9.1 Sequences

2, 4, 6, 8, 10, 12, 14, ... Infinite Sequence

$a_1 = 2$

$a_2 = 4$

$a_3 = 6$

$a_4 = 8$

$\vdots$

$a_n = n^{th}$ term

Sequence can be represented by

$$a_n = 2n$$

when $n = 1, 2, 3, \ldots$

$a_n = \left(-\frac{2}{5}\right)^n$ Geometric Sequence

$a_1 = \left(-\frac{2}{5}\right)^1 = -\frac{2}{5}$

$a_2 = \left(-\frac{2}{5}\right)^2 = \frac{4}{25}$

$a_3 = \left(-\frac{2}{5}\right)^3 = -\frac{8}{125}$

$a_4 = \left(-\frac{2}{5}\right)^4 = \frac{16}{625}$

$a_5 = \left(-\frac{2}{5}\right)^n = -\frac{32}{3125}$

$$a_n = \frac{3n}{n+4}$$

$$a_1 = \frac{3(1)}{1+4} = 3/5$$

$$a_{10} = \frac{3(10)}{10+4}$$

$$a_2 = \frac{3(2)}{2+4} = 6/6 = 1$$

$$a_3 = \frac{3(3)}{3+4} = 9/7$$

$$a_4 = \frac{3(4)}{4+4} = 12/8 = 3/2$$

$$a_5 = \frac{3(5)}{5+4} = 15/9 = 5/3$$

Sequence Recursively Defined

$a_1 = 4$

$a_{k+1} = 4a_k$

$a_2 = 4a_1 = 4(4) = 16$

$a_3 = 4a_2 = 4(16) = 64$

$a_4 = 4a_3 = 4(64) = 256$

$a_5 = 4a_4 = 4(256) = 1024$

Factorial

$0! = 1$

$1! = 1$

$2! = 2 \cdot 1 = 2$

$3! = 3 \cdot 2 \cdot 1 = 6$

$4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$

$5! = (5)(5-1)(5-2)(5-3)(5-4) = 120$

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$6! = (6)(6-1)(6-2)(6-3)(6-4)(6-5) = 720$

$7! = (7)(7-1)(7-2)(7-3)(7-4)(7-5)(7-6) = \ldots$

$$100! = (100)(100-1)(100-2)(100-3)\ldots\ldots(3)(2)(1)$$

$$120! = (120)(120-1)(120-2)(120-3)\ldots\ldots(3)(2)(1)$$

$$n! = (n)(n-1)(n-2)(n-3)\ldots\ldots(3)(2)(1)$$

$$(2n)! = (2n)(2n-1)(2n-2)(2n-3)\ldots\ldots(3)(2)(1)$$

$$(n+3)! = (n+3)(n+2)(n+1)(n)\ldots\ldots(3)(2)(1)$$

Simplify $\dfrac{n!}{(n+3)!} = \dfrac{\cancel{(n)(n-1)(n-2)(n-3)\ldots(3)(2)(1)}}{(n+3)(n+2)(n+1)\cancel{(n)(n-1)\ldots(3)(2)(1)}}$

$$= \frac{1}{(n+3)(n+2)(n+1)}$$

Simplify

$$\frac{(2n+2)!}{(2n)!} = \frac{(2n+2)(2n+1)\cancel{(2n)(2n-1)\cdots(3)(2)(1)}}{\cancel{(2n)(2n-1)(2n-2)(2n-3)\cdots(3)(2)(1)}}$$

$$= (2n+2)\cdot(2n+1)$$

$$= 4n^2 + 6n + 2$$

Limits of Sequence

$$a_n = 6 + \frac{2}{n^2}$$

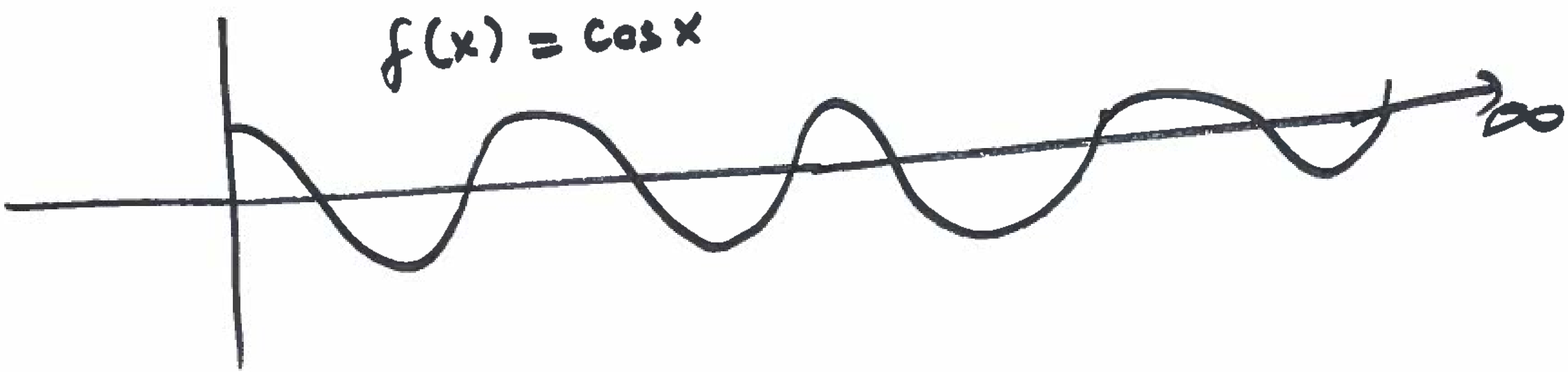
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(6 + 2/n^2\right) = 6 + 0 = 6$$

$$a_n = \cos(2/n)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(0) = 1$$

$$a_n = \cos(n)$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \cos(n) = \text{DNE}$$



Note

$$\sqrt[n]{\infty} = \infty$$

$$a_n = \frac{\sqrt[3]{n}}{\sqrt[3]{n} + 1}$$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{\sqrt[3]{n}}{\sqrt[3]{n}+1}\right) = \frac{\infty}{\infty}$$

We can apply L'Hopital's Rule.

$$L = \lim_{n\to\infty}\left(\frac{\frac{1}{3}n^{-2/3}}{\frac{1}{3}n^{-2/3}}\right) = \lim_{n\to\infty}(1) = 1$$

$$a_n = \frac{5^n}{3^n}$$

$$L = \lim_{n\to\infty} \frac{5^n}{3^n} = \frac{\infty}{\infty}$$

We can apply L'Hopital's Rule

$$L = \lim_{n\to\infty} \frac{\ln(5)\cdot 5^n}{\ln(3)\cdot 3^n} = \frac{\infty}{\infty}$$

So L'Hôpital is not useful in this case.

$$a_n = \frac{5^n}{3^n} = \left(\frac{5}{3}\right)^n$$ Geometric Sequence

$$a_1 = (5/3)^1 = 1.6$$

$$a_2 = (5/3)^2 = 2.8$$

$$a_3 = (5/3)^3 = 4.6$$

$$a_4 = (5/3)^4 = 7.7$$

$$a_5 = (5/3)^5 = 12.8$$

$\vdots$

$$a_{100} = (5/3)^{100} = 1.53 \times 10^{22}$$

$$\lim_{n\to\infty} \left(\frac{5}{3}\right)^n = \infty$$

In general for Geometric Sequence of the form $a_n = r^n$

$\lim\limits_{n\to\infty} a_n = 0$ if $-1 < r < 1$

$\lim\limits_{n\to\infty} a_n = \infty$ if $r > 1$

$\lim\limits_{n\to\infty} a_n =$ diverges if $r < -1$

$\lim\limits_{n\to\infty} a_n = 1$ if $r = 1$

$\lim\limits_{n\to\infty} a_n$ diverges if $r = -1$

$a_n = \left(\frac{1}{2}\right)^n$ $r = \frac{1}{2}$

Geometric sequence

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{1}{2}\right)^n = 0$

$a_n = \left(-\frac{1}{4}\right)^n$ $r = -\frac{1}{4}$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(-\frac{1}{4}\right)^n = 0$

$a_n = \left(\frac{5}{4}\right)^n$ $r = \frac{5}{4} = 1.25$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{5}{4}\right)^n = \infty$

$$a_n = \frac{\ln(n^3)}{2n} = \frac{3 \cdot \ln n}{2n}.$$

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{3\cdot \ln n}{2n}\right) = \frac{\infty}{\infty}$$

We can apply L'Hopital's Rule.

$$L = \lim_{n\to\infty}\left(\frac{3\cdot \frac{1}{n}}{2}\right) = \frac{3\cdot 0}{2} = 0$$

$$a_n = \frac{(n-2)!}{n!} = \frac{\cancel{(n-2)(n-3)(n-4)\cdots(3)(2)(1)}}{(n)(n-1)\cancel{(n-2)(n-3)\cdots(3)(2)(1)}}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{1}{n\cdot(n-1)}\right) = \frac{1}{\infty} = 0$$

$$a_n = -3^{-n} = -\frac{1}{3^n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(-\frac{1}{3^n}\right) = \frac{-1}{\infty} = 0$$