Improper Integrals

1) Find $\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx.$

Note: $f(x)=\frac{1}{(x-3)^{3/2}}$ is not continuous at $x=3$.

$$\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx = \lim_{b\to 3^+}\left[\int_b^4 \frac{1}{(x-3)^{3/2}}\,dx\right]$$

For $\int \frac{1}{(x-3)^{3/2}}\,dx$, let $u=x-3 \;\; \Rightarrow du=dx$

$$\int \frac{1}{(x-3)^{3/2}}\,dx = \int \frac{1}{u^{3/2}}\,du = \int u^{-3/2}du = -2u^{-1/2} = -2(x-3)^{-1/2} = \frac{-2}{(x-3)^{1/2}}$$

$$\int_3^4 \frac{1}{(x-3)^{3/2}}\,dx = \lim_{b\to 3^+}\left[\int_b^4 \frac{1}{(x-3)^{3/2}}\,dx\right] = \lim_{b\to 3^+}\left[\frac{-2}{(x-3)^{1/2}}\right]_b^4$$

$=$

$$\lim_{b\to 3^+}\left[\frac{-2}{(4-3)^{1/2}} - \frac{-2}{(b-3)^{1/2}}\right] = \frac{-2}{(4-3)^{1/2}} - (-\infty) = \frac{-2}{(4-3)^{1/2}} + (\infty) = \infty$$

2) Find $\int_{-\infty}^{0} e^{3x}\,dx.$

$$\int_{-\infty}^{0} e^{3x}\,dx = \lim_{b\to-\infty}\left[\int_{b}^{0} e^{3x}\,dx\right]$$

$$\boxed{\begin{aligned}&\text{For } \int e^{3x}\,dx, \quad \text{let } u = 3x \quad \Rightarrow du = 3dx \Rightarrow \frac{1}{3}du = dx\\ &\int e^{3x}\,dx = \frac{1}{3}\int e^{u}\,du = \frac{1}{3}e^{u} \ = \frac{1}{3}e^{3x}\end{aligned}}$$

$$\int_{-\infty}^{0} e^{3x}\,dx = \lim_{b\to-\infty}\left[\int_{b}^{0} e^{3x}\,dx\right] = \lim_{b\to-\infty}\left[\frac{1}{3}e^{3x}\right]_{b}^{0}$$

$$= \lim_{b\to-\infty}\left[\frac{1}{3}e^{0} - \frac{1}{3}e^{3b}\right] = \frac{1}{3}e^{0} - \frac{1}{3}e^{-\infty} = \frac{1}{3} - 0 = \frac{1}{3}$$

$Note$: As $b \to -\infty, e^{b} \to 0.$

3) Find $\int_{1}^{\infty} \frac{4}{\sqrt[4]{x}}\,dx.$

$$\int_{1}^{\infty} \frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\left[\int_{1}^{b} \frac{4}{\sqrt[4]{x}}\,dx\right] = \lim_{b\to\infty}\left[\int_{1}^{b} \frac{4}{x^{1/4}}\,dx\right] = \lim_{b\to\infty}\left[\int_{1}^{b} 4x^{-1/4}\,dx\right]$$

$$\boxed{\text{Note: } \int 4x^{-1/4}\,dx = 4\left(\frac{4}{3}x^{3/4}\right) = \frac{16}{3}x^{3/4}}$$

$$\int_{1}^{\infty} \frac{4}{\sqrt[4]{x}}\,dx = \lim_{b\to\infty}\left[\int_{1}^{b} 4x^{-1/4}\,dx\right] = \lim_{b\to\infty}\left[\frac{16}{3}x^{3/4}\right]_{1}^{b} = \lim_{b\to\infty}\left[\frac{16}{3}b^{3/4} - \frac{16}{3}\right]$$

$$= \infty - \frac{16}{3} = \infty$$

4) Find $\int_0^{\infty} e^{-x}\cos x dx.$

$$\int_0^{\infty} e^{-x}\cos x dx = \lim_{b\to\infty}\left[\int_0^{b} e^{-x}\cos x dx\right]$$

For $\int e^{-x}\cos x dx,$ use Formula #86 with $a=-1$ and $b=1$; $u=x \Rightarrow du=dx$

$$\int e^{-x}\cos x dx = \frac{e^{-x}}{2}\left(-\cos x+\sin x\right)$$

$$\int_0^{\infty} e^{-x}\cos x dx = \lim_{b\to\infty}\left[\int_0^{b} e^{-x}\cos x dx\right] = \lim_{b\to\infty}\left[\frac{e^{-x}}{2}\left(-\cos x+\sin x\right)\right]_0^b$$

$$=\lim_{b\to\infty}\left[\frac{e^{-b}}{2}\left(-\cos b+\sin b\right)-\frac{e^{0}}{2}\left(-\cos 0+\sin 0\right)\right]$$

$$=\lim_{b\to\infty}\left[\frac{e^{-b}}{2}\left(-\cos b+\sin b\right)-\frac{1}{2}\left(-1+0\right)\right]$$

$$=0-\frac{1}{2}\left(-1+0\right)=\frac{1}{2}$$

Note: As $b\to\infty, e^{-b}\to 0$; $b\to\infty, e^{-b}\left(-\cos b+\sin b\right)\to 0$

5) Find $\int_0^5 \frac{1}{\sqrt{25-x^2}}\,dx.$

Note: $f(x)=\frac{1}{\sqrt{25-x^2}}$ is discontinuous at $x=5$.

$$\int_0^5 \frac{1}{\sqrt{25-x^2}}\,dx=\lim_{b\to 5^-}\left[\int_0^b \frac{1}{\sqrt{25-x^2}}\,dx\right]$$

For $\int \frac{1}{\sqrt{25-x^2}}dx$, use Formula #41 with $a^2 = 25$ and $u^2=x^2$. Hence, $a=5$; $u=x$; $du=dx$ $\int \frac{1}{\sqrt{25-x^2}}dx=\arcsin\frac{x}{5}$

$$\int_0^5 \frac{1}{\sqrt{25-x^2}}\,dx=\lim_{b\to 5^-}\left[\int_0^b \frac{1}{\sqrt{25-x^2}}\,dx\right]=\lim_{b\to 5^-}\left[\arcsin\frac{x}{5}\right]_0^b$$

$$=\arcsin 1-\arcsin 0=\frac{\pi}{2}-0=\frac{\pi}{2}$$

6) Find $\int_5^{\infty} \frac{1}{x\sqrt{x^2-25}}\,dx.$

$Note:\ f(x)=\frac{1}{x\sqrt{x^2-25}}$ is discontinuous at $x=5.$

$$\int_5^{\infty} \frac{1}{x\sqrt{x^2-25}}\,dx = \lim_{b\to 5^+}\left[\int_b^{10} \frac{1}{x\sqrt{x^2-25}}\,dx\right] + \lim_{c\to\infty}\left[\int_{10}^{c} \frac{1}{x\sqrt{x^2-25}}\,dx\right]$$

For $\int \frac{1}{x\sqrt{x^2-25}}dx$, use Formula #33 with $a^2 = 25$ and $u^2 = x^2$. Hence, $a=5$; $u=x$; $du=dx$ $\int \frac{1}{x\sqrt{x^2-25}}dx = \frac{1}{5}arc\sec\frac{\lvert x\rvert}{5}$

$$\int_5^{\infty} \frac{1}{x\sqrt{x^2-25}}\,dx = \lim_{b\to 5^+}\left[\int_b^{10} \frac{1}{x\sqrt{x^2-25}}\,dx\right] + \lim_{c\to\infty}\left[\int_{10}^{c} \frac{1}{x\sqrt{x^2-25}}\,dx\right]$$

$$= \lim_{b\to 5^+}\left[\frac{1}{5}arc\sec\frac{\lvert x\rvert}{5}\right]_b^{10} + \lim_{c\to\infty}\left[\frac{1}{5}arc\sec\frac{\lvert x\rvert}{5}\right]_{10}^{c}$$

$$= \lim_{b\to 5^+}\left[\frac{1}{5}arc\sec\frac{\lvert 10\rvert}{5} - \frac{1}{5}arc\sec\frac{\lvert b\rvert}{5}\right] + \lim_{b\to 5^+}\left[\frac{1}{5}arc\sec\frac{\lvert c\rvert}{5} - \frac{1}{5}arc\sec\frac{\lvert 10\rvert}{5}\right]$$

$$= \frac{1}{5}arc\sec 2 - \frac{1}{5}arc\sec 1 + \frac{1}{5}arc\sec(\infty) - \frac{1}{5}arc\sec 2$$

$$= -\frac{1}{5}arc\sec 1 + \frac{1}{5}arc\sec(\infty) =$$

$$= 0 + \frac{1}{5}\frac{\pi}{2} = \frac{\pi}{10}$$ Note: $arc\sec(1)=0$; As $c\to\infty$, $arc\sec(c)\to\frac{\pi}{2}$

7) Find $\int_0^\infty \frac{4}{\sqrt{x}\left(x+6\right)}dx.$

$Note: \ f(x) = \frac{4}{\sqrt{x}\left(x+6\right)}$ is discontinuous at $x = 0.$

$$\int_0^\infty \frac{4}{\sqrt{x}\left(x+6\right)}dx = \lim_{b\to 0^+}\left[\int_b^1 \frac{4}{\sqrt{x}\left(x+6\right)}dx\right] + \lim_{c\to\infty}\left[\int_1^c \frac{4}{\sqrt{x}\left(x+6\right)}dx\right]$$

For $\int \frac{4}{\sqrt{x}\left(x+6\right)}dx$, let $u = \sqrt{x} \Rightarrow u^2 = \left(\sqrt{x}\right)^2 = x; \quad 2udu = dx$

$$\int \frac{4}{\sqrt{x}\left(x+6\right)}dx = 4\int \frac{1}{u\left(u^2+6\right)}2udu = 8\int \frac{1}{\left(u^2+6\right)}du$$

Using Formula 23 with $a^2 = 6$: $8\int \frac{1}{\left(u^2+6\right)}du = 8\left[\frac{1}{\sqrt{6}}\arctan\frac{u}{\sqrt{6}}\right] = 8\left[\frac{1}{\sqrt{6}}\arctan\frac{\sqrt{x}}{\sqrt{6}}\right]$

$$\int_0^\infty \frac{4}{\sqrt{x}\left(x+6\right)}dx = \lim_{b\to 0^+}\left[\int_b^1 \frac{4}{\sqrt{x}\left(x+6\right)}dx\right] + \lim_{c\to\infty}\left[\int_1^c \frac{4}{\sqrt{x}\left(x+6\right)}dx\right]$$

$$= \lim_{b\to 0^+}\left[8\left[\frac{1}{\sqrt{6}}\arctan\frac{\sqrt{x}}{\sqrt{6}}\right]_b^1\right] + \lim_{c\to\infty}\left[8\left[\frac{1}{\sqrt{6}}\arctan\frac{\sqrt{x}}{\sqrt{6}}\right]_1^c\right]$$

$$= \lim_{b\to 0^+}\left[8\left[\frac{1}{\sqrt{6}}\arctan\frac{\sqrt{1}}{\sqrt{6}}\right] - 8\left[\frac{1}{\sqrt{6}}\arctan\frac{\sqrt{b}}{\sqrt{6}}\right]\right] + \lim_{c\to\infty}\left[8\left[\frac{1}{\sqrt{6}}\arctan\frac{\sqrt{c}}{\sqrt{6}}\right] - 8\left[\frac{1}{\sqrt{6}}\arctan\frac{\sqrt{1}}{\sqrt{6}}\right]\right]$$

$$= 8\left[\frac{1}{\sqrt{6}}\arctan\frac{1}{\sqrt{6}}\right] - 8\left[\frac{1}{\sqrt{6}}\arctan 0\right] + 8\left[\frac{1}{\sqrt{6}}\arctan\frac{\infty}{\sqrt{6}}\right] - 8\left[\frac{1}{\sqrt{6}}\arctan\frac{1}{\sqrt{6}}\right]$$

$$= 8\left[\frac{1}{\sqrt{6}}\arctan\frac{1}{\sqrt{6}}\right] - 8\left[\frac{1}{\sqrt{6}}(0)\right] + 8\left[\frac{1}{\sqrt{6}}\frac{\pi}{2}\right] - 8\left[\frac{1}{\sqrt{6}}\arctan\frac{1}{\sqrt{6}}\right]$$

$$= 8\left[\frac{1}{\sqrt{6}}\arctan\frac{1}{\sqrt{6}}\right] + 8\left[\frac{1}{\sqrt{6}}\frac{\pi}{2}\right] - 8\left[\frac{1}{\sqrt{6}}\arctan\frac{1}{\sqrt{6}}\right]$$