_{x\to 0}\left(4x\right)} = \dfrac{0}{0}$, we use L'Hopital's Rule.

$$\lim_{x\to 0}\frac{\sin 6x}{4x} = \lim_{x\to 0}\frac{D_x\left(\sin 6x\right)}{D_x\left(4x\right)} = \lim_{x\to 0}\frac{\left(6\cos 6x\right)}{\left(4\right)} = \frac{6\cdot\cos 0}{4} = \frac{6}{4} = \frac{3}{2}$$

$$\boxed{\textit{Note}:\ \ D_x\left(\sin 6x\right) = \cos 6x\cdot 6 = 6\cos 6x}$$

6) Find $\lim\limits_{x\to\infty}\dfrac{x^3}{e^{x^2}}$.

Since $\lim\limits_{x\to\infty}\dfrac{x^3}{e^{x^2}}=\dfrac{\lim\limits_{x\to\infty}\left(x^3\right)}{\lim\limits_{x\to\infty}\left(e^{x^2}\right)}=\dfrac{\infty}{\infty}$, we use L'Hopital's Rule.

$$\boxed{Note:\ \ D_x\left(e^{x^2}\right)=e^{x^2}\cdot D_x\left(x^2\right)=e^{x^2}\cdot(2x)}$$

$$\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{D_x\left(x^3\right)}{D_x\left(e^{x^2}\right)}=\lim_{x\to\infty}\frac{\left(3x^2\right)}{\left(e^{x^2}\cdot(2x)\right)}=\frac{\infty}{\infty},$$ we use L'Hopital's Rule again.

$$\boxed{\begin{aligned}Note:\ \ D_x\left(e^{x^2}\cdot(2x)\right)&=e^{x^2}\cdot D_x(2x)+2x\cdot D_x\left(e^{x^2}\right)\quad \text{Product Rule for derivative}\\ &=\ e^{x^2}\cdot(2)+2x\cdot\left(e^{x^2}\cdot(2x)\right)\\ &=\ 2e^{x^2}+4x^2\cdot\left(e^{x^2}\right)\end{aligned}}$$

$$\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{D_x\left(x^3\right)}{D_x\left(e^{x^2}\right)}=\lim_{x\to\infty}\frac{\left(3x^2\right)}{\left(e^{x^2}\cdot(2x)\right)}=\lim_{x\to\infty}\frac{D_x\left(3x^2\right)}{D_x\left(e^{x^2}\cdot(2x)\right)}=\lim_{x\to\infty}\frac{(6x)}{\left[2e^{x^2}+4x^2\cdot\left(e^{x^2}\right)\right]}=\frac{\infty}{\infty},$$ we use L'Hopital's Rule again

$$\boxed{Note:\ \ D_x\left(2e^{x^2}\right)=4xe^{x^2};\qquad D_x\left(2e^{x^2}+4x^2\cdot\left(e^{x^2}\right)\right)=8x^3e^{x^2}+8xe^{x^2}}$$

$$\lim_{x\to\infty}\frac{x^3}{e^{x^2}}=\lim_{x\to\infty}\frac{\left(x^3\right)}{\left(e^{x^2}\right)}=\lim_{x\to\infty}\frac{\left(3x^2\right)}{\left(e^{x^2}\cdot(2x)\right)}=\lim_{x\to\infty}\frac{(6x)}{\left[2e^{x^2}+4x^2\cdot\left(e^{x^2}\right)\right]}=\lim_{x\to\infty}\frac{\lim\limits_{x\to\infty}(6)}{\left[8x^3e^{x^2}+8xe^{x^2}\right]}=\frac{6}{\infty}=0$$

7) Find $\lim\limits_{x\to\infty} \dfrac{x}{\sqrt{x^2+1}}$.

$$\boxed{Note: \text{ If } x \to \infty,\ \sqrt{x^2} = |x| = x \ ; \qquad \text{If } x \to -\infty,\ \sqrt{x^2} = |x| = -x}$$

For this problem, L'Hopital's Rule is not useful.

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = \lim_{x\to\infty} \frac{x/\sqrt{x^2}}{\sqrt{x^2+1}/\sqrt{x^2}} \qquad \text{divide numerator and denominator by } \sqrt{x^2}$$

$$= \lim_{x\to\infty} \frac{x/x}{\sqrt{\left(x^2+1\right)/x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x^2}} = \frac{\lim\limits_{x\to\infty} 1}{\lim\limits_{x\to\infty} \sqrt{1+1/x^2}} = \frac{1}{\sqrt{1+0}} = 1$$

1) Find $\lim_{x\to\infty} \frac{\cos x}{x}$.

For this problem, L'Hopital's Rule is not useful.

Note: $-1 \le \cos x \le 1$

$$\frac{-1}{x} \le \frac{\cos x}{x} \le \frac{1}{x}$$

$$\lim_{x\to\infty} \frac{-1}{x} \le \lim_{x\to\infty} \frac{\cos x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

Because $\lim_{x\to\infty} \frac{-1}{x} = 0$ and $\lim_{x\to\infty} \frac{1}{x} = 0$, $\lim_{x\to\infty} \frac{\cos x}{x} = 0$ by Squeeze Theorem

8) Find $\lim_{x\to 0^+} x^{1/x}$.

Let $y = \lim_{x\to 0^+} x^{1/x}$

$\ln(y) = \ln\left(\lim_{x\to 0^+} x^{1/x}\right)$

$\ln(y) = \lim_{x\to 0^+}\left(\ln x^{1/x}\right)$

$\ln(y) = \lim_{x\to 0^+}\left(\frac{1}{x}\cdot \ln x\right)$ Note: $\ln x^p = p\cdot \ln x$

$\ln(y) = \lim_{x\to 0^+}\left(\frac{1}{x}\right)\cdot \lim_{x\to 0^+}(\ln x) = (\infty)(-\infty) = -\infty$

Note: As $x \to 0$ from the right, $\ln x \to -\infty$

$\ln(y) = -\infty$

$\log_e(y) = -\infty$ Note: ln is the same $\log_e$

$y = e^{-\infty}$ Note: Property of log: $\log_b y = x \Leftrightarrow y = b^x$

$y = 0$

$\lim_{x\to 0^+}\left(\ln x^{1/x}\right) = 0$

9) Find $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x$.

Let $y=\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad\Rightarrow\quad \ln(y)=\ln\left(\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right) \quad\Rightarrow\quad \ln(y)=\lim\limits_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)^x\right)$

$\Rightarrow \quad \ln(y)=\lim\limits_{x\to\infty}\left(x\cdot\ln\left(1+\frac{1}{x}\right)\right)$ Note: $\ln x^p = p\cdot\ln x$

$$\boxed{Note: x\cdot\ln\left(1+\frac{1}{x}\right)=\frac{\ln\left(1+\frac{1}{x}\right)}{1/x}}$$

$\ln(y)=\dfrac{\lim\limits_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)\right)}{\lim\limits_{x\to\infty}(1/x)}=\dfrac{\ln(1)}{0}=\dfrac{0}{0}$ we can use L'Hopital's Rule

$$\ln(y)=\frac{\lim\limits_{x\to\infty}\left(\ln\left(1+\frac{1}{x}\right)\right)}{\lim\limits_{x\to\infty}(1/x)}=\frac{\lim\limits_{x\to\infty}\left(\frac{1}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right)\right)}{\lim\limits_{x\to\infty}\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}\frac{\left(\frac{1}{\left(1+\frac{1}{x}\right)}\left(\frac{-1}{x^2}\right)\right)}{\left(\frac{-1}{x^2}\right)}=\lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)}=\frac{1}{1}=1$$

$\ln(y)=1 \quad\Rightarrow\quad \log_e(y)=1$ Note: ln is the same $\log_e$

$y=e^1=e$ Note: Property of log: $\log_b y=x \Leftrightarrow y=b^x$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$$