

Section 9.4

Basic Comparison Test and Limit Comparison Test

Recall:

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because it a p-series with $p = 1$

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because it a p-series with $p = 2$

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges because it a p-series with $p = 3$

$\sum_{n=1}^{\infty} \frac{1}{n^4}$ converges because it a p-series with $p = 4$

Recall:

$\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by Integral Test.

$\sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges by Integral Test.

$\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by Integral Test.

$\sum_{n=1}^{\infty} \frac{\ln n}{n^4}$ converges by Integral Test.

Basic Comparison Test:

If $\sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} b_n$ converges then the series $\sum_{n=1}^{\infty} a_n$ also converges.

If $\sum_{n=1}^{\infty} b_n \leq \sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverges then the series $\sum_{n=1}^{\infty} a_n$ also diverges.

Use the Direct Comparison Test (DCT) to show that the

series $\sum_{n=1}^{\infty} \frac{1}{5 + 2^n}$ converges.

Note: $5 + 2^n > 2^n$

$$\Rightarrow \frac{1}{5 + 2^n} < \frac{1}{2^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{5 + 2^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}$$

The series $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ is a Geometric Series with $r = \frac{1}{2}$.

Hence, $\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ converges by Geometric Series Test.

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{5 + 2^n}$ converges by Direct Comparison Test

Use the Direct Comparison Test (DCT) to show that the

series $\sum_{n=1}^{\infty} \frac{1}{5 + \sqrt{n^3}}$ converges.

Note: $5 + \sqrt{n^3} > \sqrt{n^3}$

$$\Rightarrow \frac{1}{5 + \sqrt{n^3}} < \frac{1}{\sqrt{n^3}}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{5 + \sqrt{n^3}} < \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$$

The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a p -series with $p = 3/2$.

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges by p-series Test.

Therefore, the series $\sum_{n=1}^{\infty} \frac{1}{5 + \sqrt{n^3}}$ converges by Direct Comparison Test

Use the Direct Comparison Test (DCT) to show that the

series $\sum_{n=1}^{\infty} \frac{4^n}{5^n + 7}$ converges.

Note: $\frac{4^n}{5^n + 7} < \frac{4^n}{5^n}$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4^n}{5^n + 7} < \sum_{n=1}^{\infty} \frac{4^n}{5^n}$$

The series $\sum_{n=1}^{\infty} \frac{4^n}{5^n} = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ is a Geometric Series with $r = \frac{4}{5}$.

Hence, $\sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^n$ converges by Geometric Series Test.

Therefore, the series $\sum_{n=1}^{\infty} \frac{4^n}{5^n + 7}$ converges by Direct Comparison Test

Use the Direct Comparison Test (DCT) to show that the

series $\sum_{n=1}^{\infty} \frac{4}{3^n + 1}$ converges.

$$\text{Note: } \frac{4}{3^n + 1} < \frac{4}{3^n}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{4}{3^n + 1} < \sum_{n=1}^{\infty} \frac{4}{3^n}$$

The series $\sum_{n=1}^{\infty} \frac{4}{3^n} = \sum_{n=1}^{\infty} 4 \left(\frac{1}{3}\right)^n = 4 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$.

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ is a Geometric Series with $r = \frac{1}{3}$.

Hence, $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ converges by Geometric Series Test.

Which means $4 \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$ also converges.

Therefore, the series $\sum_{n=1}^{\infty} \frac{4}{3^n + 1}$ converges by Direct Comparison Test

Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-4}}$ diverges by using the Basic Comparison Test.

$$\text{Let } a_n = \frac{1}{\sqrt{n-4}} \quad \text{and} \quad b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$$

$$\text{Compare } a_n \text{ and } b_n : \frac{1}{\sqrt{n-4}} > \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}. \quad \text{Hence, } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-4}} > \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

What do we know about $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$?

$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a p-series with $p = 1/2$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges to ∞ by p-series Test.

$$\text{So, } \sum_{n=1}^{\infty} \frac{1}{\sqrt{n-4}} > \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} = \infty.$$

We have shown that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-4}} > \infty$; and therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-4}} = \infty$.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n-4}}$ diverges by Basic Comparison Test.

Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3 - 5}}$ diverges by using the Basic Comparison Test.

$$\text{Let } a_n = \frac{1}{\sqrt[4]{n^3 - 5}} \quad \text{and} \quad b_n = \frac{1}{\sqrt[4]{n^3}} = \frac{1}{n^{3/4}}$$

$$\text{Compare } a_n \text{ and } b_n : \frac{1}{\sqrt[4]{n^3 - 5}} > \frac{1}{\sqrt[4]{n^3}} = \frac{1}{n^{3/4}}. \quad \text{Hence, } \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3 - 5}} > \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$$

What do we know about $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$?

$\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ is a p-series with $p = 3/4$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$ diverges to ∞ by p-series Test.

$$\text{So, } \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3 - 5}} > \sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/4}} = \infty.$$

We have shown that $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3 - 5}} > \infty$; and therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3 - 5}} = \infty$.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n^3 - 5}}$ diverges by Basic Comparison Test.

Limit Comparison Test:

If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L > 0$, then either both series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge; or both series diverge.

Use the Limit Comparison Test (LCT) to show that the

series $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$ converges.

$$\text{Let } a_n = \frac{1}{n^2(n+3)}; \quad \text{Let } b_n = \frac{1}{n^2(n)} = \frac{1}{n^3}$$

Recall: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

$$\begin{aligned} \lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) &= \lim_{n \rightarrow \infty} \left(\frac{\frac{1}{n^2(n+3)}}{\frac{1}{n^3}} \right) = \lim_{n \rightarrow \infty} \left(\frac{n^3}{n^2(n+3)} \right) = \lim_{n \rightarrow \infty} \left(\frac{n^3}{n^3 + 3n^2} \right) \\ &= 1 \quad \text{Using L'Hopital's Rule} \end{aligned}$$

Therefore, $\sum_{n=1}^{\infty} \frac{1}{n^2(n+3)}$ converges because $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges.

Use the Limit Comparison Test (LCT) to show that the

series $\sum_{n=1}^{\infty} \frac{\ln n}{n^2 + 1}$ converges.

$$\text{Let } a_n = \frac{\ln n}{n^2 + 1}; \quad \text{Let } b_n = \frac{\ln n}{n^2}$$

Recall: $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

$$\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{\frac{\ln n}{n^2 + 1}}{\frac{\ln n}{n^2}} \right) = \lim_{n \rightarrow \infty} \left(\frac{n^2}{n^2 + 1} \right) = 1$$

Using L'Hopital's Rule

Therefore, the series $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2 + 1}$ converges because

$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{\ln n}{n^2}$ converges.

Show that the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+5}}$ diverges by using the Limit Comparison Test.

$$\text{Let } a_n = \frac{1}{\sqrt{n^2+5}} \quad \text{and} \quad b_n = \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

$$\text{Find } \lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) = \lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+5}} \cdot \frac{\sqrt{n^2}}{1} \right) = \lim_{n \rightarrow \infty} \left(\frac{\sqrt{n^2}}{\sqrt{n^2+5}} \right) = \lim_{n \rightarrow \infty} \left(\sqrt{\frac{n^2}{n^2+5}} \right) = \sqrt{\lim_{n \rightarrow \infty} \frac{n^2}{n^2+5}} = \sqrt{1} = 1 > 0$$

Limit Comparison Test says that if $\lim_{n \rightarrow \infty} \left(\frac{a_n}{b_n} \right) > 0$ then either

both series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+5}}$ and $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}}$ converge or both diverge.

What do we know about $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2}} = \sum_{n=1}^{\infty} \frac{1}{n}$?

$\sum_{n=1}^{\infty} \frac{1}{n}$ is a p-series with $p = 1$. Hence, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞ by p-series Test.

Therefore, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+5}}$ also diverges to ∞ because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞ .