

Section 9.2 Infinite Series

Let $a_n = 2n$.

Terms of Sequence a_n : $a_1 = 2$; $a_2 = 4$; $a_3 = 6$; $a_4 = 8$; $a_5 = 10$; $a_6 = 12$;

Infinite Series: $\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + 10 + 12 + \dots$

Sequence of Partial Sums:

$$S_1 = a_1 = 2$$

$$S_2 = a_1 + a_2 = 2 + 4 = 6$$

$$S_3 = a_1 + a_2 + a_3 = 2 + 4 + 6 = 12$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = 20$$

$$S_5 = a_1 + a_2 + a_3 + a_4 = 30$$

$$S_6 = a_1 + a_2 + a_3 + a_4 = 42$$

$$S_7 = a_1 + a_2 + a_3 + a_4 = 56$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$S_k = a_1 + a_2 + a_3 + a_4 + \dots + a_k$$

$S_1, S_2, S_3, S_4, S_5, S_6, S_7, \dots, S_k, \dots$ is called Sequence of Partial Sums.

Note: $\lim_{k \rightarrow \infty} S_k = \sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + 8 + 10 + 12 + \dots$

Let $a_n = \left(\frac{2}{5}\right)^n$.

Terms of Sequence a_n : $a_1 = \left(\frac{2}{5}\right)^1$; $a_2 = \left(\frac{2}{5}\right)^2$; $a_3 = \left(\frac{2}{5}\right)^3$; $a_4 = \left(\frac{2}{5}\right)^4$; $a_5 = \left(\frac{2}{5}\right)^5$; $a_6 = \left(\frac{2}{5}\right)^6$;

Infinite Series: $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \left(\frac{2}{5}\right)^6 + \dots$

Sequence of Partial Sums:

$$S_1 = a_1 = \left(\frac{2}{5}\right)^1$$

$$S_2 = a_1 + a_2 = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2$$

$$S_3 = a_1 + a_2 + a_3 = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4$$

$$S_5 = a_1 + a_2 + a_3 + a_4 = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5$$

$$S_6 = a_1 + a_2 + a_3 + a_4 = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \left(\frac{2}{5}\right)^6$$

$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$

$$S_k = a_1 + a_2 + a_3 + a_4 + \dots + a_k = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \left(\frac{2}{5}\right)^6 + \dots + \left(\frac{2}{5}\right)^k$$

$S_1, S_2, S_3, S_4, S_5, S_6, S_7, \dots, S_k, \dots$ is called Sequence of Partial Sums.

Note: $\lim_{k \rightarrow \infty} S_k = \sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \left(\frac{2}{5}\right)^6 + \dots + \left(\frac{2}{5}\right)^k + \dots$

Geometric Sequence

Geometric Sequence Review:

Let $a_n = \left(\frac{2}{5}\right)^n$. Then $r = 2/5$. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{2}{5}\right)^n = 0$ because r is between -1 and 1.

Let $a_n = \left(-\frac{2}{3}\right)^n$. Then $r = -2/3$. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(-\frac{2}{3}\right)^n = 0$ because r is between -1 and 1.

Let $a_n = (5/2)^n$. Then $r = 5/2$. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (5/2)^n = \infty$ because $r > 1$.

Let $a_n = (-5/4)^n$. Then $r = -5/4$. $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-5/4)^n$ diverges because $r < -1$.

Geometric Series

Show that the series $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ converges to $\frac{2}{3}$.

The series $\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n$ is called a Geometric Series of the form $\sum_{n=1}^{\infty} r^n$

$$\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n = \left(\frac{2}{5}\right)^1 + \left(\frac{2}{5}\right)^2 + \left(\frac{2}{5}\right)^3 + \left(\frac{2}{5}\right)^4 + \left(\frac{2}{5}\right)^5 + \left(\frac{2}{5}\right)^6 + \dots$$

The first term of the series is $a = \left(\frac{2}{5}\right)^1 = \left(\frac{2}{5}\right)$

$$r = 2/5$$

Geometric Series Theorem states that if r is between -1 and 1, then

$$\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n \text{ converges to } \frac{a}{1-r} = \frac{2/5}{1-2/5} = \frac{2/5}{3/5} = \frac{2}{3}$$

Show that the series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ converges to 4.

The series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$ is called a Geometric Series of the form $\sum_{n=1}^{\infty} r^n$

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n = \left(\frac{3}{4}\right)^0 + \left(\frac{3}{4}\right)^1 + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \left(\frac{3}{4}\right)^5 + \left(\frac{3}{4}\right)^6 + \dots$$

The first term of the series is $a = \left(\frac{3}{4}\right)^0 = 1$

$$r = \text{common ratio} = 3/4$$

Geometric Series Theorem states that if r is between -1 and 1, then

$$\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \text{ converges to } \frac{a}{1-r} = \frac{1}{1-3/4} = \frac{1}{1/4} = 4$$

Geometric Series

Show that the series $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$ diverges.

The series $\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n$ is called a Geometric Series of the form $\sum_{n=1}^{\infty} r^n$

$$\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n = \left(\frac{5}{4}\right)^0 + \left(\frac{5}{4}\right)^1 + \left(\frac{5}{4}\right)^2 + \left(\frac{5}{4}\right)^3 + \left(\frac{5}{4}\right)^4 + \left(\frac{5}{4}\right)^5 + \left(\frac{5}{4}\right)^6 + \dots$$

The first term of the series is $a = \left(\frac{5}{4}\right)^0 = 1$

$r = \text{common ratio} = 5/4$

Geometric Series Theorem states that if r is not between -1 and 1, then

$$\sum_{n=0}^{\infty} \left(\frac{5}{4}\right)^n \text{ diverges to } \infty.$$

Show that the series $\sum_{n=0}^{\infty} (-1.04)^n$ diverges.

The series $\sum_{n=0}^{\infty} (-1.04)^n$ is called a Geometric Series of the form $\sum_{n=1}^{\infty} r^n$

$$\sum_{n=0}^{\infty} (-1.04)^n = (-1.04)^0 + (-1.04)^1 + (-1.04)^2 + (-1.04)^3 + (-1.04)^4 + (-1.04)^5 + (-1.04)^6 + \dots$$

The first term of the series is $a = (-1.04)^0 = 1$

$r = \text{common ratio} = -1.04$

Geometric Series Theorem states that if r is not between -1 and 1, then

$$\sum_{n=0}^{\infty} (-1.04)^n \text{ diverges.}$$

Geometric Series

Show that the series $\sum_{n=0}^{\infty} 7(0.15)^n$ converges.

The series $\sum_{n=0}^{\infty} 7(0.15)^n$ is called a Geometric Series.

$$\sum_{n=0}^{\infty} 7(0.15)^n = 7 \sum_{n=0}^{\infty} (0.15)^n = 7 \left[(0.15)^0 + (0.15)^1 + (0.15)^2 + (0.15)^3 + (0.15)^4 + (0.15)^5 + (0.15)^6 + \dots \right]$$

The first term of the series $\sum_{n=0}^{\infty} (0.15)^n$ is $a = (0.15)^0 = 1$

$r = \text{common ratio} = 0.15$

Geometric Series Theorem states that if r is between -1 and 1, then

$$\sum_{n=0}^{\infty} (0.15)^n \text{ converges } \frac{a}{1-r} = \frac{1}{1-0.15} = \frac{1}{0.85} = \frac{20}{17}$$

Therefore, $\sum_{n=0}^{\infty} 7(0.15)^n$ converges to $7 \left[\frac{20}{17} \right] = \frac{140}{17}$

Geometric Series

Show that the series $\sum_{n=0}^{\infty} (1)^n$ diverges.

The series $\sum_{n=0}^{\infty} (1)^n$ is called a Geometric Series.

$$\sum_{n=0}^{\infty} (1)^n = \sum_{n=0}^{\infty} (1)^n = (1)^0 + (1)^1 + (1)^2 + (1)^3 + (1)^4 + (1)^5 + (1)^6 + \dots = \infty$$

Therefore, the series $\sum_{n=0}^{\infty} (1)^n$ diverges to ∞ .

Show that the series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

$$\sum_{n=1}^{\infty} (-1)^n = \sum_{n=1}^{\infty} (-1)^n = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5 + (-1)^6 + \dots$$

$$S_1 = a_1 = (-1)^1 = -1$$

$$S_2 = a_1 + a_2 = (-1)^1 + (-1)^2 = 0$$

$$S_3 = a_1 + a_2 + a_3 = (-1)^1 + (-1)^2 + (-1)^3 = -1$$

$$S_4 = a_1 + a_2 + a_3 + a_4 = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = 0$$

$$S_5 = a_1 + a_2 + a_3 + a_4 = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 = -1$$

$$S_6 = a_1 + a_2 + a_3 + a_4 = (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 + (-1)^5 + (-1)^6 = 0$$

Hence the series $\sum_{n=1}^{\infty} (-1)^n$ oscillates between 0 and -1.

Therefore, the series $\sum_{n=1}^{\infty} (-1)^n$ diverges.

Telescoping Series

Show that the telescoping series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right)$ converges.

$$\text{Note: } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = 0$$

$$a_1 = \frac{1}{1} - \frac{1}{2}; \quad a_2 = \frac{1}{2} - \frac{1}{3}; \quad a_3 = \frac{1}{3} - \frac{1}{4};$$

$$a_{k-1} = \frac{1}{k-1} - \frac{1}{k-1+1} = \frac{1}{k-1} - \frac{1}{k}$$

$$a_k = \frac{1}{k} - \frac{1}{k+1}$$

$$S_k = a_1 + a_2 + a_3 + \cdots + a_{k-1} + a_k$$

$$S_k = \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \cdots + \left(\frac{1}{k-1} - \frac{1}{k} \right) + \left(\frac{1}{k} - \frac{1}{k+1} \right)$$

$$S_k = \frac{1}{1} - \frac{1}{k+1} = 1 - \frac{1}{k+1}$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left(1 - \frac{1}{k+1} \right) = 1 - 0 = 1$$

Telescoping Series

Show that the telescoping series $\sum_{n=1}^{\infty} \ln\left(\frac{n+3}{n+4}\right)$ diverges.

$$\text{Note: } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \ln\left(\frac{n+3}{n+4}\right) = \ln\left(\lim_{n \rightarrow \infty} \left[\frac{n+3}{n+4}\right]\right) = \ln(1) = 0$$

$$\text{Note: } \ln\left(\frac{n+3}{n+4}\right) = \ln(n+3) - \ln(n+4)$$

$$a_1 = \ln(4) - \ln(5); \quad a_2 = \ln(5) - \ln(6); \quad a_3 = \ln(6) - \ln(7);$$

$$a_{k-1} = \ln(k-1+3) - \ln(k-1+4) = \ln(k+2) - \ln(k+3)$$

$$a_k = \ln(k+3) - \ln(k+4)$$

$$S_k = a_1 + a_2 + a_3 + \cdots + a_{k-1} + a_k$$

$$S_k = (\ln(4) - \ln(5)) + (\ln(5) - \ln(6)) + (\ln(6) - \ln(7)) + \cdots + (\ln(k+2) - \ln(k+3)) + (\ln(k+3) - \ln(k+4))$$

$$S_k = \ln(4) - \ln(k+4)$$

$$\sum_{n=1}^{\infty} \ln\left(\frac{n+3}{n+4}\right) = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} [\ln(4) - \ln(k+4)] = \ln(4) - \ln(\infty) = -\infty$$

Therefore, $\sum_{n=1}^{\infty} \ln\left(\frac{n+3}{n+4}\right)$ diverges to $-\infty$.

Note: Even though $\lim_{n \rightarrow \infty} a_n = 0$, $\sum_{n=1}^{\infty} \ln\left(\frac{n+3}{n+4}\right)$ still diverges to $-\infty$.

Telescoping Series

Show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)}$ converges to $\frac{1}{2}$.

Using Partial Fraction Decomposition: $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) \\ &\quad + \left(\frac{1}{6} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{8} \right) + \cdots + \left(\frac{1}{(k-1)+1} - \frac{1}{(k-1)+2} \right) \\ &\quad + \left(\frac{1}{(k)+1} - \frac{1}{(k)+2} \right) + \cdots \end{aligned}$$

$S_k =$ Sum of the first k th terms $= a_1 + a_2 + a_3 + a_4 + \cdots + a_k$

$$\begin{aligned} &= \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \left(\frac{1}{4} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{6} \right) + \left(\frac{1}{6} - \frac{1}{7} \right) + \left(\frac{1}{7} - \frac{1}{8} \right) \\ &\quad + \cdots + \left(\frac{1}{(k-1)+1} - \frac{1}{(k-1)+2} \right) + \left(\frac{1}{(k)+1} - \frac{1}{(k)+2} \right) \end{aligned}$$

$$S_k = \frac{1}{2} - \frac{1}{k+2}$$

Therefore: $\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2} \right) = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left(\frac{1}{2} - \frac{1}{k+2} \right) = \frac{1}{2} - \frac{1}{\infty} = \frac{1}{2}$

Telescoping Series

Show that the infinite series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$ converges to $\frac{1}{6}$.

Using Partial Fraction Decomposition: $\frac{1}{(2n+1)(2n+3)} = \frac{1/2}{2n+1} - \frac{1/2}{2n+3}$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} &= \sum_{n=1}^{\infty} \left(\frac{1/2}{2n+1} - \frac{1/2}{2n+3} \right) = \left(\frac{1/2}{3} - \frac{1/2}{5} \right) + \left(\frac{1/2}{5} - \frac{1/2}{7} \right) + \left(\frac{1/2}{7} - \frac{1/2}{9} \right) \\ &\quad + \left(\frac{1/2}{9} - \frac{1/2}{11} \right) + \left(\frac{1/2}{11} - \frac{1/2}{13} \right) + \left(\frac{1/2}{13} - \frac{1/2}{15} \right) \\ &\quad + \dots + \left(\frac{1/2}{2(k-1)+1} - \frac{1/2}{2(k-1)+3} \right) + \left(\frac{1/2}{2(k)+1} - \frac{1/2}{2(k)+3} \right) + \dots \end{aligned}$$

$S_k =$ Sum of the first k th terms $= a_1 + a_2 + a_3 + a_4 + \dots + a_k$

$$\begin{aligned} S_k &= \left(\frac{1/2}{3} - \frac{1/2}{5} \right) + \left(\frac{1/2}{5} - \frac{1/2}{7} \right) + \left(\frac{1/2}{7} - \frac{1/2}{9} \right) + \left(\frac{1/2}{9} - \frac{1/2}{11} \right) \\ &\quad + \left(\frac{1/2}{11} - \frac{1/2}{13} \right) + \left(\frac{1/2}{13} - \frac{1/2}{15} \right) + \dots + \left(\frac{1/2}{2(k-1)+1} - \frac{1/2}{2(k-1)+3} \right) \\ &\quad + \left(\frac{1/2}{2(k)+1} - \frac{1/2}{2(k)+3} \right) \end{aligned}$$

$$S_k = \frac{1/2}{3} - \frac{1/2}{2(k)+3}$$

$$\text{Therefore: } \sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)} = \sum_{n=1}^{\infty} \left(\frac{1/2}{2n+1} - \frac{1/2}{2n+3} \right)$$

$$= \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left(\frac{1/2}{3} - \frac{1/2}{2(k)+3} \right) = \frac{1/2}{3} - \frac{1/2}{\infty} = \frac{1/2}{3} = \frac{1}{6}$$

Telescoping Series

Show that the infinite series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$ converges to 1.

Using Partial Fraction Decomposition: $\frac{2}{4n^2 - 1} = \frac{-1}{2n+1} + \frac{1}{2n-1}$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} &= \sum_{n=1}^{\infty} \left(\frac{-1}{2n+1} + \frac{1}{2n-1} \right) = \left(\frac{-1}{3} + \frac{1}{1} \right) + \left(\frac{-1}{5} + \frac{1}{3} \right) + \left(\frac{-1}{7} + \frac{1}{5} \right) + \left(\frac{-1}{9} + \frac{1}{7} \right) \\ &\quad + \left(\frac{-1}{11} + \frac{1}{9} \right) + \left(\frac{-1}{13} + \frac{1}{11} \right) + \cdots + \left(\frac{-1}{2(k-1)+1} + \frac{1}{2(k-1)-1} \right) \\ &\quad + \left(\frac{-1}{2(k)+1} + \frac{1}{2(k)-1} \right) + \cdots\end{aligned}$$

$S_k =$ Sum of the first k th terms $= a_1 + a_2 + a_3 + a_4 + \cdots + a_k$

$$\begin{aligned}&= \left(\frac{-1}{3} + \frac{1}{1} \right) + \left(\frac{-1}{5} + \frac{1}{3} \right) + \left(\frac{-1}{7} + \frac{1}{5} \right) + \left(\frac{-1}{9} + \frac{1}{7} \right) + \left(\frac{-1}{11} + \frac{1}{9} \right) + \left(\frac{-1}{13} + \frac{1}{11} \right) \\ &\quad + \cdots + \left(\frac{-1}{2(k-1)+1} + \frac{1}{2(k-1)-1} \right) + \left(\frac{-1}{2(k)+1} + \frac{1}{2(k)-1} \right)\end{aligned}$$

$$S_k = \frac{1}{1} + \frac{-1}{2(k)+1} = 1 + \frac{-1}{2(k)+1}$$

Therefore: $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1} = \sum_{n=1}^{\infty} \frac{-1}{2n+1} + \frac{1}{2n-1} = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left(1 + \frac{-1}{2(k)+1} \right) = 1 + \frac{1}{\infty} = 1$

Telescoping Series

Show that the infinite series $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$ converges to 3.

Using Partial Fraction Decomposition: $\frac{4}{n(n+2)} = \frac{2}{n} + \frac{-2}{n+2}$

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{4}{n(n+2)} &= \sum_{n=1}^{\infty} \left(\frac{2}{n} + \frac{-2}{n+2} \right) = \left(\frac{2}{1} + \frac{-2}{3} \right) + \left(\frac{2}{2} + \frac{-2}{4} \right) + \left(\frac{2}{3} + \frac{-2}{5} \right) + \left(\frac{2}{4} + \frac{-2}{6} \right) \\ &\quad + \left(\frac{2}{5} + \frac{-2}{7} \right) + \left(\frac{2}{6} + \frac{-2}{9} \right) + \left(\frac{2}{7} + \frac{-2}{10} \right) + \left(\frac{2}{8} + \frac{-2}{11} \right) + \left(\frac{2}{9} + \frac{-2}{12} \right) \\ &\quad + \cdots + \left(\frac{2}{(k-2)} + \frac{-2}{(k-2)+2} \right) + \left(\frac{2}{(k-1)} + \frac{-2}{(k-1)+2} \right) \\ &\quad + \left(\frac{2}{(k)} + \frac{-2}{(k)+2} \right) + \left(\frac{2}{(k+1)} + \frac{-2}{(k+1)+2} \right) + \cdots\end{aligned}$$

$S_k =$ Sum of the first k th terms $= a_1 + a_2 + a_3 + a_4 + \cdots + a_k$

$$\begin{aligned}&= \left(\frac{2}{1} + \frac{-2}{3} \right) + \left(\frac{2}{2} + \frac{-2}{4} \right) + \left(\frac{2}{3} + \frac{-2}{5} \right) + \left(\frac{2}{4} + \frac{-2}{6} \right) \\ &\quad + \left(\frac{2}{5} + \frac{-2}{7} \right) + \left(\frac{2}{6} + \frac{-2}{9} \right) + \left(\frac{2}{7} + \frac{-2}{10} \right) + \left(\frac{2}{8} + \frac{-2}{11} \right) + \left(\frac{2}{9} + \frac{-2}{12} \right) \\ &\quad + \cdots + \left(\frac{2}{(k-2)} + \frac{-2}{(k-2)+2} \right) + \left(\frac{2}{(k-1)} + \frac{-2}{(k-1)+2} \right) + \left(\frac{2}{(k)} + \frac{-2}{(k)+2} \right) + \cdots\end{aligned}$$

$$S_k = \frac{2}{1} + \frac{2}{2} + \frac{-2}{(k-1)+2} + \frac{-2}{(k)+2}$$

$$\begin{aligned}\text{Therefore: } \sum_{n=1}^{\infty} \frac{4}{n(n+2)} &= \sum_{n=1}^{\infty} \left(\frac{2}{n} + \frac{-2}{n+2} \right) = \lim_{k \rightarrow \infty} S_k = \lim_{k \rightarrow \infty} \left(\frac{2}{1} + \frac{2}{2} + \frac{-2}{(k-1)+2} + \frac{-2}{(k)+2} \right) \\ &= 2 + 1 + \frac{-2}{\infty} + \frac{-2}{\infty} = 3\end{aligned}$$

Harmonic Series

The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \dots = \infty$

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞ .

nth Term Test

Use the nth Term Test to show that the series $\sum_{n=1}^{\infty} \frac{4n}{n+5}$ diverges.

$$\text{Let } a_n = \frac{4n}{n+5}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{4n}{n+5} = 4/1 = 4$$

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} \frac{4n}{n+5}$ diverges.

Use the nth Term Test to show that the series $\sum_{n=1}^{\infty} \frac{n^2}{n^2+2}$ diverges.

$$\text{Let } a_n = \frac{n^2}{n^2+2}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n^2}{n^2+2} = 1$$

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} \frac{n^2}{n^2+2}$ diverges.

nth Term Test

Use the nth Term Test to show that the series $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$ diverges.

$$\text{Let } a_n = \left(\frac{5}{2}\right)^n$$

Hence, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \left(\frac{5}{2}\right)^n = \infty$ (Note: This is a Geometric Sequence.)

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$ diverges.

Use the nth Term Test to show that the series $\sum_{n=1}^{\infty} \frac{3n^2 + 5n + 4}{n^2 + n + 4}$ diverges.

$$\text{Let } a_n = \frac{3n^2 + 5n + 4}{n^2 + n + 4}$$

Hence, $\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{3n^2 + 5n + 4}{n^2 + n + 4} = 3$

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n \neq 0$ then the series $\sum_{n=1}^{\infty} \frac{3n^2 + 5n + 4}{n^2 + n + 4}$ diverges.

nth Term Test

What does the nth Term Test say about the series $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$?

$$\text{Let } a_n = \frac{4}{n(n+2)}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{4}{n(n+2)} = 0.$$

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$ may or may not converge.

It was shown earlier that the telescoping series $\sum_{n=1}^{\infty} \frac{4}{n(n+2)}$ converges.

What does the nth Term Test say about the series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$?

$$\text{Let } a_n = \frac{1}{(2n+1)(2n+3)}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{(2n+1)(2n+3)} = 0.$$

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$ may or may not converge.

It was shown earlier that the telescoping series $\sum_{n=1}^{\infty} \frac{1}{(2n+1)(2n+3)}$ converges.

nth Term Test

What does the nth Term Test say about the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$?

$$\text{Let } a_n = \frac{2}{4n^2 - 1}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{2}{4n^2 - 1} = 0.$$

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$ may or may not converge.

It was shown earlier that the telescoping series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$ converges.

What does the nth Term Test say about the series $\sum_{n=1}^{\infty} \frac{1}{n}$?

$$\text{Let } a_n = \frac{1}{n}$$

$$\text{Hence, } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

nth Term Test states that if $\lim_{n \rightarrow \infty} a_n = 0$ then the series $\sum_{n=1}^{\infty} \frac{1}{n}$ may or may not converge.

It was shown earlier that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

nth Term Test Summary:

The series $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \rightarrow \infty} a_n \neq 0$.

The series $\sum_{n=1}^{\infty} a_n$ may or may not converge if $\lim_{n \rightarrow \infty} a_n = 0$.