Section 9.3 Integral Test and p-series Test

Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ ....$ Harmonic Series

Corresponding continuous function $f(x) = \frac{1}{x} \quad x \geq 1$

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{b \to \infty} \int_1^b \frac{1}{x} dx = \lim_{b \to \infty} \left(\ln|x| \Big|_1^b \right) = \lim_{b \to \infty} \left(\ln|b| - \ln 1 \right) = \infty = \textit{diverges}$$

The Integral Test states that if the corresponding integral $\int_1^{\infty} \frac{1}{x} dx$ diverges,

then the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Corresponding continuous function $f(x) = \frac{1}{x^2} \qquad x \geq 1$

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{b\to\infty} \int_1^b x^{-2}\,dx = \lim_{b\to\infty}\left(-x^{-1}\Big|_1^b \right) = \lim_{b\to\infty}\left(\frac{-1}{x}\Big|_1^b \right) = \lim_{b\to\infty}\left(\frac{-1}{b} - \frac{-1}{1} \right) = 1 = \text{ converges}$$

The Integral Test states that if the corresponding integral $\int_1^{\infty} \frac{1}{x^2}\,dx$ converges,

then the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Corresponding continuous function $f(x) = \frac{1}{x^3} \qquad x \geq 1$

$$\int_1^{\infty} \frac{1}{x^3}\,dx = \lim_{b\to\infty} \int_1^b x^{-3}\,dx = \lim_{b\to\infty}\left(\frac{x^{-2}}{-2}\Big|_1^b \right) = \lim_{b\to\infty}\left(-\frac{1}{2x^2}\Big|_1^b \right) = \lim_{b\to\infty}\left(\frac{-1}{2b^2} - \frac{-1}{2} \right) = 2 = \text{ converges}$$

The Integral Test states that if the corresponding integral $\int_1^{\infty} \frac{1}{x^3}\,dx$ converges,

then the series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{2}{3n+5}$ diverges.

Corresponding continuous function $f(x) = \frac{2}{3x+5} \qquad x \geq 1$

$$\int_1^{\infty} \frac{2}{3x+5}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{2}{3x+5}\,dx$$

$$Note: \int_1^{b} \frac{2}{3x+5}\,dx = 2\int_1^{b} \frac{1}{3x+5}\,dx = 2\left[\frac{1}{3}\ln|3x+5|\right]_1^b = 2\left[\frac{1}{3}\ln|3b+5|\right] - 2\left[\frac{1}{3}\ln|8|\right]$$

$$\int_1^{\infty} \frac{2}{3x+5}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{2}{3x+5}\,dx = \lim_{b\to\infty}\left[2\left[\frac{1}{3}\ln|3b+5|\right] - 2\left[\frac{1}{3}\ln|8|\right]\right]$$

$$= \infty - 2\left[\frac{1}{3}\ln|8|\right] = \infty = diverges$$

The Integral Test states that if the corresponding integral $\int_1^{\infty} \frac{2}{3x+5}\,dx$ diverges,

then the series $\sum_{n=1}^{\infty} \frac{2}{3n+5}$ diverges.

Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{n^2+25}$ converges.

Corresponding continuous function $f(x) = \frac{1}{x^2+25} \qquad x \geq 1$

$$\int_1^{\infty} \frac{1}{x^2+25}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x^2+25}\,dx$$

$$Note: \int_1^{b} \frac{1}{x^2+25}\,dx = \left[\frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right)\right]_1^b = \left[\frac{1}{5}\tan^{-1}\left(\frac{b}{5}\right)\right] - \left[\frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right]$$

$$\int_1^{\infty} \frac{2}{3x+5}\,dx = \lim_{b\to\infty}\left[\left[\frac{1}{5}\tan^{-1}\left(\frac{b}{5}\right)\right] - \left[\frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right]\right] = \left[\frac{1}{5}\tan^{-1}(\infty)\right] - \left[\frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right]$$

$$= \left[\frac{1}{5}\cdot\frac{\pi}{2}\right] - \left[\frac{1}{5}\tan^{-1}\left(\frac{1}{5}\right)\right] = \text{finite} = \text{converges} \qquad \text{Note: } \tan^{-1}(\infty) = \frac{\pi}{2}$$

The Integral Test states that if the corresponding integral $\int_1^{\infty} \frac{1}{x^2+25}\,dx$ converges,

then the series $\sum_{n=1}^{\infty} \frac{1}{n^2+25}$ diverges.

Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty}\left(\frac{5}{2}\right)^{n}$ diverges.

Corresponding continuous function $f(x)=\left(\frac{5}{2}\right)^{x} \qquad x\geq 1$

$$\int_{1}^{\infty}\left(\frac{5}{2}\right)^{x}dx=\lim_{b\to\infty}\int_{1}^{b}\left(\frac{5}{2}\right)^{x}dx$$

$$Note: \int\left(\frac{5}{2}\right)^{x}dx=\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^{x}$$

$$\int_{1}^{\infty}\left(\frac{5}{2}\right)^{x}dx=\lim_{b\to\infty}\int_{1}^{b}\left(\frac{5}{2}\right)^{x}dx=\lim_{b\to\infty}\left[\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^{b}-\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^{1}\right]$$

$$=\left[\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^{\infty}-\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)^{1}\right]=\infty-\frac{1}{\ln(5/2)}\left(\frac{5}{2}\right)=\infty$$

The Integral Test states that if the corresponding integral $\int_{1}^{\infty}\left(\frac{5}{2}\right)^{x}dx$ diverges,

then the series $\sum_{n=1}^{\infty}\left(\frac{5}{2}\right)^{n}$ diverges.

Integral Test

Show that the series $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ diverges by using the Integral Test.

$$\int_1^{\infty}\frac{\ln x}{x}dx = \lim_{b\to\infty}\int_1^{b}\ln x\cdot\frac{1}{x}dx$$

$$Note: \quad \text{Let } u = \ln x \Rightarrow du = \frac{1}{x}dx$$

$$\int\frac{\ln x}{x}dx = \int\ln x\cdot\frac{1}{x}dx = \int u\cdot du = \frac{u^2}{2} = \frac{(\ln x)^2}{2}$$

$$\int_1^{\infty}\frac{\ln x}{x}dx = \lim_{b\to\infty}\int_1^{b}\frac{\ln x}{x}dx = \lim_{b\to\infty}\left[\frac{(\ln b)^2}{2} - \frac{(\ln 1)^2}{2}\right]$$

$$= \frac{(\ln\infty)^2}{2} - \frac{(\ln 1)^2}{2} = \infty - 0 = \infty$$

The Integral Test states that if the corresponding integral $\int_1^{\infty}\frac{\ln x}{x}dx$ diverges,

then the series $\sum_{n=1}^{\infty}\frac{\ln n}{n}$ diverges.

Integral Test

Show that the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by using the Integral Test.

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{\ln x}{x^2}\,dx$$

$$Note: \int \frac{\ln x}{x^2}\,dx = \int x^{-2}\ln x\,dx = \frac{x^{-1}}{1}\left[-1+(-1)\ln x\right] \qquad \text{Using Formula 89}$$

$$= \frac{1}{x}\left[-1-1\ln x\right] = \frac{-1-\ln x}{x}$$

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\left[\frac{-1-\ln b}{b} - \frac{-1-\ln 1}{1^2}\right]$$

$$= \lim_{b\to\infty}\left[\frac{-1-\ln b}{b} - \frac{-1-\ln 1}{1^2}\right] = 0+1=1$$

$$\text{Note: } \lim_{b\to\infty}\left[\frac{-1-\ln b}{b}\right] = \lim_{b\to\infty}\left[\frac{-\frac{1}{b}}{b}\right] = \lim_{b\to\infty}\left[\frac{-1}{b^2}\right] = 0 \quad \text{Using L'Hopital's Rule}$$

The Integral Test states that if the corresponding integral $\int_1^{\infty} \frac{\ln x}{x^2}\,dx$ converges,

then the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

Integral Test

Use the Integral Test to show that the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges.

Corresponding continuous function $f(x) = \frac{\ln x}{x^3} \qquad x \geq 1$

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^3}\,dx$$

$$Note: \int \frac{\ln x}{x^3}\,dx = \int x^{-3} \ln x dx = \frac{x^{-2}}{4}\left[-1+(-2)\ln x\right] \qquad \text{Using Formula 89}$$

$$= \frac{1}{4x^2}\left[-1-2\ln x\right] = \frac{-1-2\ln x}{4x^2}$$

$$\int_1^{\infty} \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty} \int_1^{b} \frac{\ln x}{x^3}\,dx = \lim_{b\to\infty}\left[\frac{-1-2\ln b}{4b^2} - \frac{-1-2\ln 1}{4(1)^2}\right]$$

$$= \lim_{b\to\infty}\left[\frac{-1-2\ln b}{4b^2} - \frac{-1-2\ln 1}{4(1)^2}\right] = 0 - \frac{-1-2\ln 1}{4(1)^2} = \text{ finite = converges}$$

$$\text{Note: } \lim_{b\to\infty}\left[\frac{-1-\ln b}{b^2}\right] = \lim_{b\to\infty}\left[\frac{-\frac{1}{b}}{2b}\right] = \lim_{b\to\infty}\left[\frac{-1}{2b^2}\right] = 0 \quad \text{Using L'Hopital's Rule}$$

The Integral Test states that if the corresponding integral $\int_1^{\infty} \frac{\ln x}{x^3}\,dx$ converges,

then the series $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges.

p-series Test

We have already seen:

The series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges by the Integral Test.

The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the Integral Test.

The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by the Integral Test.

p-series Test:

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 1$.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

p-series Test

Use p-series Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = 1/2$; therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges.

p-series Test

Use p-series Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{x^3}}$ diverges.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{x^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ is a p-series with $p = 3/4;$ therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{x^3}}$ diverges.

p-series Test

Use p-series Test to show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{x^5}}$ converges.

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{x^5}} = \sum_{n=1}^{\infty} \frac{1}{n^{5/3}}$ is a p-series with $p = 5/3;$ therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{x^5}}$ converges.

p-series Test

Use p-series Test to show that the series $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt[3]{n^4}} + \frac{1}{n^3} \right)$ converges.

$\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt[3]{n^4}} + \frac{1}{n^3} \right) = \sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}} + \sum_{n=1}^{\infty} \frac{1}{n^3}.$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is a p-series with $p = 4/3;$ therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n^4}}$ converges.

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ is a p-series with $p = 3;$ therefore, $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

In summary: $\sum_{n=1}^{\infty} \left(\frac{1}{\sqrt[3]{n^4}} + \frac{1}{n^3} \right)$ converges.

p-series Test

Use p-series Test to show that the series $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)$ diverges.

$$\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^7}}+\sum_{n=1}^{\infty}\frac{1}{n}.$$

$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^7}} = \sum_{n=1}^{\infty}\frac{1}{n^{7/3}}$ is a p-series with $p=7/3;$ therefore, $\sum_{n=1}^{\infty}\frac{1}{\sqrt[7]{n^4}}$ converges.

$\sum_{n=1}^{\infty}\frac{1}{n}$ is a p-series with $p=1;$ therefore, $\sum_{n=1}^{\infty}\frac{1}{n}$ divverges to ∞.

In summary: $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)=\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n^7}}+\sum_{n=1}^{\infty}\frac{1}{n} = \infty;$ $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt[3]{n^7}}+\frac{1}{n}\right)$ diverges.