

Taylor Series Centered at  $c$ :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$$

$$\begin{aligned} f(x) &= \frac{f(c)}{0!} (x-c)^0 + \frac{f'(c)}{1!} (x-c)^1 + \frac{f''(c)}{2!} (x-c)^2 \\ &+ \frac{f'''(c)}{3!} (x-c)^3 + \frac{f^{(4)}(c)}{4!} (x-c)^4 + \dots \\ &+ \frac{f^{(n)}(c)}{n!} (x-c)^n + \dots \end{aligned}$$

Example 1:

Find Taylor series for the function  $f(x) = e^{4x}$  centered at  $c = 0$ ?

$$f(x) = e^{4x}$$

$$\text{a) } f(0) = e^{4x} = e^0 = 1 = 4^0$$

$$\text{b) } f'(x) = 4e^{4x}$$

$$f'(0) = 4e^{4x} = 4e^0 = 4 = 4^1$$

$$\text{c) } f''(0) = 16e^{4x}$$

$$f''(0) = 16e^{4x} = 16e^0 = 16 = 4^2$$

$$\text{d) } f'''(0) = 64e^{4x}$$

$$f'''(0) = 64e^{4x} = 64e^0 = 64 = 4^3$$

$$\text{e) } f^{(4)}(0) = 256e^{4x}$$

$$f^{(4)}(0) = 256e^{4x} = 256e^0 = 256 = 4^4$$

$$\text{f) } f^{(5)}(0) = 1024e^{4x}$$

$$f^{(5)}(0) = 1024e^{4x} = 1024e^0 = 1024 = 4^5$$

Example 1 (con't):

g) Find Taylor series for the function  $f(x) = e^{4x}$  centered at  $c = 0$ ?

Taylor Series Centered at  $c$ :

$$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 \\ + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

$$f(x) = \frac{4^0}{0!}(x-c)^0 + \frac{4^1}{1!}(x-c)^1 + \frac{4^2}{2!}(x-c)^2 + \frac{4^3}{3!}(x-c)^3 \\ + \frac{4^4}{4!}(x-c)^4 + \dots + \frac{4^n}{n!}(x-c)^n + \dots \\ = \sum_{n=0}^{\infty} \frac{4^n}{n!}(x-0)^n = \sum_{n=0}^{\infty} \frac{4^n}{n!}(x)^n$$

Example 2:

Find a Taylor series for the function  $f(x) = \sin 2x$  centered at  $c = \frac{\pi}{4}$ .

a)  $f(\pi/4) = \sin 2x = \sin(2 \cdot \pi/4) = \sin(\pi/2) = 1$

b)  $f'(\pi/4) = 2 \cos(2x)$

$$f'(\pi/4) = 2 \cos(2x) = 2 \cos(2 \cdot \pi/4) = 2 \cos(\pi/2) = 2(0) = 0$$

c)  $f''(\pi/4) = -4 \sin(2x)$

$$f''(\pi/4) = -4 \sin(2x) = -4 \sin(2 \cdot \pi/4) = -4 \sin(\pi/2) = -4$$

d)  $f'''(\pi/4) = -8 \cos(2x)$

$$f'''(\pi/4) = -8 \cos(2x) = -8 \cos(2 \cdot \pi/4) = -8 \cos(\pi/2) = 0$$

e)  $f^{(4)}(\pi/4) = 16 \sin(2x)$

$$f^{(4)}(\pi/4) = 16 \sin(2x) = 16 \sin(2 \cdot \pi/4) = 16 \sin(\pi/2) = 16$$

f)  $f^{(5)}(\pi/4) = 32 \cos(2x)$

$$f^{(5)}(\pi/4) = 32 \cos(2x) = 32 \cos(2 \cdot \pi/4) = 32 \cos(\pi/2) = 0$$

g) Taylor series for the function  $f(x) = \sin 2x$  centered at  $c = \frac{\pi}{4}$ ?

Example 2 (con't):

Find a Taylor series for the function  $f(x) = \sin 2x$  centered at  $c = \frac{\pi}{4}$ .

$$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

$$\begin{aligned} f(x) &= \frac{2^0}{0!}(x - \pi/4)^0 + 0 + \frac{-2^2}{2!}(x - \pi/4)^2 + 0 \\ &+ \frac{2^4}{4!}(x - \pi/4)^4 + 0 + \frac{-2^6}{6!}(x - \pi/4)^6 + 0 \\ &+ \frac{2^8}{8!}(x - \pi/4)^8 + 0 + \frac{-2^{10}}{10!}(x - \pi/4)^{10} + 0 \\ &+ \frac{2^{12}}{12!}(x - \pi/4)^{12} + 0 + \frac{-2^{14}}{14!}(x - \pi/4)^{14} + 0 \\ &+ \frac{2^{16}}{16!}(x - \pi/4)^{16} + 0 + \frac{-2^{18}}{18!}(x - \pi/4)^{18} + 0 \\ &+ \dots \end{aligned}$$

Example 2 (con't):

Find a Taylor series for the function  $f(x) = \sin 2x$  centered at  $c = \frac{\pi}{4}$ :

$$\begin{aligned} f(x) &= \frac{2^0}{0!} (x - \pi/4)^0 - \frac{2^2}{2!} (x - \pi/4)^2 + \frac{2^4}{4!} (x - \pi/4)^4 \\ &\quad - \frac{2^6}{6!} (x - \pi/4)^6 + \frac{2^8}{8!} (x - \pi/4)^8 - \frac{2^{10}}{10!} (x - \pi/4)^{10} \\ &\quad + \frac{2^{12}}{12!} (x - \pi/4)^{12} - \frac{2^{14}}{14!} (x - \pi/4)^{14} + \frac{2^{16}}{16!} (x - \pi/4)^{16} \\ &\quad - \frac{2^{18}}{18!} (x - \pi/4)^{18} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} (x - \pi/4)^{2n} \end{aligned}$$

Example 3:

Find Taylor series for the function  $f(x) = \frac{1}{x-1}$  centered at  $c = 2$ .

$$\text{Find } f(x) = \frac{1}{x-1}$$

$$\text{Find } f'(x) = -1 \left( \frac{1}{x-1} \right)^{-2} = -(1!) \left( \frac{1}{x-1} \right)^{-2}$$

$$\text{Find } f''(x) = 2 \left( \frac{1}{x-1} \right)^{-3} = (2!) \left( \frac{1}{x-1} \right)^{-3}$$

$$\text{Find } f'''(x) = -6 \left( \frac{1}{x-1} \right)^{-4} = -(3!) \left( \frac{1}{x-1} \right)^{-4}$$

$$\text{Find } f^{(4)}(x) = 24 \left( \frac{1}{x-1} \right)^{-5} = (4!) \left( \frac{1}{x-1} \right)^{-5}$$

$$\text{Find } f^{(5)}(x) = -120 \left( \frac{1}{x-1} \right)^{-6} = -(5!) \left( \frac{1}{x-1} \right)^{-6}$$

Example 3 (con't):

Find Taylor series for the function  $f(x) = \frac{1}{x-1}$  centered at  $c = 2$ .

$$f(2) = \frac{1}{2-1} = 1 = 0!$$

$$f'(2) = -(1!) \left( \frac{1}{2-1} \right)^{-2} = -1!$$

$$f''(2) = (2!) \left( \frac{1}{2-1} \right)^{-3} = 2!$$

$$f'''(2) = -(3!) \left( \frac{1}{2-1} \right)^{-4} = -3!$$

$$f^{(4)}(2) = (4!) \left( \frac{1}{2-1} \right)^{-5} = 4!$$

$$f^{(5)}(2) = -(5!) \left( \frac{1}{2-1} \right)^{-6} = -5!$$



Example 3 (con't):

Find Taylor series for the function  $f(x) = \frac{1}{x-1}$  centered at  $c = 2$ .

$$f(x) = \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 \\ + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots$$

$$f(x) = \frac{0!}{0!}(x-2)^0 + \frac{-1!}{1!}(x-2)^1 + \frac{2!}{2!}(x-2)^2 + \frac{-3!}{3!}(x-2)^3 \\ + \frac{4!}{4!}(x-2)^4 + \frac{-5!}{5!}(x-2)^5 + \frac{6!}{6!}(x-2)^6 + \dots$$

$$f(x) = (x-2)^0 - (x-2)^1 + (x-2)^2 - (x-2)^3 + (x-2)^4 \\ - (x-2)^5 + (x-2)^6 + \dots \\ = \sum_{n=0}^{\infty} (-1)^n (x-2)^n$$

Example 4:

Find Taylor series for the function  $f(x) = \ln(x+1)$  centered at  $c = 0$ .

a)  $f(x) = \ln(x+1)$

b)  $f'(x) = \frac{1}{x+1} = \frac{(1-1)!}{x+1}$

c)  $f''(x) = -\frac{1}{(x+1)^2} = -\frac{(2-1)!}{(x+1)^2}$

d)  $f'''(x) = \frac{2}{(x+1)^3} = \frac{(3-1)!}{(x+1)^3}$

e)  $f^{(4)}(x) = \frac{-6}{(x+1)^4} = \frac{-(4-1)!}{(x+1)^4}$

f)  $f^{(5)}(x) = \frac{24}{(x+1)^5} = \frac{(5-1)!}{(x+1)^5}$

Example 4 (con't):

Find Taylor series for the function  $f(x) = \ln(x+1)$  centered at  $c = 0$ .

$$\text{a) } f(0) = \ln(x+1) = \ln(0+1) = 0$$

$$\text{b) } f'(1) = \frac{1}{x+1} = \frac{(1-1)!}{1+1} = \frac{(1-1)!}{2^1}$$

$$\text{c) } f''(1) = -\frac{(2-1)!}{(x+1)^2} = -\frac{(2-1)!}{(1+1)^2} = -\frac{(2-1)!}{(2)^2} =$$

$$\text{d) } f'''(1) = \frac{(3-1)!}{(x+1)^3} = \frac{(3-1)!}{(1+1)^3} = \frac{(3-1)!}{(2)^3}$$

$$\text{e) } f^{(4)}(1) = \frac{-(4-1)!}{(x+1)^4} = \frac{-(4-1)!}{(1+1)^4} = \frac{-(4-1)!}{(2)^4}$$

$$\text{f) } f^{(5)}(1) = \frac{(5-1)!}{(x+1)^5} = \frac{(5-1)!}{(1+1)^5} = \frac{(5-1)!}{(2)^5}$$

Example 4 (con't):

Find Taylor series for the function  $f(x) = \ln(x+1)$  centered at  $c = 0$ .

$$\begin{aligned} f(x) &= \frac{f(c)}{0!}(x-c)^0 + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 \\ &+ \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \dots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \dots \\ &= \frac{0}{0!}(x-0)^0 + \frac{(1-1)!}{1!} \frac{(2-1)!}{(2)^2} (x-0)^1 + \frac{(2-1)!}{2!} (x-0)^2 \\ &+ \frac{(3-1)!}{(2)^3} (x-0)^3 + \frac{-(4-1)!}{4!} (x-0)^4 + \frac{(5-1)!}{5!} (x-0)^5 \\ &+ \dots \end{aligned}$$

Example 4 (con't):

Find Taylor series for the function  $f(x) = \ln(x+1)$  centered at  $c = 0$ .

$$f(x) = \ln(x+1)$$

$$= \frac{(1-1)!}{2^1 1!} (x)^1 - \frac{(2-1)!}{(2)^2 2!} (x)^2 + \frac{(3-1)!}{(2)^3 3!} (x)^3 - \frac{(4-1)!}{(2)^4 4!} (x)^4$$

$$+ \frac{(5-1)!}{(2)^5 5!} (x)^5 + \dots$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(n-1)!}{(2)^n n!} (x)^n$$